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Two approaches to the use of unbounded operators in Feynman's operational calculus

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Abstract. In this paper, we investigate two approaches to the use of unbounded operators in Feynman's operational calculus. The first involves using a functional calculus for unbounded operators introduced by A. E. Taylor in the paper [34]. The second approach uses analytic families of closed unbounded operators as discussed in [19]. For each approach, we discuss the essential properties of the operational calculus as well as continuity (or stability) properties. Finally, for the approach using the Taylor calculus, we discussion a connection between Feynman's operational calculus in this setting with the Modified Feynman Integral of M. L. Lapidus ([14, 20]).

Contents

1. Introduction

The subject of this paper is, as the reader has no doubt deduced, Feynman's operational calculus – the formation of functions of several not necessarily commuting operators (originated in [5]). The aim of this paper, again as is clear from its title, is to introduce two methods of using unbounded operators in the abstract approach to Feynman's operational calculus (originated in [8,

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9, 10, 11] and discussed in great detail in [15].) The approach to Feynman's operational calculus introduced by G. W. Johnson and B. Jefferies in the late 1990's allows only bounded operators to be used. Later work by Johnson, Jefferies and the author ([12]) introduced the (typically unbounded) generator of a (C_0) semigroup into the abstract approach to the operational calculus. The use of such generators has been, to now, the only way any unbounded operators could be used in the mathematically rigorous approach to the operational calculus. No other unbounded operators could be accommodated. (The use of unbounded semigroup generators was introduced in [12] in a way that was consistent with the abstract approach, but was somewhat informally done. The recent paper [30] incorporated the semigroup generator in a much more formal way via certain Banach algebras of entire functions.)

The first method introduced which enables unbounded operators to be used in the operational calculus comes from the 1951 paper [34] by A. E. Taylor. In this paper, Taylor introduces a functional calculus for unbounded operators using methods from complex analysis. His approach is similar to the approach he and N. Dunford (see [33], [2] and [3]) took when developing an operational calculus for bounded linear operators. This approach for bounded operators A was carried out by choosing the algebra of functions $f(z)$ which are analytic on some open set containing the spectrum $\sigma(A)$ of A. The function $f(A)$ was defined by

$$f(A) = \frac{1}{2\pi i}\int_{\Gamma} f(z)(zI - A)^{-1}dz,$$

the curve Γ being the boundary of a suitable bounded domain containing $\sigma(A)$. In [34], Taylor shows how to develop an operational calculus which can be applied to any closed linear operator whose spectrum does not cover the entire plane (i.e., the resolvent set of the operator must be nonempty). This operational calculus is defined in such a way that the contour integral above is part of the general theory for a closed operator. In fact, the corresponding expression for an unbounded closed operator B takes the form

$$f(B) = f(\infty)I + \frac{1}{2\pi i}\int_{\partial D} f(z)(zI - B)^{-1}dz,$$

where D is a particular domain in $\mathbb{C}$, called a Cauchy domain and $f(\infty)$ denotes the value of f at the point at infinity, where f is assumed to be nicely behaved on a neighborhood of $\sigma(B)$ which includes a neighborhood of infinity. The operator $f(B)$ is shown to be a bounded linear operator.

In Section 3 of the present paper, we make use of the calculus of A. E. Taylor to obtain bounded linear operators from unbounded closed operators by using certain analytic functions f as above for each closed unbounded operator under consideration. Once we have the bounded linear operators $f_1(A_1), \ldots, f_n(A_n)$, we can then use the mathematically rigorous setting for Feynman's operational calculus in [15] to form functions

$$g\left(f_1(A_1), \ldots, f_n(A_n)\right)$$

of the not necessarily commuting bounded operators $f_1(A_1), \ldots, f_n(A_n)$.

In Subsection 3.1, we apply the Feynman's operational calculus in the setting of the Taylor calculus to discuss a connection between the operational calculus and the Modified Feynman Integral of M. L. Lapidus (see [20] and [14]). In the "usual" setting of the operational calculus, this connection cannot be made; it requires the use of the Taylor calculus.

Finally, in Subsection 3.2, we investigate how the operational calculus in the Taylor calculus setting behaves when we have appropriately convergent sequences $\{f_{j,k}\}_{k=1}^{\infty}$, $j = 1, \ldots, n$, of analytic functions. It is shown that (to state the conclusion informally)

$$g\left(f_{1,k}(A_1), \ldots, f_{n,k}(A_n)\right) \longrightarrow g\left(f_1(A_1), \ldots, f_n(A_n)\right).$$

This result is in the spirit of the stability theory for Feynman's operational calculus developed by the author in the papers [15], [16], [21], [22], [23], [24], [25], [27], [28], [29] as well as [13], [14]. See Subsection 4.4 below for a short discussion of what we mean by "stability" in the context of Feynman's operational calculus. Also, see [15] for a detailed discussion of the stability theory for Feynman's operational calculus.

The second method we introduce in this paper – Section 5 – for the incorporation of unbounded operators into Feynman's operational calculus involves the use of Kato's analytic families of closed unbounded operators (see [19, Chapter 7]). Somewhat informally, an analytic family of closed unbounded operators is an analytic operator-valued function $T(z)$, $z \in D_0 \subseteq \mathbb{C}$. The definition of an analytic family gives analytic operator-valued functions $U(z)$ and $V(z)$ on D_0 into $\mathcal{L}(X)$ (i.e., "boundedly analytic families" or "boundedly analytic function") for which

$$T(z)U(z) = V(z),$$

where $U(z)$ is injective onto the domain $\mathcal{D}(T(z))$ of $T(z)$. Consequently, we are able to deal with the unbounded operator $T(z)$ by working with the bounded operator $V(z)$. Hence, if we have analytic families $T_1(z), \ldots, T_n(z)$ on D_0 we obtain boundedly analytic operator-valued functions $U_1(z), V_1(z), \ldots, U_n(z), V_n(z)$ where $T_j(z)U_j(z) = V_j(z)$. The boundedly analytic function $U_j(z)$ is injective onto the domain of $T_j(z)$. Once we have the boundedly analytic operator-valued functions $V_1(z), \ldots, V_n(z)$, we can then apply the rigorous setting for Feynman's operational calculus to obtain functions of the operators $V_1(z), \ldots, V_n(z)$ and so, indirectly, of the closed unbounded operators $T_1(z), \ldots, T_n(z)$. See Subsections 2.6 and 2.7 for a brief discussion of analytic families of operators. (For a detailed discussion, see [19, Chapter 7].)

As we did with the Taylor calculus setting, after we show how to use analytic families on the operational calculus, we turn to stability issues. Here, we choose sequences $\{T_{j,k}(z)\}_{k=1}^{\infty}$, $j = 1, \ldots, n$, of analytic families of closed unbounded operators. What is needed is the appropriate idea of convergence, as with unbounded operators we cannot consider norm convergence. The convergence we use is Kato's "generalized convergence" of closed unbounded operators [19, Chapter 4, Section 2]. If we have a sequence $\{T_n\}_{n=1}^{\infty}$ of closed and unbounded operators, we say that $\{T_n\}_{n=1}^{\infty}$ converges to the closed unbounded operator T if the "gap" between the graphs of T_n and the graph of T goes to zero as $n \to \infty$. (See Subsections 2.4 and 2.5 for a brief review of these ideas.) By using this idea of generalized convergence of sequences of unbounded closed operators, we obtain a stability-like result for the operational calculus in the setting of analytic families of operators. (We note that the generalized convergence of sequences of closed unbounded operators can be expressed in terms of the norm convergence of the resolvents of the operators in question. However, this approach was not found to be as useful as the idea of the gap between graphs of the operators.)

We now turn to a brief discussion of Feynman's operational calculus. This operational calculus dates back to the work of R. P. Feynman, in particular to his 1951 paper [5]. In his 1951 paper, Feynman discussed the computation (or "formation") of functions of several not necessarily commuting operators using three heuristic "rules":

(i) Attach time indices to the operators to specify the order of operators in products.
(ii) With time indices attached, compute the functions of these operators by treating them as though they were commuting.
(iii) Finally, "disentangle" the resulting expressions; that is, restore the conventional ordering of the operators.

Of the disentangling process, Feynman states [5, p. 110], "The process is not always easy to perform and, in fact, is the central problem of this operator calculus." One ought to note as well that Feynmman did not try to supply rigorous (or even heuristic) proofs of his results.

The obvious question to ask, when considering the heuristic rules above, is how one goes about attaching time indices to operators. Of course, it is possible that one or more of the operators under consideration may come with time indices naturally attached, which would be the case, for instance, when operators of multiplication by time-dependent potentials are present and also in connection with the Heisenberg representation in quantum mechanics. However, if an operator does not depend on time, as happens most often in quantum mechanics and in the mathematical literature, we need a mechanism for attaching time indices to such operators. Given a time independent operator A, Feynman, nearly without exception, used Lebesgue measure to attach time indices by writing

$$A = \frac{1}{t}\int_0^t A(s)\,ds,$$

where $A(s) \equiv A$ for all $s \in [0,t]$. Even though it appears artifical, this way of attaching time indices is extraordinarily useful and is crucial to the approach we will take to Feynman's operational calculus. (See [15], [12], [8, 9, 10, 11].)

To familiarize the reader with the basic ideas of the operational calculus, we will take the time to present two elementary examples. In these examples, X will be a Banach space with norm $\|\cdot\|_X$ and $\mathcal{L}(X)$ will denote the Banach space of bounded linear operators on X equipped with the norm

$$\|A\|_{\mathcal{L}(X)} = \sup\left\{\|Ax\|_X : x \in X, \|x\|_X \leq 1\right\}.$$

Example 1.1. Take $f(x,y) = xy$ and let $A, B \in \mathcal{L}(X)$. We associate Lebesgue measure ℓ on $[0,1]$ to both operators. More specifically,

$$A = \int_0^1 A(s)\,ds \text{ and } B = \int_0^1 B(s)\,ds,$$

where $A(s) \equiv A$ and $B(s) \equiv B$, for all $s \in [0,1]$. We will compute $f(A,B)$ labeling the result as $f_{\ell,\ell}(A,B)$ to make explicit the role that Lebesgue measure is playing. We have, computing heuristically,

$$\begin{aligned} f_{\ell,\ell}(A,B) &= \left\{\int_0^1 A(s)\,ds\right\}\left\{\int_0^1 B(s)\,ds\right\} \\ &= \int_0^1\int_0^1 A(s_1)B(s_2)\,ds_1 ds_2 \\ &= \int\limits_{\{(s_1,s_2):s_1<s_2\}} B(s_2)A(s_1)\,ds_1 ds_2 + \\ &\qquad \int\limits_{\{(s_1,s_2):s_2<s_1\}} A(s_1)B(s_2)\,ds_1 ds_2 \\ &= \frac{1}{2}BA + \frac{1}{2}AB = \frac{1}{2}(BA + AB). \end{aligned}$$

The first equality above follows from how we've attached time-indices to the operators. The second equality is the result of writing the product of the integrals as an iterated integral over $[0,1]^2$. It is after the third equality that the time-ordering is carried out. We write

$$\begin{aligned} [0,1]^2 = \{(s_1,s_2) : 0 < s_1 < s_2 < 1\} &\cup \{(s_1,s_2) : 0 < s_2 < s_1 < 1\} \\ &\cup \{(s_1,s_2) : s_1 = s_2\}. \end{aligned}$$

Since ℓ is a continuous measure, $(\ell \times \ell)(\{(s_1,s_2) : s_1 = s_2\}) = 0$; this is why there is no third integral in the computation above. In the union just above, because the time index s_1 is smaller (or *earlier*) than s_2, the operator product is written as $B(s_2)A(s_1)$. On the second set in the union, because the time index s_2 is smaller (or *earlier*) than s_1, the operator product is written as $A(s_1)B(s_2)$. (We are using "rule" (*ii*) here.) Finally, we evaluate the integrals and obtain the last expressions.

Example 1.2. We take $f(x,y) = xy$ and $A, B \in \mathcal{L}(X)$. We associate Lebesgue measure ℓ on $[0,1]$ to A and we associate the Dirac point mass δ_τ, $\tau \in (0,1)$, to B. So,

$$A = \int_0^1 A(s)\,ds \;\text{ and }\; B = \int_0^1 B(s)\,\delta_\tau(ds),$$

where, as before, $A(s) \equiv A$ and $B(s) \equiv B$, for all $s \in [0,1]$. Following Feynman's rules leads to

$$\begin{aligned} f_{\ell,\delta_\tau}(A,B) &= \left\{\int_0^1 A(s)\,ds\right\}\left\{\int_0^1 B(s)\,\delta_\tau(ds)\right\} \\ &= \left\{\int_{(0,\tau)} A(s)\,ds + \int_{(\tau,1)} A(s)\,ds\right\}\left\{\int_{\{\tau\}} B(s)\,\delta_\tau(ds)\right\} \\ &= \left\{\int_{\{\tau\}} B(s)\,\delta_\tau(ds)\right\}\left\{\int_{(0,\tau)} A(s)\,ds\right\} + \\ &\qquad\qquad \left\{\int_{(\tau,1)} A(s)\,ds\right\}\left\{\int_{\{\tau\}} B(s)\,\delta_\tau(ds)\right\} \\ &= B\,[\tau A] + [(1-\tau)A]\,B = \tau BA + (1-\tau)AB. \end{aligned}$$

We follow Feynman's rules in the same way as in the first example; however, there are differences here. Since δ_τ is supported on $\{\tau\}$, we break the integral for A at τ, leading to the expression after the second equality. To obtain the expression after the third equality, we follow Feynman's second rule as well as the time-ordering. Since τ comes after all time-indices in $(0,\tau)$ and before all time-indices in $(\tau,1)$, we obtain the expression after the third equality. Evaluating the integrals leads to the expression after the fourth equality.

The two examples presented above illustrate the essential ideas of the operational calculus, but the calculations carried out in both examples are not at all rigorous. To see how Feynman's ideas can be made mathematically rigorous, we briefly sketch the approach taken by G. W. Johnson and B. Jefferies (initiated in the late 1990's, appearing in print in [8, 9, 10, 11]). For a more detailed discussion, see Subsections 2.1 and 2.2 below. We take $A_1, \ldots, A_n$ to be noncommuting bounded linear operators on a Banach space X. Associate to each operator A_j a time-ordering measure λ_j on $[0,1]$ (a probability measure on $[0,1]$). We may then write

$$A_j = \int_{[0,1]} A_j(s)\,\lambda_j(ds),$$

where $A_j(s) \equiv A_j$ for all $s \in [0,1]$. Let $f(z_1, \ldots, z_n)$ be a function analytic on some polydisk P centered at the origin in $\mathbb{C}^n$ and write

$$f(z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} z_1^{m_1} \cdots z_n^{m_n}.$$

Define

$$P^{m_1,\dots,m_n}(z_1,\dots,z_n) := z_1^{m_1}\cdots z_n^{m_n}.$$

With the examples above in mind, we can compute, using Feynman's rules, the disentangling of $P^{m_1,\dots,m_n}(A_1,\dots,A_n)$ using the time-ordering directions supplied by the time-ordering measures $\lambda_1,\dots,\lambda_n$. One expects (and this is indeed the case) that the disentangled operator corresponding to the function f to take the form

$$f_{\lambda_1,\dots,\lambda_n}(A_1,\dots,A_n) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} P^{m_1,\dots,m_n}_{\lambda_1,\dots,\lambda_n}(A_1,\dots,A_n).$$

Obviously, questions of convergence arise, but the series above will turn out to converge in operator norm on $\mathcal{L}(X)$. It is also the case that the disentangled monomial

$$P^{m_1,\dots,m_n}_{\lambda_1,\dots,\lambda_n}(A_1,\dots,A_n)$$

is written as a sum of time-ordered products, much as we saw in the simple examples above. Jefferies and Johnson made these ideas mathematically rigorous by first supplying a "commutative world" – the *disentangling algebra* – in which the time-ordering (or, *disentangling*) calculations can be done in a rigorous way. (The disentangling algebra is, in fact, a commutative Banach algebra; see Subsection 2.1.1.) Once these computations are complete, the result is then mapped to the noncommuting environment of $\mathcal{L}(X)$ using the so-called *disentangling map*, see Subsection 2.2. It is this abstract approach which will be followed in the current paper. A comprehensive discussion of this approach can be found in [15] (as well as in the fundamental papers [8, 9, 10, 11]. Furthermore, the paper [12] (see also [21]) extends these ideas to the setting where the operators are time-dependent (or, operator-valued functions).

2. Necessary definitions and initial constructions

In this section we take the time in, Subsections 2.1 and 2.2, to outline the essential ideas of the abstract approach to Feynman's operational calculus which is used in this paper. In Subsection 2.3, we outline the operator theory which will play a role in our investigations. Subsection 2.4 gives a short discussion of the Taylor calculus which was introduced in the 1951 paper [34]. This calculus will play a crucial role in Section 3. In Subsection 2.5, Kato's *generalized convergence* of sequences of closed operators is discussed. The main properties of analytic families of operators are introduced in Subsections 2.6 and 2.7. Finally, in Subsection 2.8 we present a detailed discussion of the interplay between analytic families of (closed unbounded) operators and generalized convergence. This discussion will be used to obtain a stability-like result for the operational calcululs in the setting of analytic families of unbounded operators (see Subsection 4.1).

2.1. The operational calculus. In this sub-section, we describe, briefly, Feynman's operational calculus as developed in [8, 9, 10, 11], [12], [17] and [21]. A detailed discussion of this abstract approach to the operational calculus can be found in [15].

2.1.1. The commutative Banach algebras. In this subsection we will define the commutative Banach algebras that will be of use in our approach to Feynman's operational calculus. To begin, we will take X to be a separable Banach space and we will denote by $\mathcal{L}(X)$ the Banach space of bounded linear operators on X. Let $A_j : [0,T] \to \mathcal{L}(X)$, $j = 1, \ldots, n$, be strongly measurable; i.e., $A_j^{-1}(E)$ is a Borel set in $[0,T]$ for every strongly open $E \subset \mathcal{L}(X)$. Associate to each $A_j(\cdot)$, $j = 1, \ldots, n$, a Borel probability measure λ_j on $[0,T]$ and decompose λ_j as $\lambda_j = \mu_j + \eta_j$, where μ_j is a continuous Borel measure on $[0,T]$ and η_j is a finitely supported discrete measure on $[0,T]$. We refer to the measures λ_j, $j = 1, \ldots, n$, as *time-ordering measures*. As mentioned in the introduction, the time-ordering measures attached to a given operator-valued function (or operator) serve to determine where (or *when*) the operator acts in operator products. Let $\{\tau_1, \ldots, \tau_h\}$ be the union of the supports of the discrete measures $\eta_1, \ldots, \eta_n$ and assume that

$$0 < \tau_1 < \cdots < \tau_h < T. \tag{2.1}$$

We can then write

$$\eta_j = \sum_{i=1}^{h} p_{ji} \delta_{\tau_i} \tag{2.2}$$

for each $j = 1, \ldots, n$. Note that, with this definition, it may be that many of the p_{ji} are zero. However, this observation will not play any role in this paper.

With our operator-valued functions and their associated time-ordering measures in hand, we make the assumption that

$$\begin{aligned} r_j &:= \int_{[0,T]} \|A_j(s)\|_{\mathcal{L}(X)} \, \lambda_j(ds) \\ &= \int_{[0,T]} \|A_j(s)\|_{\mathcal{L}(X)} \, \mu_j(ds) + \sum_{i=1}^{h} p_{ji} \|A_j(\tau_i)\|_{\mathcal{L}(X)} < \infty \end{aligned} \tag{2.3}$$

for each $j = 1, \ldots, n$. The first commutative Banach algebra we require is defined as follows.

Definition 2.1. Let $R_1, \ldots, R_n$ be positive real numbers. We define $\mathbb{A}(R_1, \ldots, R_n)$ to be the family of all functions analytic on the open polydisk

$$P_{R_1,\ldots,R_n} := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_j| < R_j, j = 1, \ldots, n\},$$

and continuous on its boundary. For $f \in \mathbb{A}(R_1, \ldots, R_n)$, we define

$$\|f\|_{\mathbb{A}} := \sum_{m_1,\ldots,m_n=0}^{\infty} |a_{m_1,\ldots,m_n}| R_1^{m_1} \cdots R_n^{m_n} \tag{2.4}$$

where

$$f(z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} z_1^{m_1} \cdots z_n^{m_n} \tag{2.5}$$

is the Taylor series expansion of f at the origin in $\mathbb{C}^n$. The family $\mathbb{A}(R_1, \ldots, R_n)$ is a Banach algebra under pointwise operations. (See [15, Chapter 2].) We will often denote this algebra by $\mathbb{A}$.

With the real numbers $r_1, \ldots, r_n$ from (2.3), we construct the commutative Banach algebra $\mathbb{A}(r_1, \ldots, r_n)$. We now define a second, but closely related, commutative Banach algebra.

Definition 2.2. The *disentangling algebra* $\mathbb{D}\left(A_1(\cdot)^\sim, \ldots, A_n(\cdot)^\sim\right)$ is defined to be the family of all expressions

$$f\left(A_1(\cdot)^\sim, \ldots, A_n(\cdot)^\sim\right) := \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} \left(A_1(\cdot)^\sim\right)^{m_1} \cdots \left(A_n(\cdot)^\sim\right)^{m_n} \tag{2.6}$$

for which

$$\|f\|_{\mathbb{D}} := \sum_{m_1,\ldots,m_n=0}^{\infty} |a_{m_1,\ldots,m_n}| r_1^{m_1} \cdots r_n^{m_n} < \infty. \tag{2.7}$$

The objects $A_j(\cdot)^\sim$ are *formal commuting objects* which take the place of the operator functions $A_j(\cdot)$. All function- and operator-theoretic properties of $A_j(\cdot)$ are discarded from $A_j(\cdot)^\sim$ with the exception of the operator norm (or L^1-norm).

This family is a commutative Banach algebra under pointwise operations and with norm (2.7). (See [15, Chapter 2].) We will often denote this algebra by $\mathbb{D}$.

Remark 2.3. We note that we make no assumptions that the operator-valued functions (or operators) are linearly independent; however, we do assume that the formal objects $A_j(\cdot)^\sim$, $j = 1, \ldots, n$, are linearly independent.

Also, upon reflection, it becomes clear that the commutative Banach algebras $\mathbb{A}$ and $\mathbb{D}$ are, in fact, isometrically isomorphic. A detailed discussion of these algebras can be found in [15, Chapters 2 and 6].

It is the disentangling algebra above that will supply the "commutative world" which will allow us to apply Feynman's heuristic rules in a rigorous fashion. More specifically, we carry out the time-ordering computations in the disentangling algebra and then map the resulting expression into $\mathcal{L}(X)$. The resulting operator is referred to as the *disentangled operator*. We make this very brief remark precise as follows. First, to streamline our notation, for $m_1, \ldots, m_n \in \mathbb{N} \cup \{0\}$, we define

$$P^{m_1,\ldots,m_n}(z_1, \ldots, z_n) := z_1^{m_1} \cdots z_n^{m_n};$$

i.e., $P^{m_1,\ldots,m_n}(z_1, \ldots, z_n)$ is the monomial with exponents $m_1, \ldots, m_n$. Given $f \in \mathbb{D}$, we may therefore write

$$f\left(A_1(\cdot)^{\sim}, \ldots, A_n(\cdot)^{\sim}\right) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} P^{m_1,\ldots,m_n}\left(A_1(\cdot)^{\sim}, \ldots, A_n(\cdot)^{\sim}\right),$$

where f has the Taylor series given in (2.5).

2.1.2. Sets of time indices. To carry out the necessary time-ordering calculations required by Feynman's rules, we need to have certain ordered sets of time indices. First, given $t \in [0, T]$, and $m \in \mathbb{N}$, we define

$$\Delta_m(t) = \Delta_m^t := \{(s_1, \ldots, s_m) \in [0, T]^m : 0 < s_1 < \cdots < s_m < t\}. \quad (2.8)$$

Next, given $t \in [0, T]$, $m \in \mathbb{N}$ and $\pi \in S_m$, the group of permutations on m objects, we define

$$\Delta_m^t(\pi) := \left\{(s_1, \ldots, s_m) \in [0, T]^m : 0 < s_{\pi(1)} < \cdots < s_{\pi(m)} < t\right\}. \quad (2.9)$$

As is shown in [14] and [15],

$$[0, t]^m = \bigcup_{\pi \in S_m} \Delta_m^t(\pi),$$

where the union is a disjoint union. Indeed, if $\lambda_1, \ldots, \lambda_n$ are continuous measures on $[0, T]$, then the equality above holds up to a set of $\lambda_1^{m_1} \times \cdots \times \lambda_n^{m_n}$- measure zero. (We will use ν^k to denote the k-fold product measure $\nu \times \cdots \times \nu$ and, when the context is clear, often omit the superscript t or T from Δ_m^T, $\Delta_m^t(\pi)$, etc.)

To accommodate the discrete measures, we need one further set of time-indices which lets us deal with the presence of time-ordering measures with nonzero discrete parts. Let $\tau_1, \ldots, \tau_h \in [0, T]$, $0 < \tau_1 < \cdots < \tau_h < T$. Let $q \in \mathbb{N}$ and let $\theta_1, \ldots, \theta_{h+1} \in \mathbb{N}$ be such that $\theta_1 + \cdots + \theta_{h+1} = q$. We define

$$\begin{aligned}\Delta_{q;\theta_1,\ldots,\theta_{h+1}} := \{(s_1, \ldots, s_q) \in [0, T]^q &: 0 < s_1 < \cdots < s_{\theta_1} < \tau_1 < s_{\theta_1+1} < \cdots \\ &< s_{\theta_1+\theta_2} < \tau_2 < s_{\theta_1+\theta_2+1} < \cdots < s_{\theta_1+\cdots+\theta_h} < \tau_h < \\ &s_{\theta_1+\cdots+\theta_h+1} < \cdots < s_q < T\}.\end{aligned} \quad (2.10)$$

The set $\Delta_{q;\theta_1,\ldots,\theta_{h+1}}(\pi)$, for $\pi \in S_q$, is defined in the same way as (2.9). Also, if ν is a continuous probability measure on $[0, T]$, then

$$\Delta_q = \bigcup_{\theta_1+\cdots+\theta_{h+1}=q} \Delta_{q;\theta_1,\ldots,\theta_{h+1}}$$

up to a set of ν^q-measure zero (see [14, Lemma 15.2.7]) and the union above is a disjoint union.

2.1.3. Definitions concerning operators and formal objects. We need the following definitions to carry out the time-ordering calculations required by Feynman's rules. Recall that for operator-valued functions $A_1(\cdot), \ldots, A_n(\cdot)$ we discard all operator-theoretic properties of these functions and create formal commuting objects $A_1(\cdot)^\sim, \ldots, A_n(\cdot)^\sim$. We take, for every $j = 1, \ldots, n$, $A_j(s)^\sim \equiv A_j(\cdot)^\sim$ for all $s \in [0,T]$ and define

$$C_j(s)^\sim := \begin{cases} A_1(s)^\sim & \text{if } j \in \{1, \ldots, m_1\}, \\ A_2(s)^\sim & \text{if } j \in \{m_1+1, \ldots, m_1+m_2\}, \\ \vdots & \\ A_n(s)^\sim & \text{if } j \in \{m_1+\cdots+m_{n-1}+1, \ldots, m_1+\cdots+m_n\}. \end{cases} \tag{2.11}$$

We also define

$$C_j(s) := \begin{cases} A_1(s) & \text{if } j \in \{1, \ldots, m_1\}, \\ A_2(s) & \text{if } j \in \{m_1+1, \ldots, m_1+m_2\}, \\ \vdots & \\ A_n(s) & \text{if } j \in \{m_1+\cdots+m_{n-1}+1, \ldots, m_1+\cdots+m_n\}. \end{cases} \tag{2.12}$$

We note that, if the operator-valued functions $A_j(\cdot)$ are constant-valued; i.e., $A_j(s) \equiv A_j$ for all $s \in [0,T]$, then the definitions still apply, with (2.12) having only the operators $A_1, \ldots, A_n$ on the right-hand side.

2.2. Applying Feynman's rules – disentangling. We will continue with the strongly measurable operator-valued functions $A_j : [0,T] \to \mathcal{L}(X)$ (X a separable Banach space) which appeared in Subsection 2.1.1. Associate with each $A_j(\cdot)$, $j = 1, \ldots, n$, the Borel probability measure $\lambda_j = \mu_j + \eta_j$, again just as in Subsection 2.1.1. (We will use the same notation as in Section 2.1.1.)

To begin, we let $f \in \mathbb{D}$ and write (as in Definition 2.2)

$$f\left(A_1(\cdot)^\sim, \ldots, A_n(\cdot)^\sim\right) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} P^{m_1,\ldots,m_n}\left(A_1(\cdot)^\sim, \ldots, A_n(\cdot)^\sim\right).$$

We will carry out the time-ordering of the monomial first. However, we will state the result without a detailed calculation as the interested reader can consult [15, Chapters 2 and 8] as well as [17] for a detailed exposition of the time-ordering calculations.

Proposition 2.4. *Using the notation developed in this section and given* $m_1, \ldots, m_n \in \mathbb{N} \cup \{0\}$,

$$P^{m_1,\ldots,m_n}\left(A_1(\cdot)^\sim, \ldots, A_n(\cdot)^\sim\right) \tag{2.13}$$
$$= \sum_{q_{11}+q_{12}=m_1} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left(\frac{m_1! \cdots m_n!}{q_{11}! q_{12}! \cdots q_{n1}! q_{n2}!}\right) \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \cdots$$

$$\sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \left(\frac{q_{12}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \sum_{\theta_1+\cdots+\theta_{h+1}=q_{11}+\cdots+q_{n1}} \cdot$$
$$\sum_{\pi \in S_{q_{11}+\cdots+q_{n1}}} \int_{\Delta_{q_{11}+\cdots+q_{n1};\theta_1,\ldots,\theta_{h+1}}(\pi)} C_{\pi(q_{11}+\cdots+q_{n1})} \left(s_{\pi(q_{11}+\cdots+q_{n1})}\right)^{\sim} \cdots$$
$$C_{\pi(\theta_1+\cdots+\theta_h+1)} \left(s_{\pi(\theta_1+\cdots+\theta_h+1)}\right)^{\sim} \left[\prod_{\beta=0}^{n-1} \{p_{\beta h} A_{n-\beta}(\tau_h)^{\sim}\}^{j_{n-\beta,h}} \right] \cdot$$
$$C_{\pi(\theta_1+\cdots+\theta_h)} \left(s_{\pi(\theta_1+\cdots+\theta_h)}\right)^{\sim} \cdots C_{\pi(\theta_1+1)} \left(s_{\pi(\theta_1+1)}\right)^{\sim} \cdot$$
$$\left[\prod_{\beta=0}^{n-1} \{p_{\beta 1} A_{n-\beta}(\tau_1)^{\sim}\}^{j_{n-\beta,1}} \right] \cdot$$
$$C_{\pi(\theta_1)} \left(s_{\pi(\theta_1)}\right)^{\sim} \cdots C_{\pi(1)} \left(s_{\pi(1)}\right)^{\sim} (\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}}) (ds_1, \ldots, ds_{q_{11}+\cdots+q_{n1}}).$$

It follows at once that, given $f \in \mathbb{D}$ with its series representation (2.6),

$$\begin{aligned}
&f\left(A_1(\cdot)^{\sim}, \ldots, A_n(\cdot)^{\sim}\right) \\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} \sum_{q_{11}+q_{12}=m_1} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left(\frac{m_1! \cdots m_n!}{q_{11}! q_{12}! \cdots q_{n1}! q_{n2}!} \right) \cdot \\
&\sum_{j_{11}+\cdots+j_{1h}=q_{12}} \cdots \sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \left(\frac{q_{12}! \cdots q_{n2}!}{j_{11}! \cdots j_{1h}! \cdots j_{n1}! \cdots j_{nh}!} \right) \cdot \\
&\sum_{\theta_1+\cdots+\theta_{h+1}=q_{11}+\cdots+q_{n1}} \sum_{\pi \in S_{q_{11}+\cdots+q_{n1}}} \int_{\Delta_{q_{11}+\cdots+q_{n1};\theta_1,\ldots,\theta_{h+1}}(\pi)} \cdot \\
&C_{\pi(q_{11}+\cdots+q_{n1})} \left(s_{\pi(q_{11}+\cdots+q_{n1})}\right)^{\sim} \cdots C_{\pi(\theta_1+\cdots+\theta_h+1)} \left(s_{\pi(\theta_1+\cdots+\theta_h+1)}\right)^{\sim} \cdot \\
&\left[\prod_{\beta=0}^{n-1} \{p_{\beta h} A_{n-\beta}(\tau_h)^{\sim}\}^{j_{n-\beta,h}} \right] C_{\pi(\theta_1+\cdots+\theta_h)} \left(s_{\pi(\theta_1+\cdots+\theta_h)}\right)^{\sim} \cdots \\
&C_{\pi(\theta_1+1)} \left(s_{\pi(\theta_1+1)}\right)^{\sim} \left[\prod_{\beta=0}^{n-1} \{p_{\beta 1} A_{n-\beta}(\tau_1)^{\sim}\}^{j_{n-\beta,1}} \right] C_{\pi(\theta_1)} \left(s_{\pi(\theta_1)}\right)^{\sim} \cdots \\
&C_{\pi(1)} \left(s_{\pi(1)}\right)^{\sim} (\mu_1^{q_{11}} \times \cdots \times \mu_n^{q_{n1}}) (ds_1, \ldots, ds_{q_{11}+\cdots+q_{n1}}).
\end{aligned} \tag{2.14}$$

With the time-ordering in the disentangling algebra at our disposal, we can now define the *disentangling map* which will take these time-ordered expressions from the commutative setting of the disentangling algebra $\mathbb{D}$ to the noncommutative setting of $\mathcal{L}(X)$. In fact, using the time-ordering from Proposition 2.4 and the definition of the operators $C_j(s)$ from (2.12) it is straightforward to define the disentangling map.

Definition 2.5. Using notation of this section, we define the disentangling map

$$\mathcal{T}_{\lambda_1,\ldots,\lambda_n} : \mathbb{D}\left(A_1(\cdot)^{\sim},\ldots,A_n(\cdot)^{\sim}\right) \to \mathcal{L}(X)$$

as follows. For $m_1,\ldots,m_n \in \mathbb{N}\cup\{0\}$, we define the action of the disentangling map on

$$P^{m_1,\ldots,m_n}\left(A_1(\cdot)^{\sim},\ldots,A_n(\cdot)^{\sim}\right)$$

by

$$\begin{aligned}
&\mathcal{T}_{\lambda_1,\ldots,\lambda_n} P^{m_1,\ldots,m_n}\left(A_1(\cdot)^{\sim},\ldots,A_n(\cdot)^{\sim}\right) \\
&= P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(A_1(\cdot),\ldots,A_n(\cdot)\right) \\
&= \sum_{q_{11}+q_{12}=m_1} \cdots \sum_{q_{n1}+q_{n2}=m_n} \left(\frac{m_1!\cdots m_n!}{q_{11}!q_{12}!\cdots q_{n1}!q_{n2}!}\right) \sum_{j_{11}+\cdots+j_{1h}=q_{12}} \cdots \\
&\sum_{j_{n1}+\cdots+j_{nh}=q_{n2}} \left(\frac{q_{12}!\cdots q_{n2}!}{j_{11}!\cdots j_{1h}!\cdots j_{n1}!\cdots j_{nh}!}\right) \sum_{\theta_1+\cdots+\theta_{h+1}=q_{11}+\cdots+q_{n1}} \cdot \\
&\sum_{\pi\in S_{q_{11}+\cdots+q_{n1}}} \int_{\Delta_{q_{11}+\cdots+q_{n1};\theta_1,\ldots,\theta_{h+1}}(\pi)} C_{\pi(q_{11}+\cdots+q_{n1})}\left(s_{\pi(q_{11}+\cdots+q_{n1})}\right)\cdots
\end{aligned} \tag{2.15}$$

$$C_{\pi(\theta_1+\cdots+\theta_h+1)}\left(s_{\pi(\theta_1+\cdots+\theta_h+1)}\right)\left[\prod_{\beta=0}^{n-1}\{p_{\beta h}A_{n-\beta}(\tau_h)\}^{j_{n-\beta,h}}\right]\cdot$$

$$C_{\pi(\theta_1+\cdots+\theta_h)}\left(s_{\pi(\theta_1+\cdots+\theta_h)}\right)\cdots C_{\pi(\theta_1+1)}\left(s_{\pi(\theta_1+1)}\right)\left[\prod_{\beta=0}^{n-1}\{p_{\beta 1}A_{n-\beta}(\tau_1)\}^{j_{n-\beta,1}}\right]\cdot$$

$$C_{\pi(\theta_1)}\left(s_{\pi(\theta_1)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\left(\mu_1^{q_{11}}\times\cdots\times\mu_n^{q_{n1}}\right)\left(ds_1,\ldots,ds_{q_{11}+\cdots+q_{n1}}\right).$$

Given $f \in \mathbb{D}$ with the representation (2.6), we define the action of the disentangling map on this element by

$$\begin{aligned}
\mathcal{T}_{\lambda_1,\ldots,\lambda_n} f\left(A_1(\cdot)^{\sim},\ldots,A_n(\cdot)^{\sim}\right) &= f_{\lambda_1,\ldots,\lambda_n} f\left(A_1(\cdot),\ldots,A_n(\cdot)\right) \\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(A_1(\cdot),\ldots,A_n(\cdot)\right);
\end{aligned} \tag{2.16}$$

i.e., we define the action of $\mathcal{T}_{\lambda_1,\ldots,\lambda_n}$ on f term-by-term. We often refer to the operator obtained from the action of the disentangling map as the *disentangled operator*.

The following theorem can be found in [15, Chapter 8]. (A thorough discussion of the disentangling map and its properties can be found in Chaper 8 of [15] and [17] when the time-ordering measures have non-zero discrete parts and in Chapter 2 of [15] when the time-ordering measures are continuous.

Theorem 2.6. *The disentangling map $\mathcal{T}_{\lambda_1,\dots,\lambda_n}$ is a bounded linear operator from $\mathbb{D}$ into $\mathcal{L}(X)$. Moreover, $\|\mathcal{T}_{\lambda_1,\dots,\lambda_n}\| \le 1$; i.e., $\mathcal{T}_{\lambda_1,\dots,\lambda_n}$ is a linear contraction on $\mathbb{D}$.*

It is also worth recording the form the disentangling map takes when the time-ordering measures are purely discrete and finitely supported. (See Corollary 8.4.3 of [15].)

Theorem 2.7. *If the time-ordering measures $\lambda_1,\dots,\lambda_n$ are purely discrete and finitely supported with the union of the supports being $\{\tau_1,\dots,\tau_h\}$ with $0 < \tau_1 < \cdots < \tau_h < T$ and*

$$\lambda_j = \sum_{i=1}^{h} p_{ji}\delta_{\tau_i},$$

then

$$\begin{aligned}
&\mathcal{T}_{\lambda_1,\dots,\lambda_n} f(A_1(\cdot)^\sim,\dots,A_n(\cdot)^\sim)\\
&= \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} \sum_{j_{11}+\cdots+j_{1h}=m_1} \cdots\\
&\sum_{j_{n1}+\cdots+j_{nh}=m_n} \left(\frac{m_1!\cdots m_n!}{j_{11}!\cdots j_{1h}!\cdots j_{n1}!\cdots j_{nh}!}\right)\\
&\left\{\prod_{\beta=0}^{n-1} [p_{n-\beta,h}A_{n-\beta}(\tau_h)]^{j_{n-\beta,h}}\right\}\cdots\left\{\prod_{\beta=0}^{n-1} [p_{n-\beta,1}A_{n-\beta}(\tau_1)]^{j_{n-\beta,1}}\right\}.
\end{aligned}$$

Finally, if all of our time-ordering measures are continuous, then we have the following. (See Chapter 2 of [15]. See also [8])

Theorem 2.8. *If the time-ordering measures $\lambda_1,\dots,\lambda_n$ are continuous, then*

$$\begin{aligned}
&\mathcal{T}_{\lambda_1,\dots,\lambda_n} f\left(A_1(\cdot)^\sim,\dots,A_n(\cdot)^\sim\right)\\
&= \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1,\dots,m_n} \sum_{\pi\in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)})\cdots \qquad (2.17)\\
&\qquad C_{\pi(1)}(s_{\pi(1)})\,(\lambda_1^{m_1}\times\cdots\times\lambda_n^{m_n})(ds_1,\dots,ds_m).
\end{aligned}$$

Note that the disentangled monomial, in the case that our measures are continuous, is

$$\sum_{\pi\in S_m} \int_{\Delta_m(\pi)} C_{\pi(m)}(s_{\pi(m)})\cdots C_{\pi(1)}(s_{\pi(1)})(\lambda_1^{m_1}\times\cdots\times\lambda_n^{m_n})(ds_1,\dots,ds_m).$$

Remark 2.9. While our time-ordering measures in the presentation above have been probability measures, it turns out that the disentangling takes the same form when the time-ordering measures are non-probability measures. See [8] or [15, Section 3.2]. Such measures are of use when considering evolution problems, for example; see [30]. We do not consider such problems in this paper, and so the reader can safely assume that all time-ordering measures used below are probability measures on $[0, T]$.

2.3. Some basic facts from operator theory. We present here some of the basic ideas of operator theory which will be needed in this paper. A detailed discussion of the ideas presented below can be found in many standard references, for example [19], [31], [32], [4]. Here, we closely follow the presentation of [19].

As the reader is no doubt aware, unlike with bounded linear operators from a Banach space X to a Banach space Y, with unbounded operators having domains not equal to all of X, various difficulties arise. As is well-known, if S and T are unbounded linear operators from X to Y with domains $\mathcal{D}(S)$ and $\mathcal{D}(T)$, respectively, linear combinations $\alpha S + \beta T$ may not be defined and the operator product ST (or TS) may not be defined. Both of these pathologies arise in Feynman's operational calculus if one attempts to use unbounded operators in a naive way.

We will turn our attention to closed unbounded linear operators from a Banach space X to a Banach space Y. (Below, X and Y will always denote Banach spaces.)

Definition 2.10. Let T be a linear operator from X to Y with domain $\mathcal{D}(T)$. If $\{\phi_n\}_{n=1}^{\infty}$ is a sequence from $\mathcal{D}(T)$ such that $\phi_n \to \phi$ as $n \to \infty$ and such that $T\phi_n \to \psi \in Y$, then $\phi \in \mathcal{D}(T)$ and $T\phi = \psi$, then we say that T is *closed*. In paticular, if we define the *graph* of T by

$$\mathcal{G}(T) := \{(\phi, T\phi) : \phi \in \mathcal{D}(T)\},$$

then T is closed if and only if $\mathcal{G}(T)$ is a closed subspace of $X \times Y$.

Remark 2.11. The family of all closed linear operators from X to Y will be denoted by $\mathscr{C}(X, Y)$.

If a linear operator from X to Y is not closed, it may still be *closable*.

Definition 2.12. A linear operator T from X to Y is *closable* if the closure $\overline{\mathcal{G}(T)}$ of $\mathcal{G}(T)$ is a graph; i.e., there is a $T_0 \in \mathscr{C}(X, Y)$ with $\mathcal{G}(T_0) = \overline{\mathcal{G}(T)}$. We call T_0 the closure of T and T_0 is the smallest closed extension of T.

Now, we take the time to remind the reader of the resolvent of an operator and of the resolvent set of an operator.

Definition 2.13. Let T be a closed operator on X. Then the same is true for $\xi - T$, for any $\xi \in \mathbb{C}$. (We follow the convention that, when writing $\xi - T$, we interpret ξ as ξI where I is the identity operator.) If $\xi - T$ is invertible with

$$R(\xi; T) := (\xi - T)^{-1} \in \mathcal{L}(X),$$

ξ is said to belong to the *resolvent set* of T and we denote the resolvent set by $\rho(T)$. The operator-valued function $R(\cdot; T) : \rho(T) \longrightarrow \mathcal{L}(X)$ is called the *resolvent* of T.

As is well-known, an operator T commutes with its resolvent function; i.e., $R(\xi;T)T \subset TR(\xi;T)$ for $\xi \in \rho(T)$. Just as well-known is the fact that $\rho(T)$ is an open subset of $\mathbb{C}$ and $R(\xi;T)$ is a (piecewise) analytic function for $\xi \in \rho(T)$.

Definition 2.14. For a closed operator T on X with resolvent set $\rho(T)$, the *spectrum* $\sigma(T)$ of T is the complement of $\rho(T)$: $\sigma(T) = \mathbb{C}\backslash\rho(T)$. So, if $\zeta \in \sigma(T)$, $\zeta - T$ is not invertible or is invertible but has range smaller than X.

When we address the use of the Taylor calculus for unbounded operators in Feynman's operational calculus, we will need to refer specifically to the point at infinity. The following theorem is Theorem III.6.13 of [19].

Theorem 2.15. *Let $T \in \mathscr{C}(X)$ and let $\rho(T)$ contain the exterior of a circle. Then we have the alternatives*:

(*i*) *$T \in \mathcal{L}(X)$; $R(\xi;T)$ is analytic at $\xi = \infty$ and $R(\infty;T) = 0$.*

(*ii*) *$R(\xi;T)$ has an essential singularity at $\xi = \infty$.*

In view of this theorem, we include $\xi = \infty$ in the resolvent set of T if $T \in \mathcal{L}(X)$ and in $\sigma(T)$ otherwise. Hence, an unbounded operator always has $\xi = \infty$ in its spectrum and, if it is an isolated point, it is an essential singularity of the resolvent function.

We now undertake a brief discussion of unbounded quadratic forms, which will be useful to us below. The presentation in [14, Section 10.3] will be followed here. Also, $\mathcal{H}$ will be a complex Hilbert space throughout this discussion.

Definition 2.16. A *sesquilinear form* is a map $q : Q(q) \times Q(q) \longrightarrow \mathbb{C}$, where $Q(q)$ is a dense subspace of $\mathcal{H}$, the *form domain* of q, such that q is linear in the first variable and conjugate linear in the second variable. If $q(\phi,\psi) = \overline{q(\psi,\phi)}$ for every $\phi,\psi \in Q(q)$, the form q is said to be symmetric. For notational symplicity, we will often write $q(\phi)$ for $q(\phi,\phi)$ when $\phi \in Q(q)$. If there is a number $c \geq 0$ such that

$$q(\phi) \geq -c\|\phi\|^2_{\mathcal{H}}, \tag{2.18}$$

for $\phi \in Q(q)$, we say that q is *semibounded* or *bounded below.* If we can take $c = 0$, then q is said to be *nonnegative.*

Let q be a semibounded quadratic form and let $c \geq 0$ be such that (2.18) holds for all $\phi \in Q(q)$. If we define

$$\langle\phi,\psi\rangle_{+1} := q(\phi,\psi) + (c+1)\langle\phi,\psi\rangle_{\mathcal{H}}, \tag{2.19}$$

then $\langle\cdot,\cdot\rangle_{+1}$ is an inner product on $Q(q)$. The associated norm $\|\cdot\|_{+1}$ is

$$\|\phi\|_{+1} := \{q(\phi) + (c+1)\langle\phi,\phi\rangle_{\mathcal{H}}\}^{1/2} \tag{2.20}$$

for $\phi \in Q(q)$.

Definition 2.17. A semibounded quadratic form q as above is called *closed* if $Q(q)$ is complete under the norm $\|\cdot\|_{+1}$ given by (2.20). If q is closed, and if D is a subspace of $Q(q)$ which is dense in $Q(q)$ in the norm $\|\cdot\|_{+1}$, then D is called a *form core* for q.

Now, let A be a (unbounded) self-adjoint operator on $\mathcal{H}$. There is always a quadratic form q_A associated with A (q_A will not in general be semibounded). The form domain $Q(q_A)$ and the form q_A can be described in terms of a spectral representation for A. This representation involves both the Hilbert space $\mathcal{H}$ and a unitary operator U from $\mathcal{H}$ onto the representing space. However, one normally suppresses reference to U and acts as though $\mathcal{H}$ were the representing space.

The form q_A is called the *quadratic form associated with* A, and we sometimes write $Q(q_A) = Q(A)$. The subspace $Q(A)$ is called the *form domain of the operator* A. It follows from this definition that $q_A(\phi, \psi) = (\phi, A\psi)$ for all $\phi \in Q(A)$ and all $\psi \in \mathcal{D}(A)$.

We need the definition of a semibounded symmetric operator.

Definition 2.18. A symmetric operator $A : \mathcal{D}(A) \to \mathcal{H}$ is said to be *semibounded* if there is a $c \geq 0$ such that

$$(\phi, A\phi) \geq -c\|\phi\|_{\mathcal{H}}^2$$

for all $\phi \in \mathcal{D}(A)$.

We note that if A is semibounded, then the quadratic form associated to A is semibounded and closed. (Proposition 10.3.8 of [14].)

We now outline the idea of the form sum of two self-adjoint operators. The reader will recall that the ordinary operator sum of self-adjoint operators A and B is defined on $\mathcal{D}(A + B) := \mathcal{D}(A) \cap \mathcal{D}(B)$ and, for $\phi \in \mathcal{D}(A + B)$, $(A + B)\phi := A\phi + B\phi$. As the reader no doubt knows, however, it may happen that $\mathcal{D}(A) \cap \mathcal{D}(B)$ is not dense in $\mathcal{H}$ and, in fact, it may be $\{0\}$. If q_A and q_B are the quadratic forms associated to A and B, respectively, the idea is to consider $q_A + q_B$. Under appropriate conditions, there will be a unique densely defined, semibounded, self-adjoint operator associated to $q_A + q_B$ and this operator, called the *form sum of* A *and* B and denoted by $A \dot{+} B$.

In most applications, the self-adjoint operator A will be nonnegative and so we restrict ourselves to this case. We will write the self-adjoint operator B as $B = B_+ - B_-$, using the spectral theorem for self-adjoint operators. We also appeal to the spectral theorem for the operators $A^{1/2}, B^{1/2}$. Indeed, according to the spectral theorem in the multiplication operator form (see Theorem 10.1.8 of [14]), the self-adjoint operator A is unitarily equivalent to the multiplication operator M_f on $L^2(\mu)$; i.e., $A = U^{-1}M_f$ for a unitary operator $U : \mathcal{H} \to L^2(\mu)$. We can then write $A_\pm = U^{-1}M_{f_\pm}U$ with $\mathcal{D}(A_\pm) =$

$\{\phi \in L^2(\mu) : f_\pm \cdot \phi \in L^2(\mu)\}$. Also,

$$A_\pm^{1/2} = U^{-1} M_{f_\pm^{1/2}} U$$

with domain defined similarly. Note that, if A is nonnegative, $A_+ = A$.

Definition 2.19. We say that B_- is *relatively form bounded with respect to* A ("B_- is A-form bounded") with bound less than 1 if $Q(A) \subseteq Q(B_-)$ and there are positive constants $\gamma < 1$ and δ such that

$$\|B_-^{1/2}\phi\|_{\mathcal{H}}^2 \leq \gamma \|A^{1/2}\phi\|_{\mathcal{H}}^2 + \delta \|\phi\|_{\mathcal{H}}^2.$$

The infimum of all such positive numbers γ is called the *A-form bound* of B.

The form sum is determined by the following theorem [14, Theorem 10.3.19].

Theorem 2.20. *Let A and B be self-adjoint operators with A nonnegative. Suppose also that B_- is A-form bounded with bound less than 1 and assume that $Q(A) \cap Q(B_+)$ is dense in $\mathcal{H}$. Then the quadratic form $q_A + q_B$ given by*

$$q_A(\phi, \psi) + q_B(\phi, \psi) = (A^{1/2}\phi, A^{1/2}\psi) + (B_+^{1/2}\phi, B_+^{1/2}\psi) - (B_-^{1/2}\phi, B_-^{1/2}\psi)$$

for $\phi, \psi \in Q(A) \cap Q(B_+)$ (this form is bounded below) is closed and semi-bounded and so there is a unique semi-bounded self-adjoint operator $A \dot{+} B$, called the form sum of A and B, such that $q_{A\dot{+}B} = q_A + q_B$ with $Q(A \dot{+} B) = Q(A) \cap Q(B) = Q(A) \cap Q(B_+)$. The operator $A \dot{+} B$ is a self-adjoint extension of the algebraic sum $A + B$.

2.4. Functions of unbounded operators - The calculus of A. E. Taylor. This section serves to outline the essential details of the functional calculus of A. E. Taylor found in the 1951 paper [34]. This calculus serves as the basis for Section 4 and we follow [34] closely in this subsection.

Let X be a complex Banach space. Throughout this section, T will be a closed operator on X with domain $\mathcal{D}(T)$. We will also let $\rho(T)$ be the resolvent set of T (see Subsection 2.3) and we will denote, for $\lambda \in \rho(T)$, the resolvent operator $(\lambda - T)^{-1}$ by $R(\lambda; T)$. The spectrum $\sigma(T)$ consists of all complex numbers λ not in $\rho(T)$.

We know that $\rho(T)$ is open in $\mathbb{C}$ and so $\sigma(T)$ is closed. When $T \in \mathcal{L}(X)$, $\sigma(T)$ is bounded and nonempty. Just as well-known is the fact that, in the general case, $\sigma(T)$ may be empty, unbounded or equal to all of $\mathbb{C}$. Also, when $\rho(T) \neq \emptyset$, the resolvent operator $R(\lambda; T)$ is an analytic $\mathcal{L}(X)$-valued function on $\rho(T)$.

We now need a definition of the certain subsets of the complex plane.

Definition 2.21. A set $D \subseteq \mathbb{C}$ is called a *Cauchy Domain* if the following conditions are satisfied:

(i) D is open in $\mathbb{C}$;
(ii) D has a finite number of components, the closures of any two of which are disjoint;
(iii) the boundary of D is composed of a finite number of closed rectifiable Jordan curves, no two of which intersect.

Remark 2.22. We note that a component of a Cauchy domain is also a Cauchy domain. Furthermore, if the Cauchy domain D is unbounded, it has just one unbounded component. This unbounded component contains a neighborhood of the point at infinity (meaning all points outside a sufficiently large disk), and has as its boundary a finite number of closed rectifiable Jordan curves which do not intersect and no one of these curves is inside any other.

If D is a Cauchy domain and if C is one of the curves making up its boundary, we will, as is customary, define the positive orientation of C as part of the boundary ∂D of D.

The following theorem is fundamental. (It is Theorem 3.3 of [34].)

Theorem 2.23. *Let F and Δ be subsets of $\mathbb{C}$. Suppose that F is closed and Δ is open and assume that $F \subseteq \Delta$. Suppose further that $\partial\Delta$ is nonempty and bounded. Then there is a Cauchy domain D such that*:

(i) $F \subset D$,
(ii) $\overline{D} \subset \Delta$,
(iii) *the curves forming ∂D are polygons and*
(iv) *D is unbounded if Δ is unbounded.*

We now move onto the functional calculus of Taylor. To this end, we take $f(z)$ to be a complex-valued function of the complex variable z, we denote by $\Delta(f)$ the domain of f and we will assume that $\Delta(f) \neq \emptyset$ (but it may not necessarily be connected) and finally that $f(z)$ is single-valued and analytic on $\Delta(f)$.

Definition 2.24. We let $\mathscr{G}(T)$ be the family of all analytic functions f for which

(i) $\sigma(T) \subseteq \Delta(f)$,
(ii) $\Delta(f)$ contains a neighborhood of $z = \infty$ in $\mathbb{C}$ and f is regular at $z = \infty$. (By "regular," we mean that f is analytic in a neighborhood of $z = \infty$; i.e., analytic on the complement of some open disk.)

When $f \in \mathscr{G}(T)$, we denote the limiting value of $f(z)$ as $z \to \infty$ by $f(\infty)$.

Remark 2.25. If $\sigma(T) = \mathbb{C}$, the Taylor calculus cannot be defined. Hence, we will assume from here on that $\rho(T)$ is never empty for any operator under consideration.

The primary theorem which allows the Taylor calculus to be used is Theorem 4.1 of [34], which we will state here for the reader's convenience.

Theorem 2.26. *Suppose that $f \in \mathscr{G}(T)$. There is an unbounded Cauchy domain D such that $\sigma(T) \subset D$ and $\overline{D} \subset \Delta(f)$. The integral*

$$\frac{1}{2\pi i}\int_{\partial D} f(z)R(z;T)\,dz \tag{2.21}$$

defines an element of $\mathcal{L}(X)$ which is the same for any choice of Cauchy domain D satisfying the conditions stated above.

We now define $f(T)$ using Theorem 2.26.

Definition 2.27. When $f \in \mathscr{G}(T)$ we define

$$f(T) := f(\infty)I + \frac{1}{2\pi i}\int_{\partial D} f(z)R(z;T)\,dz. \tag{2.22}$$

2.5. Generalized convergence of closed operators. In this subsection, we follow Kato's monograph [19]. We start with the idea of the "gap" between subspaces. Let X and Y be Banach spaces and let $\mathscr{C}(X,Y)$ be the space of all closed densely defined linear operators from X to Y. If $T, S \in \mathscr{C}(X,Y)$, then their graphs $\mathcal{G}(T)$ and $\mathcal{G}(S)$ are closed subspaces of $X \times Y$. In [19], the "distance" between T and S is measured by the "gap" between the closed subspaces $\mathcal{G}(T)$ and $\mathcal{G}(S)$. In order to measure this gap, we proceed as follows. (We follow Kato's notation below.)

Let Z be a Banach space and let M, N be closed subspaces of Z. Define

$$\delta(M,N) := \sup_{u \in S_M} \operatorname{dist}(u,N), \tag{2.23}$$

where S_M denotes the unit sphere of M and where $\operatorname{dist}(u,N)$ denotes the distance from u to N; i.e., $\operatorname{dist}(u,N) = \inf\{\|u-a\|_Z : a \in N\}$. We next define

$$\hat{\delta}(M,N) := \max\{\delta(M,N), \delta(N,M)\}, \tag{2.24}$$

and we define $\delta(0,N) = 0$. Also, $\delta(M,0) = 1$ if $M \neq \emptyset$. We also note that $\delta(M,N)$ is the smallest nonnegative real number δ for which

$$\operatorname{dist}(u,N) \leq \delta\|u\|, \tag{2.25}$$

for all $u \in M$. (Indeed, (2.25) will play a crucial role in Subsections 2.8 and 4.1.) The number $\hat{\delta}(M,N)$ is called the *gap* between M and N. Kato makes note of the following properties:

$$\delta(M,N) = 0 \text{ if and only if } M \subseteq N,$$

$$\hat{\delta}(M,N) = 0 \text{ if and only if } M = N,$$

$$\hat{\delta}(M,N) = \hat{\delta}(N,M),$$

$$0 \leq \delta(M,N) \leq 1 \text{ and } 0 \leq \hat{\delta}(M,N) \leq 1.$$

We now use these ideas to discuss the gap between closed operators (see Section IV.4 of [19]). Given $T, S \in \mathscr{C}(X,Y)$, the number $\hat{\delta}(\mathcal{G}(T), \mathcal{G}(S)) =: \hat{\delta}(T,S)$ is called the *gap* between T and S. Next, suppose that $\{T_n\}_{n=1}^{\infty}$ is a sequence from $\mathscr{C}(X,Y)$ and suppose that $T \in \mathscr{C}(X,Y)$.

Definition 2.28. (See Page 202 of [19]) We say that $T_n \longrightarrow T$ in the *generalized sense* if and only if $\hat{\delta}(T_n, T) \longrightarrow 0$ as $n \to \infty$.

Note that if $\hat{\delta}(T_n, T) \longrightarrow 0$, then $\delta(T_n, T) \longrightarrow 0$. Also, in the case that $Y = X$, there is a criterion for generalized convergence, Theorem IV.2.25 of [19].

Theorem 2.29. *Let $T \in \mathscr{C}(X)$ have a non-empty resolvent set $\rho(T)$. In order that a sequence $\{T_n\}_{n=1}^{\infty}$ from $\mathscr{C}(X)$ to converge to T, it is necessary that $\rho(T) \subseteq \rho(T_n)$ for sufficiently large n and*

$$\|R(\zeta; T_n) - R(\zeta; T)\|_{\mathcal{L}(X)} \longrightarrow 0$$

as $n \to \infty$, while it is sufficient that this be true for some $\zeta \in \rho(T)$.

A sufficient condition for generalized convergence is Theorem IV.2.29 of [19].

Theorem 2.30. *Let $T_n, T \in \mathscr{C}(X, Y)$. Let Z be a third Banach space and let $U_n, U \in \mathcal{L}(Z, X)$ and $V_n, V \in \mathcal{L}(Z, Y)$ be such that U_n, U map Z onto $\mathcal{D}(T_n), \mathcal{D}(T)$ injectively, respectively and $T_n U_n = V_n$, $TU = V$. If $\|U_n - U\| \to 0$ and $\|V_n - V\| \to 0$ as $n \to \infty$, then $T_n \to T$ in the generalized sense.*

2.6. Bounded analytic families of operators. Let X and Y be Banach spaces. We are interested in families $T(z)$ (notation following [19], Chapter VII, Section 1) of operators from X to Y where $T(z)$ is an analytic function in a domain D of $\mathbb{C}$. First, recall that a X-valued function $u(z)$ on a domain D of $\mathbb{C}$ is analytic if it is differentiable at each $z \in D$; it does not matter whether the derivative is a strong or weak derivative. (See [19] or Section VI.3 of [31].) Therefore, $u(z)$ is analytic if and only if $(u(z), x^*)$ is analytic for every $x^* \in X^*$. (We use the standard notation $(\cdot, \cdot)$ for the duality bracket.)

For operator-valued analytic functions $T(z)$, we say that $T(z)$ is analytic in a domain D of $\mathbb{C}$ if and only if $T(z)$ is norm differentiable for all $z \in D$. We can say that $T(z) \in \mathcal{L}(X, Y)$ is analytic if and only if each z has a neighborhood in which $T(z)$ is bounded and $(T(z)x, y^*)$ is analytic for any x in a fundamental subset of X and every y^* in a fundamental subset of Y^*. Note that, if $Y = X$, then $T(z)$ is analytic if and only if each z has a neighborhood in which $T(z)$ is bounded and $(T(z)x, x^*)$ is analytic for all x in a fundamental subset of X and every x^* in a fundamental subset of X^*. These statements follow from Theorem III.3.12 of [19]:

Theorem 2.31. *Let $T(z) \in \mathcal{L}(X, Y)$ be defined in a domain Δ of $\mathbb{C}$ and suppose that $(T(z)x, y^*)$ is analytic in $z \in \Delta$ for each $x \in X$ and $y^* \in Y^*$. Then $T(z)$ is analytic in Δ in the sense of norm differentiability.*

2.7. Analytic families of unbounded operators. Subsection 2.6 states the relevant definitions and criteria for analytic families (functions) from $\mathbb{C}$

into $\mathcal{L}(X,Y)$. (The so-called *bounded analytic families* or *boundedly analytic families.*) However, norm differentiability will not work when looking at functions from $\mathbb{C}$ into $\mathscr{C}(X,Y)$. Following Section VII.2 of [19], we extend the idea of analytic family to unbounded operators using the idea of generalized convergence of closed operators.

Definition 2.32. A family $T(z) \in \mathscr{C}(X,Y)$ defined in a neighborhood of $z=0$ in $\mathbb{C}$ is said to be holomorphic at $z=0$ (in the generalized sense) if there is a third Banach space Z and two boundedly analytic functions $U(z) \in \mathcal{L}(Z,X)$ and $V(z) \in \mathcal{L}(Z,Y)$ (in the sense of Subsection 2.2) at $z=0$ such that $U(z)$ maps Z onto $\mathcal{D}(T(z))$ injectively and

$$T(z)U(z) = V(z). \tag{2.26}$$

The function $T(z)$ is analytic in a domain D of the complex plane if it is analytic at every $z \in D$. Finally, if $T(z)$ is analytic, then it is continuous in the generalized sense that if $z \to z_0$, then $T(z) \to T(z_0)$ in the generalized sense; see Theorem 2.30 above.

When $Y = X$, we have Theorem VII.1.3 of [19].

Theorem 2.33. *Let $T(z) \in \mathscr{C}(X)$ be defined in a neighborhood of $z=0$ and let $\zeta \in \rho(T(0))$. Then $T(z)$ is analytic at $z=0$ if and only if $\zeta \in \rho(T(z))$ and the resolvent $R(\zeta;T(z)) = (\zeta - T(z))^{-1}$ is boundedly analytic for sufficiently small $|z|$.*

2.8. Analytic families of unbounded operators and generalized convergence. In this subsection we state a definition which will be used to assist our investigations of the operational calculus in the setting of analytic families. In particular, this definition and the discussion which follows will be of crucial use in the study of the stability of the operational caluclus with respect to sequences of analytic families.

Definition 2.34. Suppose that $T(z)$ is an analytic family of unbounded operators on the domain $D \subseteq \mathbb{C}$ in the sense of Subsection 2.7. We will say that $T(z)$ *is uniform at* $z_0 \in D$ if, for any $\epsilon > 0$, there is a $\iota > 0$ such that given any $(\phi, T(z_0)\phi) \in \mathcal{G}(T(z_0))$ and $(\psi, T(z)\psi) \in \mathcal{G}(T(z))$ with $|z - z_0| < \iota$, then

$$\|(\phi, T(z_0)\phi) - (\psi, T(z)\psi)\|_{X\times X} < \epsilon. \tag{2.27}$$

We will say that $T(z)$ is uniform on $D \subseteq \mathbb{C}$ if is uniform at every point in D.

We apply this definition to a sequence $\{T_n(z)\}_{n=1}^{\infty}$ of analytic families of closed unbounded operators on X on the domain $D \subseteq \mathbb{C}$ each of which satisfy Definition 2.34 on D and which are such that $T_n(z) \longrightarrow T(z)$ in the generalized sense for each $z \in D$. We will also assume that the analytic family $T(z)$ satisfies Definition 2.34 at each point $z \in D$.

Write, as in Subsection 2.7, $T(z)U(z) = V(z)$ where $U(z)$ is boundedly analytic on D and injective from a Banach space Z onto $\mathcal{D}(T(z)) \subseteq X$

and $V(z)$ is boundedly analytic on D and maps Z into X. Similarly, write $T_n(z)U_n(z) = V_n(z)$ for each $n \in \mathbb{N}$, where $U_n(z)$ is boundedly analytic on D and injective from Z onto $\mathcal{D}(T_n(z)) \subseteq X$ and $V_n(z)$ is boundedly analytic on D from Z into X. Note that we may write

$$\begin{aligned}\mathcal{G}(T(z)) &= \{(\phi, T(z)\phi) : \phi \in \mathcal{D}(T(z))\} \\ &= \{(U(z)\psi, V(z)\psi) : \psi \in Z\}\end{aligned} \tag{2.28}$$

and, similarly,

$$\mathcal{G}(T_n(z)) = \{(U_n(z)\psi, V_n(z)\psi) : \psi \in Z\}. \tag{2.29}$$

Fix a $z \in D$ and let $w(z) \in \mathcal{G}(T(z))$. For each positive integer n, there is a $w_n(z) \in \mathcal{G}(T_n(z))$ for which

$$\begin{aligned}\|w(z) - w_n(z)\|_{X\times X} &< \text{dist}\,(w(z), \mathcal{G}(T_n(z))) + \frac{1}{n} \\ &\le \delta_n(z)\|w(z)\|_{X\times X} + \frac{1}{n} \longrightarrow 0\end{aligned} \tag{2.30}$$

as $n \to \infty$ (see (2.25) in Subsection 2.5). Since $w(z) \in \mathcal{G}(T(z))$,

$$w(z) = (U(z)\phi(z), V(z)\phi(z)) \tag{2.31}$$

for some $\phi(z) \in Z$ and since $w_n(z) \in \mathcal{G}(T_n(z))$,

$$w_n(z) = (U_n(z)\psi_n(z), V_n(z)\psi_n(z)) \tag{2.32}$$

for some $\psi_n(z) \in Z$. Because $\|w(z) - w_n(z)\|_{X\times X} \to 0$ as $n \to \infty$, the definition of the norm on $X \times X$ ($\|(a,b)\|_{X\times X} := \left\{\|a\|_X^2 + \|b\|_X^2\right\}^{1/2}$), tells us that

$$\|U(z)\phi(z) - U_n(z)\psi_n(z)\|_X \to 0 \tag{2.33}$$

and

$$\|V(z)\phi(z) - V_n(z)\psi_n(z)\|_X \to 0 \tag{2.34}$$

as $n \to \infty$.

We now ask what happens when we move from $z \in D$ to $z_1 \in D$. Given $\epsilon > 0$, choose $\iota > 0$ according to Definition 2.34. If $z_1 \in D$ satisfies $|z_1 - z| < \iota$, then

$$\begin{aligned}&\|w(z) - w(z_1)\|_{X\times X} \\ &= \|(U(z)\phi(z), V(z)\phi(z)) - (U(z_1)\phi(z_1), V(z_1)\phi(z_1))\|_{X\times X} < \epsilon.\end{aligned} \tag{2.35}$$

In particular, then, it follows from the definition of the norm on $X \times X$ that

$$\|U(z)\phi(z) - U(z_1)\phi(z_1)\|_{X\times X} < \epsilon \tag{2.36}$$

and

$$\|V(z)\phi(z) - V(z_1)\phi(z_1)\|_{X\times X} < \epsilon. \tag{2.37}$$

If we define $W : D \longrightarrow X$ by

$$W(z) := V(z)\phi(z), \tag{2.38}$$

we see that W is continuous at every $z \in D$. Proceeding in the same way, we see that, for each $n \in \mathbb{N}$, the map $W_n : D \longrightarrow X$ defined by

$$W_n(z) := V_n(z)\psi_n(z) \tag{2.39}$$

is continuous at every $z \in D$. Moreover, from (2.34), we see that

$$W_n(z) \longrightarrow W(z) \tag{2.40}$$

in X-norm as $n \to \infty$ for each $z \in D$.

We will use the continuity and limits discussed above in Subsection 4.1 below, just after the definitions (5.18) and (5.19).

3. Feynman's operational calculus on Banach algebras and some stability theory

In this section we first outline how to carry out Feynman's operational calculus in a Banach algebra setting. Indeed, usual setting for the operational calculus is $\mathcal{L}(X)$, a Banach algebra. We will use this formalism in Section 4 of this paper when we study stability theory when using analytic families of closed unbounded linear operators. We will also use the stability result, Theorem 3.16, when considering joint stability of the operational calculus in the setting of analytic families of operators.

3.1. Comments on measurability. Before proceeding further, we will make some comments concerning measurability of functions taking values in a Banach space. We will follow the notation/terminology found in [7]. In particular, let $(S, \mathscr{A})$ be a measurable space and let X be a Banach space. The first definition of *measurable* which comes to mind is that, given a function $f : S \to X$, f is measurable provided that $f^{-1}(B) \in \mathscr{A}$ for every Borel set $B \subset X$. As the reader is no doubt aware, this definition is, in general, not too useful as the Borel class $\mathscr{B}(X)$ is 'too large'. Indeed, the σ-algebra generated by all continuous linear functions on X may be strictly smaller than $\mathscr{B}(X)$ and this is an obstruction to the application of the usual tools of functional analysis. Nevertheless, we state the following definition.

Definition 3.1. Let $(S, \mathscr{A})$ be a measurable space and let X be a Banach space. A function $f : S \to X$ is said to be measurable if $f^{-1}(B) \in \mathscr{A}$ for every Borel set $B \subset X$.

It is the case, however, that if X is a separable Banach space, the problems presented by $\mathscr{B}(X)$ being too large disappear. Indeed, if X is separable and Y is a weak* dense linear subspace of X^* (i.e., Y separates the points of X) then, according to Proposition 1.1.1 of [7],

$$\sigma(Y) = \sigma(X^*) = \mathscr{B}(X),$$

where $\sigma(Y)$ is, by definition, the smallest σ-algebra for which every $y^* \in Y^*$ is measurable. (It can be shown that $\sigma(Y)$ is generated by all sets of the form $\{x \in X : (\langle x, x_1^*\rangle, \dots, \langle x, x_n^*\rangle) \in B\}$ with $n \geq 1$, $x_1^*, \dots, x_n^* \in Y$ and $B \in \mathscr{B}(\mathbb{C}^n)$.) The following corollary (Corollary 1.1.2 of [7]) is useful.

Corollary 3.2. *If X is separable, then for a function $f : S \to X$ the following are equivalent.*

(i) *f is measurable*;
(ii) *$\langle f, x^* \rangle$ is measurable (as a scalar function) for all $x^* \in X^*$.*

Next, we state the following definition, which is Definition 1.1.4 of [7].

Definition 3.3. A function $f : S \to X$ is *strongly measurable* if there is a sequence of simple functions $f_n : S \to X$ such that $f_n \to f$ pointwise on S. When we wish to stress the underlying σ-algebra, we will write "strongly $\mathscr{A}$-measurable."

The result we are interested in is Corollary 1.1.10 of [7] to the Pettis measurability theorem.

Corollary 3.4. *For a function $f : S \to X$, the following assertions are equivalent*:
(i) f is strongly measurable;
(ii) f is separably valued and measurable.

Consequently, if X is separable, then an X-valued function f is strongly measurable if and only if it is measurable.

We now move to considering X-valued functions on the measure space $(S, \mathscr{A}, \mu)$. The definition of strong μ-measurability is Definition 1.1.14 of [7].

Definition 3.5. A function $f : S \to X$ is *strongly μ-measurable* if there is a sequence $f_n : S \to X$ of μ-simple functions converging to f μ-almost everywhere, where a μ-simple function is a simple function defined using elements of $\mathscr{A}$ (i.e., of the form $\sum_{j=1}^{N} \alpha_j \chi_{A_j}$, for $A_j \in \mathscr{A}$ and $\mu(A_j) < \infty$).

A proposition relating the ideas of 'strong measurability' and 'μ - measurability' is Proposition 1.1.16 of [7].

Proposition 3.6. *Consider a function $f : S \to X$.*

(i) If f is strongly μ-measurable, then f is μ-almost everywhere equal to a strongly measurable function.

(ii) If μ is σ-finite and if f is μ-almost everywhere equal to a strongly measurable function, then f is strongly μ-measurable.

Finally, we state the Pettis measurability theorem, Theorem 1.1.20 of [7].

Theorem 3.7. *For a function $f : S \to X$, the following are equivalent*:
(i) f is strongly μ-measurable;
(ii) f is μ-essentially separably valued and weakly μ-measurable;
(iii) f is μ-essentially separably valued and there exists a weak-$*$ dense subspace Y of X^* such that $\langle f, x^* \rangle$ is μ-measurable for all $x^* \in Y$.

Since Feynman's operational calculus involves operator-valued functions, it will be useful to note the relevant definition of strong measurability (respectively, of strong μ-measurability).

Definition 3.8. A function $f : S \to \mathcal{L}(X,Y)$ (X and Y Banach spaces) is called strongly measurable (respectively, strongly μ-measurable) if for all $x \in X$, the Y-valued function $f_x : S \to Y$ defined by $f_x(s) = f(s)x$ is strongly measurable (respectively strongly μ-measurable). The reader will notice that it is more accurate to refer to these functions as being strongly (μ-) measurable with respect to the strong operator topology, however, we will usually omit reference to the strong operator topology.

3.2. Feynman's operational calculus on a Banach algebra. Let X be a Banach algebra and assume, for convenience, that X is separable. As seen in the previous subsection, the assumption that X is separable simplifies discussions of measurability. (See Theorem 3.7 and Corollary 3.4 above.) For each $j = 1, \ldots, n$, let $W_j : [0,T] \to X$ and associate to $W_j(\cdot)$ a Borel probability measure λ_j on $[0,T]$. We assume that $W_j(\cdot)$, $j = 1, \ldots, n$, is strongly λ_j-measurable. As is well-known, $W_j(\cdot)$ is strongly λ_j-measurable if and only if $W_j(\cdot)$ is measurable and λ_j-almost separably valued. (See Corollary 3.4 and, for a detailed discussion, [4, p. 147] or [7].) It is here that our separability assumption on X plays its role. Since X is assumed to be separable, we can therefore say that $W_j(\cdot)$ is strongly λ_j-measurable if and only if $W_j(\cdot)$ is measurable.

To use Feynman's operational calculus in this setting, we have very little to do – all necessary work has already been done earlier. However, we sketch out some details. To start, define, for each $j = 1, \ldots, n$,

$$R_j := \int_{[0,T]} \|W_j(s)\|_X \, \lambda_j(ds)$$

and assume that this number is finite. Construct the commutative Banach algebras $\mathbb{A}(R_1, \ldots, R_n)$ and $\mathbb{D}\left((W_1(\cdot), \lambda_1)^{\sim}, \ldots, (W_j(\cdot), \lambda_j)^{\sim}\right)$ just as before. (A moments' reflection should convince the reader that the formal objects $(W_j(\cdot), \lambda_j)^{\sim}$, $j = 1, \ldots, n$, are defined in essentially the same way as the formal objects for $\mathcal{L}(X)$-valued functions.) Using the same notation as above, we can write for $f \in \mathbb{A}(R_1, \ldots, R_n) \cong \mathbb{D}\left(W_1(\cdot)^{\sim}, \ldots, W_n(\cdot)^{\sim}\right)$, written as a Taylor series centered at $0 \in \mathbb{C}^n$,

$$\begin{aligned} &\mathcal{T}_{\lambda_1,\ldots,\lambda_n} f\left(W_1(\cdot)^{\sim}, \ldots, W_n(\cdot)^{\sim}\right) && (3.1)\\ &= f_{\lambda_1,\ldots,\lambda_n}\left(W_1(\cdot), \ldots, W_n(\cdot)\right) && (3.2)\\ &= \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(W_1(\cdot), \ldots, W_n(\cdot)\right) && (3.3) \end{aligned}$$

where $P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(W_1(\cdot), \ldots, W_n(\cdot)\right)$ takes exactly the same form as in (2.13), earlier, with $W_1(\cdot), \ldots, W_n(\cdot)$ replacing $A_1(\cdot), \ldots, A_n(\cdot)$, all other notation being exactly the same. The properties of the disentangling map $\mathcal{T}_{\lambda_1,\ldots,\lambda_n}$ also remain the same.

3.3. Two analytic lemmas. We now establish two lemmas which will be crucial to our study of the stability theory of the operational calculus using analytic families of operators in Section 4, below. However, before stating and proving our lemmas, we take a moment to remind the reader of the definition of weak convergence of sequences of probability measures.

Definition 3.9. Let S be a metric space and suppose that $\{\lambda_k\}_{k=1}^{\infty}$ be a sequence of Borel probability measures on S. We say that $\{\lambda_k\}_{k=1}^{\infty}$ is *weakly convergent* to the Borel probability measure λ (and write $\lambda_k \rightharpoonup \lambda$) if

$$\lim_{k\to\infty} \int_S f\, d\lambda_k = \int_S f\, d\lambda$$

for every bounded continuous real-valued function f on S.

The first lemma addresses the first ingredient for joint stability.

Lemma 3.10. *Let S be a separable metric space and let $f : S \to X$, X a separable Banach algebra, be strongly λ-measurable where λ is a Borel probability measure on S. Suppose that $\{\lambda_k\}_{k=1}^{\infty}$ is a sequence of Borel probability measures on S for which $\lambda_k \rightharpoonup \lambda$ as $k \to \infty$ and assume that, for every Borel set $E \subseteq S$ and every $k \in \mathbb{N}$, $\lambda_k(E) \le \lambda(E)$. Let $\{f_k\}_{k=1}^{\infty}$ be a sequence of strongly λ_k-measurable functions for each $k \in \mathbb{N}$ and for which $f_k(\cdot) \to f(\cdot)$ λ-almost everywhere on S. Furthermore, we assume that*

$$C_1 := \sup\{\|f(s)\|_X : s \in S\} < \infty \tag{3.4}$$

and that

$$C_2 := \sup\{\|f_k(s)\|_X : s \in S, k \in \mathbb{N}\} < \infty. \tag{3.5}$$

It then follows that

$$\int_S f_k(s)\, \lambda_k(ds) \longrightarrow \int_S f(s)\, \lambda(ds) \tag{3.6}$$

as $k \to \infty$ in norm on X.

Proof. We may write, for any $k, \ell \in \mathbb{N}$,

$$\begin{aligned}
&\left\| \int_S f_k(s)\, \lambda_k(ds) - \int_S f(s)\, \lambda(ds) \right\|_X \\
&\le \left\| \int_S f_k(s)\lambda_k(ds) - \int_S f_\ell(s)\lambda_k(ds) \right\|_X + \left\| \int_S f_\ell(s)\lambda_k(ds) - \int_S f_\ell(s)\lambda(ds) \right\|_X \\
&+ \left\| \int_S f_\ell(s)\, \lambda(ds) - \int_S f(s)\, \lambda(ds) \right\|_X \\
&=: I + II + III.
\end{aligned} \tag{3.7}$$

We treat these terms individually. For term I, we first note that, given $\epsilon > 0$, Egorov's theorem (see, for instance, [6]), supplies a Borel set $A \subseteq S$ such that $f_k(\cdot) \to f(x)$ uniformly on $S \backslash A$ and

$$\lambda(A) < \frac{\epsilon}{12 \max(C_1, C_2)}. \tag{3.8}$$

We may now write

$$I = \left\| \int_{S\backslash A} f_k(s)\lambda_k(ds) - \int_{S\backslash A} f_\ell(s)\,\lambda_k(ds) + \int_A f_k(s)\,\lambda_k(ds) - \int_A f_\ell(s)\,\lambda_k(ds) \right\|_X$$
$$\leq \int_{S\backslash A} \|f_k(s) - f_\ell(s)\|_X\,\lambda_k(ds) + \int_A \|f_k(s)\|_X \lambda_k(ds) + \int_A \|f_\ell(s)\|_X \lambda_k(ds). \tag{3.9}$$

Because $f_k(\cdot) \to f(\cdot)$ uniformly on $S\backslash A$ as $k \to \infty$, $\{f_k\}_{k=1}^{\infty}$ is uniformly Cauchy. There is, then, a $N_1 \in \mathbb{N}$, for which

$$\sup\left\{\|f_k(s) - f_\ell(s)\|_X : s \in S\backslash A\right\} < \frac{\epsilon}{6} \tag{3.10}$$

whenever $k, \ell \geq N_1$. Therefore, for $k, \ell \geq N_1$,

$$\begin{aligned}
&\int_{S\backslash A} \|f_k(s) - f_\ell(s)\|_X\,\lambda_k(ds)\\
&\leq \sup\left\{\|f_k(s) - f_\ell(s)\|_X : s \in S\backslash A\right\}\lambda_k(S\backslash A)\\
&\leq \frac{\epsilon}{6}\cdot \lambda_k(S\backslash A)\\
&\leq \frac{\epsilon}{6}.
\end{aligned} \tag{3.11}$$

Also, using (3.8), we note that

$$\int_A \|f_k(s)\|_X\,\lambda_k(ds) + \int_{[0,T]} \|f_\ell(s)\|_X\,\lambda_k(ds)$$
$$< C_1\left(\frac{\epsilon}{12\max(C_1, C_2)}\right) + C_1\left(\frac{\epsilon}{12\max(C_1, C_2)}\right)$$
$$\leq C_1\left(\frac{\epsilon}{12C_1}\right) + C_1\left(\frac{\epsilon}{12C_1}\right) \tag{3.12}$$
$$= \frac{\epsilon}{6}. \tag{3.13}$$

Therefore, for $k, \ell \geq N_1$, we have

$$I < \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}. \tag{3.14}$$

Turning to term II, for any $\ell \in \mathbb{N}$, we can find $N_2 \in \mathbb{N}$ such that if $k \geq N_2$,

$$\left\| \int_S f_\ell(s)\,\lambda_k(ds) - \int_S f_\ell(s)\,\lambda(ds) \right\|_X < \frac{\epsilon}{3}. \tag{3.15}$$

Note that the integer N_2 depends on epsilon and on ℓ. However, just below, we will be fixing a value of ℓ.

Finally, for term III, we follow our analysis of term I with a change to accommodate the last term. Indeed, we may use (3.8) to write

$$\begin{aligned}
III &= \left\| \int_{S\setminus A} f_\ell(s)\,\lambda(ds) - \int_{S\setminus A} f(s)\,\lambda(ds) + \int_A f_\ell(s)\,\lambda(ds) - \int_A f(s)\,\lambda(ds) \right\|_X \\
&\leq \int_{S\setminus A} \|f_\ell(s) - f(s)\|_X\,\lambda(ds) + \int_A \|f_\ell(s)\|_X\,\lambda(ds) + \int_A \|f(s)\|_X\,\lambda(ds) \\
&< \int_{S\setminus A} \|f_\ell(s) - f(s)\|_X\,\lambda(ds) + C_1\left(\frac{\epsilon}{12\max(C_1, C_2)}\right) \\
&\quad + C_2\left(\frac{\epsilon}{12\max(C_1, C_2)}\right) \\
&\leq \int_{S\setminus A} \|f_\ell(s) - f(s)\|_X\,\lambda(ds) + \frac{\epsilon}{12} + \frac{\epsilon}{12} \\
&= \int_{S\setminus A} \|f_\ell(s) - f(s)\|_X\,\lambda(ds) + \frac{\epsilon}{6}. \qquad (3.16)
\end{aligned}$$

Choose $N_3 \in \mathbb{N}$ such that if $\ell \geq N_3$,

$$\sup\{\|f_\ell(s) - f(s)\|_X : s \in S\setminus A\} < \frac{\epsilon}{6}. \qquad (3.17)$$

Hence, for $\ell \geq N_3$,

$$III \leq \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}.$$

We now put things together. We start by fixing $\ell = \max(N_1, N_3)$ in the sum $I + II + III$ and let $k \geq \max(N_1, N_2)$. It then follows that

$$I + II + III \leq 3\left(\frac{\epsilon}{3}\right) = \epsilon$$

and the proof is complete. □

Our next lemma is the second ingredient which will allow us to establish joint stability for the operational calculus in the setting of analytic families of operators.

Lemma 3.11. *Let X be a separable Banach algebra. Further, let $A : [0,T] \longrightarrow X$ and let, for $k \in \mathbb{N}$, $A_k : [0,T] \longrightarrow X$. Assume that A is strongly λ-measurable where λ is a Borel probability measure on $[0,T]$ and assume that A_k is strongly λ_k-measurable where λ_k is a Borel probability measure on $[0,T]$ for each $k \in \mathbb{N}$. Assume that $\lambda_k \rightharpoonup \lambda$ as $k \to \infty$ and that $A_k(\cdot) \to A(\cdot)$ pointwise in X-norm on $[0,T]$. Finally, assume that*

$$C_1 := \sup\{\|A(s)\|_X : s \in [0,T]\} < \infty \qquad (3.18)$$

and that

$$C_2 := \sup\{\|A_k(s)\|_X : s \in [0,T], k \in \mathbb{N}\} < \infty. \qquad (3.19)$$

For any $m \in \mathbb{N}$, we have

$$\int_{\Delta_m(T)} A_k(s_m)\cdots A_k(s_1)\,\lambda_k^m(ds_1,\ldots,ds_m) \to \int_{\Delta_m(T)} A(s_m)\cdots A(s_1)\,\lambda^m(ds_1,\ldots,ds_m) \quad (3.20)$$

in X-norm as $k\to\infty$.

Proof. Given $m\in\mathbb{N}$, we may write, using the definition of the set $\Delta_m(T)$,

$$\begin{aligned}
&\int_{\Delta_m(T)} A_k(s_m)\cdots A_k(s_1)\,\lambda_k^m(ds_1,\ldots,ds_m) \\
&\qquad - \int_{\Delta_m(T)} A(s_m)\cdots A(s_1)\,\lambda^m(ds_1,\ldots,ds_m) \\
&= \int_0^T\int_0^{s_m}\cdots\int_0^{s_3}\int_0^{s_2} A_k(s_m)A_k(s_{m-1})\cdots A_k(s_2)A_k(s_1)\,\lambda_k^m(ds_1,\ldots,ds_m) \\
&\qquad - \int_0^T\int_0^{s_m}\cdots\int_0^{s_3}\int_0^{s_2} A(s_m)A(s_{m-1})\cdots A(s_2)A(s_1)\,\lambda^m(ds_1,\ldots,ds_m).
\end{aligned} \quad (3.21)$$

We use (3.21) to write, successively,

$$\begin{aligned}
&\int_0^T\int_0^{s_m}\cdots\int_0^{s_3}\int_0^{s_2} A_k(s_m)A_k(s_{m-1})\cdots A_k(s_2)A_k(s_1)\,\lambda_k^m(ds_1,\ldots,ds_m) \\
&\qquad - \int_0^T\int_0^{s_m}\cdots\int_0^{s_3}\int_0^{s_2} A(s_m)A(s_{m-1})\cdots A(s_2)A(s_1)\,\lambda^m(ds_1,\ldots,ds_m) \\
&= \int_0^T\int_0^{s_m}\cdots\int_0^{s_3}\int_0^{s_2} A_k(s_m)A_k(s_{m-1})\cdots A_k(s_2)A_k(s_1)\lambda_k^m(ds_1,\ldots,ds_m) \\
&- \int_0^T\int_0^{s_m}\cdots\int_0^{s_3}\int_0^{s_2} A_k(s_m)A_k(s_{m-1})\cdots A_k(s_2)A(s_1) \\
&\quad \lambda(ds_1)\lambda_k^{m-1}(ds_2,\ldots,ds_m) \\
&+ \int_0^T\int_0^{s_m}\cdots\int_0^{s_3}\int_0^{s_2} A_k(s_m)A_k(s_{m-1})\cdots A_k(s_2)A(s_1) \\
&\quad \lambda(ds_1)\lambda_k^{m-1}(ds_2,\ldots,ds_m) \\
&- \int_0^T\int_0^{s_m}\cdots\int_0^{s_3}\int_0^{s_2} A(s_m)A(s_{m-1})\cdots A(s_2)A(s_1)\,\lambda^m(ds_1,\ldots,ds_m) \\
&= \int_0^T\int_0^{s_m}\cdots\int_0^{s_3} A_k(s_m)A_k(s_{m-1})\cdots A_k(s_2)\left\{\int_0^{s_2} A_k(s_1)\,\lambda_k(ds_1)\right. \\
&\left. - \int_0^{s_2} A(s_1)\,\lambda(ds_1)\right\}\lambda_k^{m-1}(ds_2,\ldots,ds_m)
\end{aligned}$$

$$+ \int_0^T \int_0^{s_m} \cdots \int_0^{s_3} A_k(s_m) A_k(s_{m-1}) \cdots A_k(s_3) \left\{ A_k(s_2) \int_0^{s_2} A(s_1)\, \lambda(ds_1) \right\} \cdot \lambda_k^{m-1}(ds_2, \ldots, ds_m)$$
$$- \int_0^T \int_0^{s_m} \cdots \int_0^{s_3} A(s_m) A(s_{m-1}) \cdots A(s_3) \left\{ A(s_2) \int_0^{s_2} A(s_1)\, \lambda(ds_1) \right\} \cdot \lambda^{m-1}(ds_2, \ldots, ds_m). \tag{3.22}$$

Define

$$A_k^{(1)}(s_2) := A_k(s) \int_0^{s_2} A(s_1)\, \lambda(ds_1) \tag{3.23}$$

and

$$A^{(1)}(s_2) := A(s_2) \int_0^{s_2} A(s_1)\, \lambda(ds_1). \tag{3.24}$$

It is clear that $A_k^{(1)}(s_2) \to A^{(1)}(s_2)$ in X-norm pointwise on $[0,T]$. The expression after the final equality in (3.22) can then be written as

$$\int_0^T \int_0^{s_m} \cdots \int_0^{s_3} A_k(s_m) A_k(s_{m-1}) \cdots A_k(s_2) \left\{ \int_0^{s_2} A_k(s_1)\, \lambda_k(ds_1) - \int_0^{s_2} A(s_1)\, \lambda(ds_1) \right\} \lambda_k^{m-1}(ds_2, \ldots, ds_m)$$
$$+ \int_0^T \int_0^{s_m} \cdots \int_0^{s_3} A_k(s_m) A_k(s_{m-1}) \cdots A_k(s_3) A_k^{(1)}(s_2)\, \lambda_k^{m-1}(ds_2, \ldots, ds_m)$$
$$- \int_0^T \int_0^{s_m} \cdots \int_0^{s_3} A(s_m) A(s_{m-1}) \cdots A(s_3) A^{(1)}(s_2)\, \lambda^{m-1}(ds_2, \ldots, ds_m) \tag{3.25}$$

We continue just as in (3.22), defining

$$A_k^{(j)}(s_{j+1}) := A_k(s_{j+1}) \int_0^{s_{j+1}} A(s_j)\, \lambda(ds_j) \tag{3.26}$$

and

$$A^{(j)}(s_{j+1}) := A(s_{j+1}) \int_0^{s_{j+1}} A(s_j)\, \lambda(ds_j) \tag{3.27}$$

for $j = 2, 3, \ldots, m-1$. We can then the difference of iterated integrals in (3.21) as

$$\int_0^T \int_0^{s_m} \cdots \int_0^{s_3} \int_0^{s_2} A_k(s_m) A_k(s_{m-1}) \cdots A_k(s_2) A_k(s_1)\, \lambda_k^m(ds_1, \ldots, ds_m)$$
$$- \int_0^T \int_0^{s_m} \cdots \int_0^{s_3} \int_0^{s_2} A(s_m) A(s_{m-1}) \cdots A(s_2) A(s_1)\, \lambda^m(ds_1, \ldots, ds_m)$$
$$= \int_0^T \cdots \int_0^{s_3} A_k(s_m) \cdots A_k(s_2) \left\{ \int_0^{s_2} A_k(s_1)\, \lambda_k(ds_1) - \int_0^{s_2} A(s_1)\, \lambda(ds_1) \right\} \cdot \lambda_k^{m-1}(ds_2, \ldots, ds_m) \tag{3.28}$$

$$+\int_0^T\cdots\int_0^{s_4} A_k(s_m)\cdots A_k(s_3)\left\{\int_0^{s_3} A_k^{(1)}(s_2)\,\lambda_k(ds_2)-\int_0^{s_3} A^{(1)}(s_2)\,\lambda(ds_2)\right\}\cdot$$
$$\lambda_k^{m-2}(ds_3,\ldots,ds_m)$$
$$+\int_0^T\cdots\int_0^{s_5} A_k(s_m)\cdots A_k(s_4)\left\{\int_0^{s_4} A_k^{(2)}(s_3)\,\lambda_k(ds_3)-\int_0^{s_4} A^{(2)}(s_3)\,\lambda(ds_3)\right\}\cdot$$
$$\lambda_k^{m-3}(ds_4,\ldots,ds_m)$$
$$+\int_0^T\cdots\int_0^{s_6} A_k(s_m)\cdots A_k(s_5)\left\{\int_0^{s_5} A_k^{(3)}(s_4)\,\lambda_k(ds_4)-\int_0^{s_5} A(s_4)\,\lambda(ds_4)\right\}\cdot$$
$$\lambda_k^{m-4}(ds_5,\ldots,ds_m)$$
$$+\cdots+\int_0^T A_k(s_m)\left\{\int_0^{s_m} A_k^{(m-2)}(s_{m-1})\,\lambda_k(ds_{m-1})\right.$$
$$\left.-\int_0^{s_m} A^{(m-2)}(s_{m-1})\,\lambda(ds_{m-1})\right\}\lambda_k(ds_m)$$
$$+\left\{\int_0^T A_k^{(m-1)}(s_m)\lambda_k(ds_m)-\int_0^T A^{(m-1)}(s_m)\,\lambda(ds_m)\right\}.$$

To continue, then, we need to look at the norms of each term above:

$$\left\|\int_0^T\cdots\int_0^{s_{j+1}} A_k(s_m)\cdots A_k(s_j)\left\{\int_0^{s_j} A_k^{(j-2)}(s_{j-1})\,\lambda_k(ds_{j-1})\right.\right.$$
$$\left.\left.-\int_0^{s_j} A^{(j-2)}(s_{j-1})\,\lambda(ds_{j-1})\right\}\lambda_k^{m-(j-1)}(ds_j,\ldots,ds_m)\right\|_X. \tag{3.29}$$

We can bound this term by

$$\int_0^T\cdots\int_0^{s_{j+1}} \|A_k(s_m)\|_X\cdots\|A_k(s_j)\|_X\left\{\left\|\int_0^{s_j} A_k^{(j-2)}(s_{j-1})\,\lambda_k(ds_{j-1})\right.\right.$$
$$\left.\left.-\int_0^{s_j} A^{(j-2)}(s_{j-1})\,\lambda(ds_{j-1})\right\|_X\right\}\lambda_k^{k-(j-1)}(ds_j,\ldots,ds_m) \tag{3.30}$$
$$\le C_2^{m-(j-1)}\left\|\int_0^{s_j} A_k^{(j-2)}(s_{j-1})\,\lambda_k(ds_{j-1})-\int_0^{s_j} A^{(j-2)}(s_{j-1})\,\lambda(ds_{j-1})\right\|_X$$

using (3.5) and the fact that each of our measures is a probability measure on $[0,T]$.

Now, by definition,

$$A_k^{(j-2)}(s_{j-1})=A_k(s_{j-1})\int_0^{s_{j-1}} A^{(j-3)}(s_{j-2})\,\lambda(ds_{j-2})$$

and

$$A^{(j-2)}(s_{j-1})=A(s_{j-1})\int_0^{s_{j-1}} A^{(j-3)}(s_{j-2})\,\lambda(ds_{j-2}).$$

Since $A_k(\cdot) \to A(\cdot)$ pointwise in X-norm and since

$$\int_0^{s_{j-1}} A^{(j-3)}(s_{j-2})\, \lambda(ds_{j-2})$$

is norm bounded, it follows that $A_k^{(j-2)}(\cdot) \to A^{(j-2)}(\cdot)$ pointwise in X-norm. Using Lemma 3.10, we are able to conclude that

$$\left\| \int_0^{s_j} A_k^{(j-2)}(s_{j-1})\, \lambda_k(ds_{j-1}) - \int_0^{s_j} A^{(j-2)}(s_{j-1})\, \lambda(ds_{j-1}) \right\|_X \to 0$$

as $k \to \infty$. The same lemma shows that

$$\left\| \int_0^{s_2} A_k(s_1)\, \lambda_k(ds_1) - \int_0^{s_2} A(s_1)\, \lambda(ds_1) \right\|_X \to 0$$

and that

$$\left\| \int_0^{T} A_k^{(m-1)}(s_m)\, \lambda_k(ds_m) - \int_0^{T} A^{(m-1)}(s_m)\, \lambda(ds_m) \right\|_X \to 0$$

as $k \to \infty$. It follows, then, from the triangle inequality and the inequality (3.30), that

$$\left\| \int_{\Delta_m(T)} A_k(s_m) \cdots A_k(s_1) \lambda_k^m(ds_1, \ldots, ds_m) - \int_{\Delta_m(T)} A_k(s_m) \cdots A_k(s_1) \lambda_k^m(ds_1, \ldots, ds_m) \right\|_X \to 0$$

as $k \to 0$. □

We now consider how to apply the lemma above to the disentangling of

$$\left(\int_0^T A(s)\, \lambda(ds) \right)^m, \tag{3.31}$$

where

$$\lambda := \mu + \sum_{i=1}^{h} p_i \delta_{\tau_i}$$

and where μ is a continuous measure. Using the definition of the disentangling map, we may write the disentangling of the expression (3.31) as

$$\sum_{q_{11}+q_{12}=m} \frac{m!}{q_{11}!q_{12}!} \sum_{j_1+\cdots+j_h=q_{12}} \frac{q_{12}!}{j_1!\cdots j_h!} \sum_{\theta_1+\cdots+\theta_{h+1}=q_{11}} \sum_{\pi\in S_{q_{11}}} \cdot \tag{3.32}$$

$$\int_{\Delta_{q_{11};\theta_1,\ldots,\theta_{h+1}}(\pi)} A(s_{\pi(q_{11})}) \cdots A(s_{\pi(\theta_1+\cdots+\theta_h+1)})\, [p_h A(\tau_h)]^{j_h}\, A(s_{\pi(\theta_1+\cdots+\theta_h)})$$

$$\cdots A(s_{\pi(\theta_1+1)})\, [p_1 A(\tau_1)]^{j_1}\, A(s_{\pi(\theta_1)}) \cdots A(s_{\pi(1)})\, \mu^{q_{11}}(ds_1, \ldots, ds_{q_{11}}).$$

Now, while one does not obtain (3.32) directly from the integral

$$\int_{\Delta_m(T)} A(s_m)\cdots A(s_1)\lambda_k^m(ds_1,\ldots,ds_m),$$

in the particular case we are addressing, the integrals in the disentangling of

$$\left(\int_0^T A(s)\,\lambda(ds)\right)^m,$$

can be obtained from the integral

$$\int_{\Delta_m(T)} A(s_m)\cdots A(s_1)\,\lambda^m(ds_1,\ldots,ds_m)$$

when careful attention is paid to preserving the time-ordering of the $A(s_i)$. Indeed, when the time-ordering measure λ has the form $\lambda = \mu + \omega\delta_\tau$ for $\tau \in (0,T)$, a tedious calculation of

$$\begin{aligned}\int_{\Delta_m(T)} & A(s_m)\cdots A(s_1)\ (\mu+\omega\delta_\tau)^m\,(ds) \\ &= \sum_{j=0}^m \int_{\Delta_{m;j}(T)} A(s_m)\cdots A(s_1)\ (\mu+\omega\delta_\tau)^m\,(ds_1,\ldots,ds_m)\end{aligned} \tag{3.33}$$

where we modify, for our purposes here, the definition of $\Delta_{m;j}(T)$ to

$$\begin{aligned}\Delta_{m;j}(T) := \{(s_1,\ldots,s_m)\in[0,T]^m : 0 < s_1 < \cdots < s_j \le \tau < s_{j+1} < \cdots \\ < s_m < T\},\end{aligned}$$

for $j = 0,\ldots,m$. The "end" cases are $j=0$, where then $\tau < s_1$ and $j = m$, where $s_m < \tau$. One then writes each term of the sum in (3.33) as an iterated integral and evaluate each integral using the expansion of $(\mu+\omega\delta_\tau)^m$. For instance, when $m = 3$, we write

$$(\mu+\omega\delta_\tau)^3 = \mu^3 + \mu\times\mu\times\omega\delta_\tau + \mu\times\omega\delta_\tau\times\mu + \omega\delta_\tau\times\mu\times\mu + \cdots + (\omega\delta_\tau)^3$$

and evalutate the integral over $\Delta_{m;j}(T)$ as an iterated integral, keeping careful eye on the ordering of the integrand. The resulting integrals turn out to be the same integrals as seen in (3.32) although the multinomial coefficients are missing. Nevertheless, our attention is focussed on the integrals using the sequences $\{A_k(\cdot)\}_{k=1}^\infty$ and $\{\lambda_k\}_{k=1}^\infty$ and these integals come from decomposing the integral over $\Delta_m(T)$ as in (3.33). By the lemma, we can write, where the limits are in X-norm,

$$\begin{aligned}\lim_{k\to\infty} & \int_{\Delta_m(T)} A_k(s_m)\cdots A_k(s_1)\,\lambda_k^m(ds_1,\ldots,ds_m) \\ &= \sum_{j=0}^m \lim_{k\to\infty}\int_{\Delta_{m;j}(T)} A_k(s_m)\cdots A_k(s_1)\,\lambda_k^m(ds_1,\ldots,ds_m)\end{aligned}$$

and so the limit of each term of the sum exists. Consequently, we can use the lemma to conclude that

$$P^m_{\lambda_k}(A_k(\cdot)) \to P^m_{\lambda}(A(\cdot))$$

in X-norm as $k \to \infty$, where $P^m_{\lambda_k}(A_k(\cdot))$ and $P^m_{\lambda}(A(\cdot))$ are written as in (3.32).

3.4. The setting and necessary constructions for joint stability theory. In this section, we introduce the necessary setting which will allow us to investigate joint stability of the operational calculus in the setting of analytic families of operators. (This setting also allows for an improvement of the stability theory in the usual setting of the operational calculus – see the discussion at the end of this paper.)

We take X to be a separable Banach algebra. Select maps $Y_j : [0,T] \to X$, $j = 1, \ldots, n$. Associate to each of the maps $Y_j(\cdot)$ a Borel probability measure λ_j on $[0,T]$. We will assume that each $Y_j(\cdot)$ is strongly λ_j-measurable. Next, assume that there is, for each $j = 1, \ldots, n$, a $R_j > 0$ for which

$$\sigma\left(Y_j(s)\right) \subsetneq B\left(0, R_j\right), \tag{3.34}$$

for all $s \in [0,T]$, where $\sigma(x)$ denotes the spectrum of the element $x \in X$ and where $B\left(0, r\right)$ is the open ball in $\mathbb{C}$ centered at 0 with radius r. Construct, as in Section 2, the commutative Banach algebra $\mathbb{A}\left(R_1, \ldots, R_n\right)$.

For each $j = 1, \ldots, n$, assume that $Y_j(\cdot)$ is in $L^1\left([0,T], X, \lambda_j\right)$ and define positive real numbers $r_1, \ldots, r_n$ by

$$r_j := \int_{[0,T]} \|Y_j(s)\|_X \, \lambda_j(ds).$$

Observe that, in view of (3.34),

$$\sup\left\{\|Y_j(s)\|_X : s \in [0,T]\right\} \leq R_j$$

and so

$$\begin{aligned} r_j &= \int_{[0,T]} \|Y_j(s)\|_X \, \lambda_j(ds) \\ &\leq \sup\left\{\|Y_j(s)\|_X : s \in [0,T]\right\} \lambda_j([0,T]) \\ &\leq R_j \end{aligned} \tag{3.35}$$

for $j = 1, \ldots, n$. Construct the commutative Banach algebras $\mathbb{A}\left(r_1, \ldots, r_n\right)$ and $\mathbb{D}\left(Y_1(\cdot)^\sim, \ldots, Y_n(\cdot)^\sim\right)$. In view of (3.35), it follows that

$$\mathbb{A}\left(R_1, \ldots, R_n\right) \subseteq \mathbb{A}\left(r_1, \ldots, r_n\right). \tag{3.36}$$

We define the disentangling algebra $\mathbb{D}\left(R_1, \ldots, R_n\right)$ (note the different notation here) as the family of all expressions

$$f\left(Y_1(\cdot)^\sim, \ldots, Y_n(\cdot)^\sim\right) = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n}\left(Y_1(\cdot)^\sim\right)^{m_1} \cdots \left(Y_n(\cdot)^\sim\right)^{m_n} \tag{3.37}$$

for which

$$\|f\|_{\mathbb{D}(R_1,\ldots,R_n)} := \sum_{m_1,\ldots,m_n=0}^{\infty} |a_{m_1,\ldots,m_n}| R_1^{m_1} \cdots R_n^{m_n} < \infty. \tag{3.38}$$

The proof that $\mathbb{D}(R_1,\ldots,R_n)$ is a commutative Banach algebra is the same as that seen in [15, Chapter 2] as is the proof that $\mathbb{A}(R_1,\ldots,R_n)$ is isometrically isomorphic to $\mathbb{D}(R_1,\ldots,R_n)$. Note that

$$\mathbb{D}(R_1,\ldots,R_n) \subseteq \mathbb{D}(Y_1(\cdot)^{\sim},\ldots,Y_n(\cdot)^{\sim}). \tag{3.39}$$

Now, let $\ell \in \mathbb{N}$ and let $j \in \{1,\ldots,n\}$. Define

$$p_j^{\ell}(Y_1(\cdot)^{\sim},\ldots,Y_n(\cdot)^{\sim}) := [Y_j(\cdot)^{\sim}]^{\ell}. \tag{3.40}$$

It is clear that $p_j^{\ell} \in \mathbb{D}(R_1,\ldots,R_n)$ and that

$$\|p_j^{\ell}\|_{\mathbb{D}(R_1,\ldots,R_n)} = R_j^{\ell} \left(= \left\| (Y_j(\cdot)^{\sim})^{\ell} \right\|_{\mathbb{D}(R_1,\ldots,R_n)} \right). \tag{3.41}$$

Knowing this relation, it is clear that, for each $j = 1,\ldots,n$,

$$\operatorname{spr}(p_j^1) = \lim_{\ell\to\infty} \left\| p_j^{\ell} \right\|_{\mathbb{D}(R_1,\ldots,R_n)}^{1/\ell} = \lim_{\ell\to\infty} \left(R_j^{\ell} \right)^{1/\ell} = R_j,$$

where $\operatorname{spr}(x)$ denotes the spectral radius of an element $x \in X$. From the definition of R_j, $j = 1,\ldots,n$, it follows that

$$R_j \geq \sup\{\operatorname{spr}(Y_j(s)) : s \in [0,T]\} \tag{3.42}$$

and so

$$\operatorname{spr}(Y_j(\cdot)^{\sim}) \geq \operatorname{spr}(Y_j(s)) \tag{3.43}$$

for all $s \in [0,T]$.

With an eye towards using Cauchy's integral theorm in our formalism, we begin by considering the function

$$h : \rho(Y_1(\cdot)^{\sim}) \times \cdots \times \rho(Y_n(\cdot)^{\sim}) \longrightarrow \mathbb{D}(R_1,\ldots,R_n)$$

by

$$h(\xi_1,\ldots,\xi_n; Y_1(\cdot)^{\sim},\ldots,Y_n(\cdot)^{\sim}) := \xi_1 \cdots \xi_n (\xi_1 - Y_1(\cdot)^{\sim})^{-1} \cdots (\xi_n - Y_n(\cdot)^{\sim})^{-1}. \tag{3.44}$$

Note that we may write

$$h(\xi_1,\ldots,\xi_n; Y_1(\cdot)^{\sim},\ldots,Y_n(\cdot)^{\sim}) = \sum_{m_1,\ldots,m_n=0}^{\infty} \xi^{-m_1} \cdots \xi_n^{-m_n} (Y_1(\cdot)^{\sim})^{m_1} \cdots (Y_n(\cdot)^{\sim})^{m_n} \tag{3.45}$$

and this series converges because

$$\xi_j > \|Y_j(\cdot)^{\sim}\|_{\mathbb{D}(R_1,\ldots,R_n)} = R_j$$

and

$$R_j > \sup \left\{ \|Y_j(s)\|_X : s \in [0,T] \right\}$$

for each $j = 1, \ldots, n$. So, for $\xi_1, \ldots, \xi_n \in \mathbb{C}$, $|\xi_j| > \|Y_j(\cdot)^\sim\|_{\mathbb{D}(R_1,\ldots,R_n)}$, $j = 1, \ldots, n$,

$$h\left(\xi_1, \ldots, \xi_n : Y_1(\cdot)^\sim, \ldots, Y_n(\cdot)^\sim\right) \in \mathbb{D}\left(R_1, \ldots, R_n\right).$$

It is clear that h is continuous.

Now, let

$$P_0 := \{(z_1, \ldots, z_n) : |z_j| \le R_j, j = 1, \ldots, n\} \tag{3.46}$$

and choose sequences $\{\epsilon_{j,k}\}_{k=1}^{\infty}$, $j = 1, \ldots, n$, of positive real numbers for which

$$\epsilon_{j,k} \searrow 0 \tag{3.47}$$

as $k \to \infty$ for each $j = 1, \ldots, n$. (Note: we use the symbol $a_k \searrow a$ to specify that the sequence $\{a_k\}_{k=1}^{\infty}$ is a decreasing sequence which converges to a.) Given $j \in \{1, \ldots, n\}$ and $k \in \mathbb{N}$, we define

$$R_{j,k} := R_j + \epsilon_{j,k}. \tag{3.48}$$

We use these numbers to define, for $k \in \mathbb{N}$,

$$P_k := \{(z_1, \ldots, z_n) : |z_j| \le R_{j,k}, j = 1, \ldots, n\}. \tag{3.49}$$

Clearly $P_{k+1} \subseteq P_k$ for all $k \in \mathbb{N}$ and so $\mathbb{A}_k \subseteq \mathbb{A}_{k+1}$,where, for any $\ell \in \mathbb{N}$,

$$\mathbb{A}_\ell := \mathbb{A}\left(R_{1,\ell}, \ldots, R_{n,\ell}\right). \tag{3.50}$$

For $k, \ell \in \mathbb{N}$ with $k < \ell$, the map $g_{\ell k} : \mathbb{A}_k \to \mathbb{A}_\ell$ defined by

$$g_{\ell k}(f) := f|_{P_\ell}. \tag{3.51}$$

It is clear that $g_{\ell k}$ is linear and bounded. Indeed, linearity is obvious and we have

$$\|g_{\ell k}(f)\|_{\mathbb{A}_\ell} = \|f|_{P_\ell}\|_{\mathbb{A}_\ell} \le \|f\|_{\mathbb{A}_k}.$$

This system of maps and algebras is an inductive system (see [1], for example) and we define

$$\mathbb{A}_\infty := \lim_{\rightarrow} g_{\ell k}(\mathbb{A}_k), \tag{3.52}$$

the inductive limit of the inductive system determined by the maps $g_{\ell k}$ and the Banach algebras $\mathbb{A}_k$. Since the construction of an inductive system is categorical, $\mathbb{A}_\infty$ is itself a commutative Banach algebra. with norm

$$\|f\|_{\mathbb{A}_\infty} = \sup\left\{\|f\|_{\mathbb{A}_k} : k \in \mathbb{N}\right\}. \tag{3.53}$$

Note that, if $f \in \mathbb{A}_\infty$, f is analytic on a polydisk containing P_0 and there is a least integer $k_0 \in \mathbb{N}$ for which $f \in \mathbb{A}_{k_0}$ and $f \in \mathbb{A}_k$ for all $k \ge k_0$. The same construction with the associated (and isometrically isomorphic) disentangling algebras allows us to define

$$\mathbb{D}_\infty := \lim_{\rightarrow} g_{\ell k}(\mathbb{D}_k). \tag{3.54}$$

Definition 3.12. For each $k \in \mathbb{N}$, let $\mathcal{T}^{(k)}_{\mu_1,\ldots,\mu_n}$ denote the disentangling map with domain $\mathbb{D}_k$ using time-ordering directions supplied by the time-ordering measures $\mu_1, \ldots, \mu_n$.

Definition 3.13. Given $f \in \mathbb{D}_\infty$, let $k_0 \in \mathbb{N}$ be the least integer for which $f \in \mathbb{D}_{k_0}$. Define

$$\mathcal{T}_{\mu_1,\ldots,\mu_n} f := \mathcal{T}^{(k_0)}_{\mu_1,\ldots,\mu_n} f, \tag{3.55}$$

with the disentangling is done using time-ordering directions supplied by the time-ordering measures $\mu_1, \ldots, \mu_n$. Note that, if $k \geq k_0$, then

$$\mathcal{T}^{(k)}_{\mu_1,\ldots,\mu_n} f = \mathcal{T}^{(k_0)}_{\mu_1,\ldots,\mu_n} f.$$

Also, it follows from Proposition IV.5.6 of [1] (concerning inductive systems) that $\mathcal{T}_{\mu_1,\ldots,\mu_n}$ is continuous from $\mathbb{D}_\infty$ to X.

Remark 3.14. In each of the definitions above, we can use the algebras $\mathbb{A}_k$ and $\mathbb{A}_\infty$, which are isometrically isomorphic to the corresponding disentangling algebras $\mathbb{D}_k$ and $\mathbb{D}_\infty$.

We now take note of how we can use Cauchy's integral theorem in the setting of this section. Given $f \in \mathbb{D}_\infty$ (or $\mathbb{A}_\infty$), we let k_0 be the least positive integer for which $f \in \mathbb{D}_{k_0}$. Returning to our X-valued functions $Y_j(\cdot)$ and their associated time-ordering measures λ_j and using [26], we may write

$$\begin{aligned}
&\mathcal{T}_{\lambda_1,\ldots\lambda_n} f\left(Y_1(\cdot)^\sim, \ldots, Y_n(\cdot)^\sim\right) \\
&= \mathcal{T}^{(k_0)}_{\lambda_1,\ldots,\lambda_n} f\left(Y_1(\cdot)^\sim, \ldots, Y_n(\cdot)^\sim\right) \\
&= (2\pi i)^{-n} \int\limits_{|\xi_1| = R_{1,k_0}} \cdots \int\limits_{|\xi_n| = R_{n,k_0}} f(\xi_1, \ldots, \xi_n) \mathcal{T}^{(k_0+1)}_{\lambda_1,\ldots,\lambda_n} \Big((\xi_1 - Y_1(\cdot)^\sim)^{-1} \cdots \\
&\qquad (\xi_n - Y_n(\cdot)^\sim)^{-1} \Big)\, d\xi_1 \cdots d\xi_n.
\end{aligned} \tag{3.56}$$

Define real numbers M_j, $j = 1, \ldots, n$, by

$$M_j := R_j + \epsilon_{j,1} \tag{3.57}$$

and modify, if necessary, $\epsilon_{j,1}$ so that $M_j > 1$. Next, define

$$\begin{aligned}
\mathcal{L}_{M_1,\ldots,M_n} := \{&(B_1(\cdot), \ldots, B_n(\cdot)) : [0,T] \to X^n : B_j : [0,T] \to X \text{ is strongly} \\
&\lambda_j\text{-measurable and } \sup\{\|B_j(s)\|_X : s \in [0,T]\} < M_j, j = 1, \ldots, n\}.
\end{aligned} \tag{3.58}$$

Construct the commutative Banach algebra $\mathbb{A}(M_1, \ldots, M_n)$. For $(B_1(\cdot), \ldots, B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$, let

$$\beta_j^B := \int_{[0,T]} \|B_j(s)\|_X \, \lambda_j(ds) \tag{3.59}$$

for $j = 1, \ldots, n$. Construct the commutative Banach algebra $\mathbb{A}\left(\beta_1^B, \ldots, \beta_n^B\right)$. We will, now and in the sequel, associate the Borel probability measure λ_j

on $[0,T]$ to $B_j(\cdot)$, the j^{th} component of the n-tuple $(B_1(\cdot),\ldots,B_n(\cdot))$. Note that, because $\beta_j^B \le M_j$ for each $j=1,\ldots,n$,

$$\mathbb{A}\left(M_1,\ldots,M_n\right) \subseteq \mathbb{A}\left(\beta_1^B,\ldots,\beta_n^B\right). \tag{3.60}$$

This containment holds for any n-tuple $(B_1(\cdot),\ldots,B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$. Since, for every $(B_1(\cdot),\ldots,B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$, $\mathbb{A}\left(\beta_1^B,\ldots,\beta_n^B\right)$ is isometrically isomorphic to

$$\mathbb{D}\left((B_1(\cdot),\lambda_1)^{\sim},\ldots,(B_n(\cdot),\lambda_n)^{\sim}\right),$$

we can consider $\mathbb{A}\left(M_1,\ldots,M_n\right)$ to be a subalgebra of every disentangling algebra $\mathbb{D}\left((B_1(\cdot),\lambda_1)^{\sim},\ldots,(B_n(\cdot),\lambda_n)^{\sim}\right)$ with $(B_1(\cdot),\ldots,B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$. Hence, any $f \in \mathbb{A}\left(M_1,\ldots,M_n\right)$ can be taken to be an element of

$$\mathbb{D}\left((B_1(\cdot),\lambda_1)^{\sim},\ldots,(B_n(\cdot),\lambda_n)^{\sim}\right)$$

for every n-tuple $(B_1(\cdot),\ldots,B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$.

We will now, in the definition (3.58), replace the phrase "strongly λ_j-measurable" with "continuous." To be explicit, we now take

$$\mathcal{L}_{M_1,\ldots,M_n} := \{(B_1(\cdot),\ldots,B_n(\cdot)) : [0,T] \to X^n : B_j : [0,T] \to X \text{ is continuous and } \sup\{\|B_j(s)\|_X : s \in [0,T]\} < M_j, j=1,\ldots,n\}.$$

Now, let

$$b := (B_1(\cdot),\ldots,B_n(\cdot)),\ c := (C_1(\cdot),\ldots,C_n(\cdot))$$

be elements of $\mathcal{L}_{M_1,\ldots,M_n}$. Also, select sequences $\{B_{j,k}(\cdot)\}_{k=1}^{\infty}, \{C_{j,k}(\cdot)\}_{k=1}^{\infty}$ for which $B_{j,k}(\cdot) \to B_j(\cdot)$ and $C_{j,k}(\cdot) \to C_j(\cdot)$ λ_j-almost everywhere in norm for each $j=1,\ldots,n$ and assume that

$$b_k := (B_{1,k}(\cdot),\ldots,B_{n,k}(\cdot)),\ c_k := (C_{1,k}(\cdot),\ldots,C_{n,k}(\cdot))$$

are elements of $\mathcal{L}_{M_1,\ldots,M_n}$ for all $k \in \mathbb{N}$. We also select sequences $\{\lambda_{j,k}\}_{k=1}^{\infty}$, $j=1,\ldots,n$, of Borel probability measures on $[0,T]$ such that $\lambda_{j,k} \rightharpoonup \lambda_j$ as $k \to \infty$. Our goal is to connect the convergence of the sequences

$$\left\{f_{\lambda_{1,k},\ldots,\lambda_{n,k}}(b_k)\right\}_{k=1}^{\infty} \text{ and } \left\{f_{\lambda_{1,k},\ldots,\lambda_{n,k}}(c_k)\right\}_{k=1}^{\infty}.$$

We will continue using the notation $b = (B_1(\cdot),\ldots,B_n(\cdot))$ for elements of $\mathcal{L}_{M_1,\ldots,M_n}$ and we will use $\tilde{b}$ for the corresponding n-tuple of formal objects. Define

$$\mathbb{D}_{\oplus} := \bigoplus_{b \in \mathcal{L}_{M_1,\ldots,M_n}} \mathbb{D}(\tilde{b}). \tag{3.61}$$

Then $\mathbb{D}_{\oplus}$ is a commutative Banach algebra under the norm

$$\left\|\{f_b\}_{b\in\mathcal{L}_{M_1,\ldots,M_n}}\right\|_{\mathbb{D}_{\oplus}} = \sup\left\{\|f_b\|_{\mathbb{D}(\tilde{b})} : b \in \mathcal{L}_{M_1,\ldots,M_n}\right\}. \tag{3.62}$$

Next, define

$$\mathscr{C} := \left\{\theta_f \in \mathbb{D}_{\oplus} : \theta_f = \{f_b\}_{b\in\mathcal{L}_{M_1,\ldots,M_n}}, f_b \equiv f \in \mathbb{A}\left(M_1,\ldots,M_n\right) \text{ for all } b \in \mathcal{L}_{M_1,\ldots,M_n}\right\}. \tag{3.63}$$

Clearly $\mathscr{C}$ is a subalgebra of $\mathbb{D}_{\oplus}$ and, for $\theta_f \in \mathscr{C}$,

$$\|\theta_f\|_{\mathbb{D}_{\oplus}} = \|f\|_{\mathbb{A}(M_1,\ldots,M_n)}. \tag{3.64}$$

Define $\Gamma : \mathscr{C} \to \mathbb{A}(M_1,\ldots,M_n)$ by

$$\Gamma(\theta_f) := f. \tag{3.65}$$

Clearly Γ is a linear bijection and, in view of (3.65), is an isometry. The same is true for Γ^{-1} and so Γ is an isometric isomorphism.

To continue, define $\Pi : \mathcal{L}_{M_1,\ldots,M_n} \times \mathscr{C} \to \mathbb{A}(M_1,\ldots,M_n)$ by

$$\Pi(b,\theta_f) := \Gamma(\theta_f)(\tilde{b}) = f(\tilde{b}). \tag{3.66}$$

For each fixed $b \in \mathcal{L}_{M_1,\ldots,M_n}$, $\Pi(b,\cdot)$ is the canonical projection on $\mathbb{D}_{\oplus}$ restricted to $\mathscr{C}$. Hence $\Pi(b,\cdot)$ is continuous. For fixed θ_f, in view of the identification of $\mathbb{A}(M_1,\ldots,M_n)$ as a subalgebra of $\mathbb{D}(\tilde{b})$ for every $b \in \mathcal{L}_{M_1,\ldots,M_n}$, $\Pi(\theta_f,\cdot)$ is constant-valued and so continuous.

We now put time-ordering measures into the mix. Let $\mathscr{P}_{\text{Borel}}([0,T])$ be the space of Borel probability measures on $[0,T]$ with the topology of weak convergence. Let $\mathscr{P}_{\text{finite}}([0,T])$ be the subspace of $\mathscr{P}_{\text{Borel}}([0,T])$ consisting of Borel probability measures with finitely supported discrete parts (we will take $\mathscr{P}_{\text{finite}}([0,T])$ to contain all continuous Borel probability measures on $[0,T]$). Define

$$T : [\mathscr{P}_{\text{finite}}([0,T])]^n \times \mathcal{L}_{M_1,\ldots,M_n} \times \mathscr{C} \to X$$

by

$$T\left((\lambda_1,\ldots,\lambda_n), b, \theta_f\right) := f_{\lambda_1,\ldots,\lambda_n}(b). \tag{3.67}$$

It is clear that

$$T = \mathcal{T}_{\lambda_1,\ldots,\lambda_n} \circ \Pi.$$

We now come to the following, crucial, proposition.

Proposition 3.15. *For each $1 \le j \le n$ let $\{\lambda_{j,k}\}_{k=1}^{\infty}$ be a sequence of Borel probability measures from $\mathscr{P}_{finite}([0,T])$ for which $\lambda_{j,k} \rightharpoonup \lambda_j$ as $k \to \infty$, where $\lambda_j \in \mathscr{P}_{finite}([0,T])$. Let $(B_{1,k}(\cdot),\ldots,B_{n,k}(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$ for every $k \in \mathbb{N}$ and let $(C_{1,k}(\cdot),\ldots,C_{n,k}(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$ for every $k \in \mathbb{N}$. Assume that $B_{j,k}(\cdot) \to B_j(\cdot)$ as $k \to \infty$ and that $C_{j,k}(\cdot) \to C_j(\cdot)$ as $k \to \infty$ λ_j-almost everywhere in $[0,T]$. We will denote our n-tuples as b, b_k, c, c_k. For $f \in \mathbb{A}(M_1,\ldots,M_n)$,*

$$f_{\lambda_{1,k},\ldots,\lambda_{n,k}}(b_k) \to f_{\lambda_1,\ldots,\lambda_n}(b) \ \textit{ iff } f_{\lambda_{1,k},\ldots,\lambda_{n,k}}(c_k) \to f_{\lambda_1,\ldots,\lambda_n}(c). \tag{3.68}$$

Proof. We may write, for $f \in \mathbb{A}(M_1,\ldots,M_n)$,

$$\begin{aligned}
&f_{\lambda_{1,k},\ldots,\lambda_{n,k}}(b_k) - f_{\lambda_1,\ldots,\lambda_n}(b)\\
&= \mathcal{T}_{\lambda_{1,k},\ldots,\lambda_{n,k}}\left(\Pi(b_k,\theta_f)\right) - \mathcal{T}_{\lambda_1,\ldots,\lambda_n}\left(\Pi(b,\theta_f)\right)\\
&= \mathcal{T}_{\lambda_{1,k},\ldots,\lambda_{n,k}}\left(\Pi(b_k,\theta_f) - \Pi(c_k,\theta_f)\right) + \mathcal{T}_{\lambda_{1,k},\ldots,\lambda_{n,k}}\Pi(c_k,\theta_f)\\
&\qquad - \mathcal{T}_{\lambda_1,\ldots,\lambda_n}\Pi(c,\theta_f) - \mathcal{T}_{\lambda_1,\ldots,\lambda_n}\left(\Pi(c,\theta_f) - \Pi(b,\theta_f)\right)\\
&= \mathcal{T}_{\lambda_{1,k},\ldots,\lambda_{n,k}}\Pi(c_k,\theta_f) - \mathcal{T}_{\lambda_1,\ldots,\lambda_n}\Pi(c,\theta_f),
\end{aligned}$$

where we've use the fact that $\Pi(\cdot, \theta_f)$ is constant-valued for each θ_f. $\square$

We now apply Cauchy's integral theorem (as in [26]) using the definition of the numbers $M_1, \ldots, M_n$. Let $f \in \mathbb{A}(M_1, \ldots, M_n)$; we know that $f \in \mathbb{A}_\infty$ (see (3.52), above). Also, let $b \in \mathcal{L}_{M_1,\ldots,M_n}$. We have

$$f_{\lambda_1,\ldots,\lambda_n}(b) = (2\pi i)^{-n} \int\limits_{|\xi_1|=M_1} \cdots \int\limits_{|\xi_n|=M_n} f(\xi_1, \ldots, \xi_n) \mathcal{T}^{(2)}_{\lambda_1\ldots,\lambda_n} \Big((\xi_1 - B_1(\cdot)^{\sim})^{-1} \cdots (\xi_n - B_n(\cdot)^{\sim})^{-1} \Big) d\xi_1 \cdots d\xi_n. \tag{3.69}$$

We set

$$B_0 := \{(w_1, \ldots, w_n) : |w_1| < 1, \ldots, |w_n| < 1\}. \tag{3.70}$$

Fix $f \in \mathbb{A}(M_1, \ldots, M_n) \subseteq \mathbb{A}_\infty$ and define

$$\mathscr{D}^B_f := \{\mathcal{T}_{\lambda_1,\ldots,\lambda_n} f(b) : (\lambda_1, \ldots, \lambda_n) \in [\mathscr{P}_{\text{finite}}([0,T])]^n, b \in \mathcal{L}_{M_1,\ldots,M_n}\}. \tag{3.71}$$

Given $(\lambda_1, \ldots, \lambda_n) \in [\mathscr{P}_{\text{finite}}([0,T])]^n$ and $b \in \mathcal{L}_{M_1,\ldots,M_n}$ we define

$$g(\lambda_1, \ldots, \lambda_n; b; f; \cdot) : B_0 \to X$$

by

$$\begin{aligned} &g(\lambda_1, \ldots, \lambda_n; b; f; w_1, \ldots, w_n) \\ &= (2\pi i)^{-n} \int\limits_{|\xi_1|=M_1} \cdots \int\limits_{|\xi_n|=M_n} f(w_1\xi_1, \ldots, w_n\xi_n) \mathcal{T}_{\lambda_1,\ldots,\lambda_n} \Big((\xi_1 - B_1(\cdot)^{\sim})^{-1} \cdots \\ &\qquad (\xi_n - B_n(\cdot)^{\sim})^{-1} \Big) d\xi_1 \cdots d\xi_n. \end{aligned} \tag{3.72}$$

Note that

$$g(\lambda_1, \ldots, \lambda_n; b; f; 1, \ldots, 1) = \mathcal{T}_{\lambda_1,\ldots,\lambda_n} f(b). \tag{3.73}$$

We can, since f is analytic, differentiate under the integral sign and so, for $(\lambda_1, \ldots, \lambda_n) \in [\mathscr{P}_{\text{finite}}([0,T])]^n$ and for $(B_1(\cdot), \ldots, B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$ it follows that $g(\lambda_1, \ldots, \lambda_n; B_1(\cdot), \ldots, B_n(\cdot); f; \cdot)$ is analytic as a function on B_0. Now define

$$\begin{aligned} \mathscr{F}_{\mathcal{D}^B_f} := \{&g(\lambda_1, \ldots, \lambda_n; B_1(\cdot), \ldots, B_n(\cdot); f; \cdot) : \\ &(\lambda_1, \ldots, \lambda_n) \in [\mathscr{P}_{\text{finite}}([0,T])]^n, (B_1(\cdot), \ldots, B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}\} \end{aligned} \tag{3.74}$$

We claim that $\mathscr{F}_{\mathcal{D}^B_f}$ is a normal family of X-valued functions.

To establish this claim, we start by computing, for $(\lambda_1, \ldots, \lambda_n) \in [\mathscr{P}_{\text{finite}}([0,T])]^n$ and $(B_1(\cdot), \ldots, B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$, a bound for

$$\left\| \mathcal{T}^{(2)}_{\lambda_1,\ldots,\lambda_n} \Big((\xi_1 - B_1(\cdot)^{\sim})^{-1} \cdots (\xi_n - B_n(\cdot)^{\sim})^{-1} \Big) \right\|_X.$$

We write (without yet specifying where the complex numbers $\xi_1, \ldots, \xi_n$ are located)

$$(\xi_1 - B_1(\cdot)^{\sim})^{-1} \cdots (\xi_n - B_n(\cdot)^{\sim})^{-1}$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \xi_1^{-m_1-1}\cdots\xi_n^{-m_n-1}P^{m_1,\ldots,m_n}\left(B_1(\cdot)^{\sim},\ldots,B_n(\cdot)^{\sim}\right)$$

and so

$$\begin{aligned}&\mathcal{T}^{(2)}_{\lambda_1,\ldots,\lambda_n}\left((\xi_1-B_1(\cdot)^{\sim})^{-1}\cdots(\xi_n-B_n(\cdot)^{\sim})^{-1}\right)\\&=\sum_{m_1,\ldots,m_n=0}^{\infty}\xi_1^{-m_1-1}\cdots\xi_n^{-m_n-1}P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(B_1(\cdot),\ldots,B_n(\cdot)\right).\end{aligned}\tag{3.75}$$

Using the norm on $\mathbb{A}_2$ (because we're applying $\mathcal{T}^{(2)}$) we have

$$\left\|\mathcal{T}^{(2)}_{\lambda_1,\ldots,\lambda_n}P^{m_1,\ldots,m_n}\left(B_1(\cdot)^{\sim},\ldots,B_n(\cdot)^{\sim}\right)\right\|_X \le (R_1+\epsilon_{1,2})^{m_1}\cdots(R_n+\epsilon_{n,2})^{m_n}.$$

It then follows that

$$\begin{aligned}&\left\|\mathcal{T}^{(2)}_{\lambda_1,\ldots,\lambda_n}\left((\xi_1-B_1(\cdot)^{\sim})^{-1}\cdots(\xi_n-B_n(\cdot)^{\sim})^{-1}\right)\right\|_X\\&\le\sum_{m_1,\ldots,m_n=0}^{\infty}|\xi_1|^{-m_1-1}\cdots|\xi_n|^{-m_n-1}(R_1+\epsilon_{1,2})^{m_1}\cdots(R_1+\epsilon_{n,2})^{m_n}.\end{aligned}$$

Recalling now the definition of the numbers $M_1,\ldots,M_n$ (see (3.57), above), we take $\xi_1,\ldots,\xi_n\in\mathbb{C}$ such that $|\xi_j|=M_j$, $j=1,\ldots,n$, we therefore have the claimed norm bound:

$$\begin{aligned}&\left\|\mathcal{T}^{(2)}_{\lambda_1,\ldots,\lambda_n}\left((\xi_1-B_1(\cdot)^{\sim})^{-1}\cdots(\xi_n-B_n(\cdot)^{\sim})^{-1}\right)\right\|_X\\&\le\sum_{m_1,\ldots,m_n=0}^{\infty}|\xi_1|^{-m_1-1}\cdots|\xi_n|^{-m_n-1}(R_1+\epsilon_{1,2})^{m_1}\cdots(R_1+\epsilon_{n,2})^{m_n}\\&\le\sum_{m_1,\ldots,m_n=0}^{\infty}\left(\frac{R_1+\epsilon_{1,2}}{R_1+\epsilon_{1,1}}\right)^{m_1}\cdots\left(\frac{R_n+\epsilon_{n,2}}{R_n+\epsilon_{n,1}}\right)^{m_n}\\&=:C_0.\end{aligned}\tag{3.76}$$

We can now use the norm bound C_0 to compute, since f is analytic on the interior of the polydisk $P_{M_1,\ldots,M_n}$,

$$\begin{aligned}&\|g(\lambda_1,\ldots,\lambda_n;B_1(\cdot),\ldots,B_n(\cdot);f;w_1,\ldots,w_n)\|_X\\&\le(2\pi)^{-n}C_0\sup_{(w_1,\ldots,w_n)\in B_0}\int_{|\xi_1|=M_1}\cdots\int_{|\xi_n|=M_n}|f(w_1\xi_1,\ldots,w_n\xi_n)|\,d\xi_1\cdots d\xi_n\\&\le(2\pi)^{-n}C_0C_1,\end{aligned}\tag{3.77}$$

where C_1 is a positive constant for which

$$\sup_{(w_1,\ldots,w_n)\in B_0}\int_{|\xi_1|=M_1}\cdots\int_{|\xi_n|=M_n}|f(w_1\xi_1,\ldots,w_n\xi_n)|\,d\xi_1\cdots d\xi_n\le C_1.\tag{3.78}$$

If follows that $\mathscr{F}_{\mathcal{D}_f^B}$ is a normal family of X-valued analytic functions.

We now assume, if necessary, that $R_j > 1$ for all $j = 1, \dots, n$. Fix an $f \in \mathbb{A}(M_1, \dots, M_n)$. Let $\{\lambda_{j,k}\}_{k=1}^{\infty}$, $j = 1, \dots, n$, be sequences from $\mathscr{P}_{\text{finite}}([0,T])$ which converge weakly to $\lambda_j \in \mathscr{P}_{\text{finite}}([0,T])$ for each $j = 1, \dots, n$. Next, let $\{B_{j,k}(\cdot)\}_{k=1}^{\infty}$, $j = 1, \dots, n$, be sequences of continuous X-valued functions which converge pointwise to continuous X-valued functions $B_j(\cdot)$, $j = 1, \dots, n$. Assume that the n-tuple $(B_{1,k}(\cdot), \dots, B_{n,k}(\cdot))$ is an element of $\mathcal{L}_{M_1,\dots,M_n}$ for every $k \in \mathbb{N}$ and assume that the n-tuple

$$(B_1(\cdot), \dots, B_n(\cdot)) \in \mathcal{L}_{M_1,\dots,M_n}.$$

These n-tuples induce the sequence

$$\{g(\lambda_{1,k}, \dots, \lambda_{n,k}; B_{1,k}(\cdot), \dots, B_{n,k}(\cdot); f; \cdot)\}_{k=1}^{\infty} \tag{3.79}$$

from the normal family $\mathscr{F}_{\mathcal{D}_f^B}$. Because $\mathscr{F}_{\mathcal{D}_f^B}$ is a normal family of X-valued analytic functions, there is a subsequence

$$\{g(\lambda_{1,k_\ell}, \dots, \lambda_{n,k_\ell}; B_{1,k_\ell}(\cdot), \dots, B_{n,k_\ell}(\cdot); f; \cdot)\}_{\ell=1}^{\infty} \tag{3.80}$$

which converges uniformly on compact subsets of B_0. Since

$$\begin{aligned} &g(\lambda_{1,k_\ell}, \dots, \lambda_{n,k_\ell}; B_{1,k_\ell}(\cdot), \dots, B_{n,k_\ell}(\cdot); f; w_1, \dots, w_n) \\ &= (2\pi i)^{-n} \int\limits_{|\xi_1|=M_1} \cdots \int\limits_{|\xi_n|=M_n} f(w_1\xi_1, \dots, w_n\xi_n) \cdot \\ &\mathcal{T}^{(2)}_{\lambda_{1,k_\ell},\dots,\lambda_{n,k_\ell}} \left((\xi_1 - B_{1,k_\ell}(\cdot)^{\sim})^{-1} \cdots (\xi_n - B_{n,k_\ell}(\cdot)^{\sim})^{-1} \right) d\xi_1 \cdots d\xi_n, \end{aligned} \tag{3.81}$$

if we apply Lebesgue's dominated convergence theorem for Bochner integrals, we are able to write

$$\begin{aligned} &\lim_{\ell\to\infty} (2\pi i)^{-n} \int\limits_{|\xi_1|=M_1} \cdots \int\limits_{|\xi_n|=M_n} f(w_1\xi_1, \dots, w_n\xi_n) \cdot \\ &\qquad \mathcal{T}^{(2)}_{\lambda_{1,k_\ell},\dots,\lambda_{n,k_\ell}} \left((\xi_1 - B_{1,k_\ell}(\cdot)^{\sim})^{-1} \cdots (\xi_n - B_{n,k_\ell}(\cdot)^{\sim})^{-1} \right) d\xi_1 \cdots d\xi_n \\ &= (2\pi i)^{-n} \int\limits_{|\xi_1|=M_1} \cdots \int\limits_{|\xi_n|=M_n} f(w_1\xi_1, \dots, w_n\xi_n) \\ &\left[\lim_{\ell\to\infty} \mathcal{T}^{(2)}_{\lambda_{1,k_\ell},\dots,\lambda_{n,k_\ell}} \left((\xi_1 - B_{1,k_\ell}(\cdot)^{\sim})^{-1} \cdots (\xi_n - B_{n,k_\ell}(\cdot)^{\sim})^{-1} \right) \right] d\xi_1 \cdots d\xi_n \end{aligned}$$

where the limit is in X-norm on compact subsets of B_0. However, this being said, it is easy to see from the limit above, that it therefore must be that the sequence

$$\left\{ \mathcal{T}^{(2)}_{\lambda_{1,k_\ell},\dots,\lambda_{n,k_\ell}} \left((\xi_1 - B_{1,k_\ell}(\cdot)^{\sim})^{-1} \cdots (\xi_n - B_{n,k_\ell}(\cdot)^{\sim})^{-1} \right) \right\}_{\ell=1}^{\infty}$$

converges in norm on X. But, by inspecting the disentangling of the Cauchy kernel (see (3.75)), it must be that the sequence

$$\left\{ P^{m_1,\ldots,m_n}_{\lambda_{1,k_\ell},\ldots,\lambda_{n,k_\ell}} \left(B_{1,k_\ell}(\cdot),\ldots,B_{n,k_\ell}(\cdot) \right) \right\}_{\ell=1}^{\infty}$$

must converge in X-norm for each $m_1,\ldots,m_n \in \mathbb{N}\cup\{0\}$. It remains to determine the limit of this sequence.

Let $m_j \neq 0$ and $m_i = 0$ for $i \neq j$, $1 \leq i,j \leq n$. We can write, in view of Lemma 3.11,

$$\begin{aligned} &\lim_{\ell\to\infty} P^{0,\ldots,0,m_j,0,\ldots,0}_{\lambda_{1,k_\ell},\ldots,\lambda_{j-1,k_\ell},\lambda_{j,k_\ell},\lambda_{j+1,k_\ell},\ldots,\lambda_{n,k_\ell}} \left(B_{1,k_\ell}(\cdot),\ldots,B_{n,k_\ell}(\cdot) \right) \\ &= P^{0,\ldots,0,m_j,0,\ldots,0}_{\lambda_1,\ldots,\lambda_{j-1},\lambda_j,\lambda_{j+1},\ldots,\lambda_n} \left(B_1(\cdot),\ldots,B_n(\cdot) \right) \end{aligned}$$

in X-norm. To proceed further, we observe that, for any Borel probability measures $\nu_1,\ldots,\nu_n$ on $[0,T]$ and for any $m_1,\ldots,m_n \in \mathbb{N}\cup\{0\}$,

$$\begin{aligned} &P^{m_1,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_n}_{\nu_1,\ldots,\nu_{j-1},\nu_j,\nu_{j+1},\ldots,\nu_n} \left(I,\ldots,I,B_j,I,\ldots,I \right) \\ &= P^{0,\ldots,0,m_j,0,\ldots,0}_{\nu_1,\ldots,\nu_{j-1},\nu_j,\nu_{j+1},\ldots,\nu_n} \left(B_1(\cdot),\ldots,B_{j-1}(\cdot),B_j(\cdot),B_{j+1}(\cdot),\ldots,B_n(\cdot) \right), \end{aligned}$$

and so, in norm on X,

$$\begin{aligned} &\lim_{\ell\to\infty} P^{m_1,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_n}_{\lambda_{1,k_\ell},\ldots,\lambda_{j-1,k_\ell},\lambda_{j,k_\ell},\lambda_{j+1,k_\ell},\ldots,\lambda_{n,k_\ell}} \left(I,\ldots,I,B_{j,k_\ell}(\cdot),I,\ldots,I \right) \\ &= P^{m_1,\ldots,m_{j-1},m_j,m_{j+1},\ldots,m_n}_{\lambda_1,\ldots,\lambda_{j-1},\lambda_j,\lambda_{j+1},\ldots,\lambda_n} \left(I,\ldots,I,B_j(\cdot),I,\ldots,I \right). \end{aligned} \tag{3.82}$$

It therefore follows, from Proposition 3.15 that, in X-norm,

$$\lim_{\ell\to\infty} P^{m_1,\ldots,m_n}_{\lambda_{1,k_\ell},\ldots,\lambda_{n,k_\ell}} \left(B_{1,k_\ell}(\cdot),\ldots,B_{n,k_\ell}(\cdot) \right) = P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n} \left(B_1(\cdot),\ldots,B_n(\cdot) \right).$$

We have therefore shown that, for our n-tuples $(B_{1,k}(\cdot),\ldots,B_{n,k}(\cdot))$, $(B_1(\cdot), \ldots, B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$,

$$\begin{aligned} &\lim_{\ell\to\infty} g\left(\lambda_{1,k_\ell},\ldots,\lambda_{n,k_\ell}; B_{1,k_\ell}(\cdot),\ldots,B_{n,k_\ell}(\cdot); f; \cdot\right) \\ &= g\left(\lambda_1,\ldots,\lambda_n; B_1(\cdot),\ldots,B_n(\cdot); f; \cdot\right) \end{aligned}$$

uniformly on compact subsets of B_0. But the limit is only taken in monomials and is therefore independent of $w_1,\ldots,w_n$. Consequently this limit is uniform on all of B_0. However, in view of (3.68) with k replacing k_ℓ, it follows from Proposition 3.15 that

$$\begin{aligned} &\lim_{k\to\infty} P^{m_1,\ldots,m_n}_{\lambda_{1,k},\ldots,\lambda_{n,k}} \left(B_{1,k}(\cdot),\ldots,B_{n,k}(\cdot) \right) \\ &= P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n} \left(B_1(\cdot),\ldots,B_n(\cdot) \right) \end{aligned}$$

uniformly in X-norm on B_0.

Now fix for the moment $k \in \mathbb{N}$. By definition of the disentangling map we may write

$$\begin{aligned} & g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; w_1, \ldots, w_n\right) \\ & = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} w_1^{m_1} \cdots w_n^{m_n} P^{m_1,\ldots,m_n}_{\lambda_{1,k},\ldots,\lambda_{n,k}} \left(B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot)\right). \end{aligned}$$

From the Lebesgue dominated convergence theorem (for Bochner integrals) it follows that

$$\begin{aligned} & \lim_{\substack{(w_1,\ldots,w_n)\to(1,\ldots,1) \\ (w_1,\ldots,w_n)\in B_0}} g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; w_1, \ldots, w_n\right) \\ & = \sum_{m_1,\ldots,m_n=0}^{\infty} a_{m_1,\ldots,m_n} P^{m_1,\ldots,m_n}_{\lambda_{1,k},\ldots,\lambda_{n,k}} \left(B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot)\right) \\ & = g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; 1, \ldots, 1\right). \end{aligned} \tag{3.83}$$

This limit is clearly uniform in $k \in \mathbb{N}$. Also, as we have shown,

$$\begin{aligned} & \lim_{k\to\infty} g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; w_1, \ldots, w_n\right) \\ & = g\left(\lambda_1, \ldots, \lambda_n; B_1(\cdot), \ldots, B_n(\cdot); f; w_1, \ldots, w_n\right) \end{aligned}$$

for each $(w_1, \ldots, w_n) \in B_0$. As we have the limit of the g's with respect to the $(w_1, \ldots, w_n)$ being uniform in $k \in \mathbb{N}$ and the limit on $k \in \mathbb{N}$ existing for each $(w_1, \ldots, w_n)$, we wish to apply Moore's theorem on iterated limits [4, p. 128]. To do this, we select arbitrary sequences $\{\alpha_{j,p}\}_{p=1}^{\infty}$, $j = 1, \ldots, n$, in $B(0,1)$ for which $\alpha_{j,p} \to 1$ as $p \to \infty$. Then, because we have

$$\begin{aligned} & g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; \alpha_{1,p}, \ldots, \alpha_{n,p}\right) \\ & \xrightarrow[p\to\infty]{} g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; 1, \ldots, 1\right) \end{aligned}$$

uniformly in $k \in \mathbb{N}$ and also, because we have

$$\begin{aligned} & g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; \alpha_{1,p}, \ldots, \alpha_{n,p}\right) \\ & \xrightarrow[k\to\infty]{} g\left(\lambda_1, \ldots, \lambda_n; B_1(\cdot), \ldots, B_n(\cdot); f; \alpha_{1,p}, \ldots, \alpha_{n,p}\right) \end{aligned}$$

for each $p \in \mathbb{N}$, we may apply Moore's theorem to show that

$$\begin{aligned} & \lim_{\substack{k\to\infty \\ p\to\infty}} g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; \alpha_{1,p}, \ldots, \alpha_{n,p}\right) \\ & = \lim_{k\to\infty} \lim_{p\to\infty} g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; \alpha_{1,p}, \ldots, \alpha_{n,p}\right) \\ & = \lim_{p\to\infty} \lim_{k\to\infty} g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; \alpha_{1,p}, \ldots, \alpha_{n,p}\right) \\ & = g\left(\lambda_1, \ldots, \lambda_n; B_1(\cdot), \ldots, B_n(\cdot); f; 1, \ldots, 1\right) \\ & = f_{\lambda_1,\ldots,\lambda_n}\left(B_1(\cdot), \ldots, B_n(\cdot)\right). \end{aligned} \tag{3.84}$$

Because the sequences $\{\alpha_{j,p}\}_{p=1}^{\infty}$, $j = 1, \ldots, n$, from $B(0,1)$ are arbitrary, we can conclude that

$$\begin{aligned}
&\lim_{\substack{k\to\infty \\ (w_1,\ldots,w_n)\to(1,\ldots,1) \\ (w_1,\ldots,w_n)\in B_0}} g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; w_1, \ldots, w_n\right) \\
&= \lim_{k\to\infty} \lim_{\substack{(w_1,\ldots,w_n)\to(1,\ldots,1) \\ (w_1,\ldots,w_n)\in B_0}} g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; w_1, \ldots, w_n\right) \\
&= \lim_{\substack{(w_1,\ldots,w_n)\to(1,\ldots,1) \\ (w_1,\ldots,w_n)\in B_0}} \lim_{k\to\infty} g\left(\lambda_{1,k}, \ldots, \lambda_{n,k}; B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot); f; w_1, \ldots, w_n\right) \\
&= g\left(\lambda_1, \ldots, \lambda_n; B_1(\cdot), \ldots, B_n(\cdot); f; 1, \ldots, 1\right).
\end{aligned} \tag{3.85}$$

If we translate to the standard notation, (3.85) can be characterized as

$$\lim_{k\to\infty} f_{\lambda_{1,k},\ldots,\lambda_{n,k}}\left(B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot)\right) = f_{\lambda_1,\ldots,\lambda_n}\left(B_1(\cdot), \ldots, B_n(\cdot)\right) \tag{3.86}$$

in norm on X.

Consequently, we have proven the following stability result using the notation introduced here.

Theorem 3.16. *Let $(B_1(\cdot), \ldots, B_n(\cdot)) \in \mathcal{L}_{M_1,\ldots,M_n}$ and associate to each $B_j(\cdot)$, $j = 1, \ldots, n$, a Borel probability measure λ_j on $[0,T]$. (If λ_j has a discrete part, we assume that it is finitely supported.) Select sequences $\{\lambda_{j,k}\}_{k=1}^{\infty}$, $j = 1, \ldots, n$, of Borel probability measures on $[0,T]$ (with finitely supported discrete parts, if a discrete part is present) such that $\lambda_{j,k} \rightharpoonup \lambda_j$ as $k \to \infty$. We also select continuous X-valued functions $B_j(\cdot)$ on $[0,T]$, $j = 1, \ldots, n$, and sequences $\{B_{j,k}(\cdot)\}_{k=1}^{\infty}$, $j = 1, \ldots, n$, of continuous X-valued functions such that $B_{j,k}(s) \to B_j(s)$ pointwise on $[0,T]$. It is assumed that the n-tuples $(B_1(\cdot), \ldots, B_n(\cdot))$ and $(B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot))$, $k \in \mathbb{N}$, are elements of $\mathcal{L}_{M_1,\ldots,M_n}$. Finally, let $f \in \mathbb{A}(M_1, \ldots, M_n)$. Then, in norm on X,*

$$\lim_{k\to\infty} f_{\lambda_{1,k},\ldots,\lambda_{n,k}}\left(B_{1,k}(\cdot), \ldots, B_{n,k}(\cdot)\right) = f_{\lambda_1,\ldots,\lambda_n}\left(B_1(\cdot), \ldots, B_n(\cdot)\right). \tag{3.87}$$

4. FOC for unbounded operators using the Taylor calculus

In this section we address our first approach to using unbounded operators in Feynman's operational calculus. Here, we will use the results of the paper [34], the Taylor calculus and discussed briefly above in Subsection 2.4. We begin by letting A_j, $j = 1, \ldots, n$, be closed densely defined unbounded operators on the Banach space X. Fix, for the moment, $j \in \{1, \ldots, n\}$. Following the paper [34], we let $\mathscr{G}(A_j)$ be the family of all analytic functions f which satisfy

(1) $\sigma(A_j) \subseteq \Delta(f)$

and

(2) $\Delta(f)$ contains a neighborhood of infinity and f is regular at infinity. We will denote

$$\lim_{|\xi|\to\infty} f(\xi) =: f(\infty).$$

Furthermore, we will assume that the resolvent set, $\rho(A_j)$, of A_j, is nonempty.

Using Theorem 4.1 of [34], there is an unbounded Cauchy domain D_j (see Subsection 2.4 above) such that $\sigma(A_j) \subseteq D_j \subseteq \overline{D}_j \subseteq \Delta(f)$. The (Bochner) integral

$$\frac{1}{2\pi i}\int_{\partial D_j} f(\xi)R(\xi; A_j)\, d\xi$$

defines a bounded linear operator which is the same for any choice of D_j satisfying the stated conditions.

For $f \in \mathscr{G}(A_j)$, we write, as in Definition 2.27,

$$f(A_j) := f(\infty)I + \frac{1}{2\pi i}\int_{\partial D_j} f(\xi)R(\xi; A_j)\, d\xi. \tag{4.1}$$

We fix a $f_j \in \mathscr{G}(A_j)$ and so fix, using (4.1), a bounded linear operator $f_j(A_j)$. Doing this for each $j = 1, \ldots, n$, we obtain bounded linear operators $f_1(A_1), \ldots, f_n(A_n)$ on X. Note that, for each $j = 1, \ldots, n$, we have

$$\|f_j(A_j)\|_{\mathcal{L}(X)} \le |f_j(\infty)| + \frac{1}{2\pi}\int_{\partial D_j} |f_j(\xi)|\, \|R(\xi; A_j)\|_{\mathcal{L}(X)}\, d\xi. \tag{4.2}$$

As noted above in Subsection 2.3, the resolvent operator $R(\xi; A_j)$ is, for each $j \in \{1, \ldots, n\}$ and each $\xi \in D_j \subseteq \rho(A_j)$, a bounded linear operator on X and is analytic on $\rho(A_j)$. Furthermore, D_j, $j \in \{1, \ldots, n\}$, is a Cauchy domain (following [34] and Subsection 2.4) and ∂D_j consists of finitely many rectifiable curves. So, $\|R(\xi; A_j)\|_{\mathcal{L}(X)}$ is such that

$$\sup\left\{\|R(\xi; A_j)\|_{\mathcal{L}(X)} : \xi \in \partial D_j\right\} =: M_j < \infty \tag{4.3}$$

for each $j \in \{1, \ldots, n\}$. It follows that, for each $j \in \{1, \ldots, n\}$,

$$\|f_j(A_j)\|_{\mathcal{L}(X)} \le |f_j(\infty)| + \frac{M_j}{2\pi}\int_{\partial D_j} |f_j(\xi)|\, d\xi. \tag{4.4}$$

As in Subsection 2.1, we construct the commutative Banach algebra $\mathbb{A}\,(R_1, \ldots, R_n)$ with radii

$$R_j := |f_j(\infty)| + \frac{M_j}{2\pi}\int_{\partial D_j} |f_j(\xi)|\, d\xi. \tag{4.5}$$

4.1. Using the Taylor calculus in Feynman's operational calculus. We are now ready to develop the operational calculus for this setting. Note that, in our setting, using the closed unbounded operators $A_1, \ldots, A_n$ and the functions $f_1, \ldots, f_n$, we obtain the fixed (i.e., *time independent*) operators $f_1(A_1), \ldots, f_n(A_n)$. This immediately puts us in the setting of the time independent operational calculus. To continue, we associate Borel probability measures $\lambda_1, \ldots, \lambda_n$ on $[0, T]$ to $f_1(A_1), \ldots, f_n(A_n)$, respectively. (They are, of course, the *time-ordering measures.*) Just above, we constructed the commutative Banach algebra $\mathbb{A}\left(R_1, \ldots, R_n\right)$ and we now construct the associated disentangling algebra $\mathbb{D}\left(f_1(A_1)^\sim, \ldots, f_n(A_n)^\sim\right)$ as in Subsection 2.1. Given an element $g \in \mathbb{D}\left(f_1(A_1)^\sim, \ldots, f_n(A_n)^\sim\right)$, we write g as a Taylor series centered at $(0, \ldots, 0)$; i.e.,

$$g\left(f_1(A_1)^\sim, \ldots, f_n(A_n)^\sim\right) = \sum_{m_1, \ldots, m_n = 0}^{\infty} a^g_{m_1, \ldots, m_n} \left[f_1(A_1)^\sim\right]^{m_1} \cdots \left[f_n(A_n)^\sim\right]^{m_n}.$$

The disentangling map from $\mathbb{D}\left(f_1(A_1)^\sim, \ldots, f_n(A_n)^\sim\right)$ to $\mathcal{L}(X)$ is defined by

$$\begin{aligned} &\mathcal{T}_{\lambda_1, \ldots, \lambda_n} g\left(f_1(A_1)^\sim, \ldots, f_n(A_n)^\sim\right) \\ &:= g_{\lambda_1, \ldots, \lambda_n}\left(f_1(A_1), \ldots, f_n(A_n)\right) \\ &= \sum_{m_1, \ldots, m_n = 0}^{\infty} a^g_{m_1, \ldots, m_n} P^{m_1, \ldots, m_n}_{\lambda_1, \ldots, \lambda_n}\left(f_1(A_1), \ldots, f_n(A_n)\right), \end{aligned} \tag{4.6}$$

exactly as in Subsection 2.2. Consequently, the operational calculus defined in terms of the Taylor calculus has all of the same properties as the operational calculus developed in [15] and outlined in Subsections 2.1 and 2.2. It is the choice of radii in (4.5) which allows us to show that the disentangling map is a contraction (see Theorem 2.6, above.) The particular form of the disentangled operator depends on whether or not the time-ordering measures have nonzero (finitely supported) discrete parts. See Equations (2.5) and (2.17) above for the form that the disentangled monomial takes depending on whether or not the time-ordering measures have a non-zero discrete part. Though it seems anticlimatic, applying the Taylor calculus in this way allows us to use the operational calculus in ways that the abstract approach of [15] does not permit. So, while the operational calculus in the setting of the Taylor calculus seems to give nothing new, this approach does indeed provide a useful extension of the operational calculus to the incorporation of unbounded operators into the abstract approach to Feynman's operational calculus.

The reader will note that using Taylor's calculus allows us to "hide" the unbounded operators $A_1, \ldots, A_n$ within the functions $f_1, \ldots, f_n$ and use the resulting bounded operators $f_1(A_1), \ldots, f_n(A_n)$ in the operational calculus in the usual way (see Subsection 2.1) to obtain the disentangled operator. However, the approach using Taylor's calculus allows us to see a connection between the abstract approach to the operational calculus seen in [15] and the "modified Feynman integral" of Lapidus ([20], [14, Chapter 11]). Indeed, this connection is the subject of the next subsection. It is worth noting that this example can not be addressed in the setting of [15]; it is the Taylor calculus which allows the example of the next subsection to be addressed.

4.2. Example: using the Taylor calculus in FOC to obtain the modified Feynman integral. In this subsection we apply the formalism developed above using the Taylor calculus to the so-called Modified Feynman Integral of Lapidus (see [20], [14]). To begin, we outline the essential setup for the "product formula for imaginary resolvents" (see Section 11.3 of [14]) and, in particular, Theorem 11.3.1). Suppose that A and B are unbounded self-adjoint operators on the Hilbert space $\mathcal{H}$ with A nonnegative. Let B_+ and B_- be the positive and negative parts of B (see Subsection 2.3 above) defined by via the spectral theorem; then B_+ and B_- are nonnegative self-adjoint operators and $B = B_+ - B_-$. We assume that B_- is relatively form bounded with respect to A with relative bound smaller than 1; i.e.,

$$Q(A) \subseteq Q(B_-)$$

and there are constants $\gamma, \delta > 0$ with $\gamma < 1$ such that

$$\|B_-^{1/2}\phi\|_{\mathcal{H}}^2 \le \gamma \|A^{1/2}\phi\|_{\mathcal{H}} + \delta\|\phi\|_{\mathcal{H}}^2$$

for all $\phi \in Q(A)$. We assume, for simplicity, that $Q(A) \cap Q(B_+)$ is dense in $\mathcal{H}$. For all $\psi \in \mathcal{H}$,

$$\lim_{k\to\infty} \left(\left[I + \frac{it}{k}A \right]^{-1} \left[I + \frac{it}{k}B \right]^{-1} \right)^k \psi = e^{-it(A\dot{+}B)}\psi \tag{4.7}$$

and this limit is uniform in t on all bounded subsets of $\mathbb{R}$. This is the "Product Formula for Imaginary Resolvents," Theorem 11.3.1 of [14]. (See also [20].)

We now put this product formula into the setting of Feynman's operational calculus using the Taylor calculus. We take, for $n \in \mathbb{N}$ and for $t \ge 0$,

$$f_k(z) := \left(1 + \frac{it}{k}z\right)^{-1}.$$

We note that $f_k(z)$ is regular at infinity and $f_k(\infty) = 0$ for all $k \in \mathbb{N}$. Also, $f_k(z)$ has a simple pole at $z = \frac{ik}{t}$ on the imaginary axis. We further observe that $\sigma(A), \sigma(B) \subseteq \Delta(f_k)$ ($\Delta(g)$ is the domain of g) and $\Delta(f_k)$ contains neighborhoods of infinity. Referring to Subsection 2.4, we see that $f_k \in \mathscr{G}(A)$ and $f_k \in \mathscr{G}(B)$ for all $k \in \mathbb{N}$. Hence, we obtain the bounded linear operators $f_1(A)$ and $f_2(B)$ on $\mathcal{H}$. We can therefore construct the commutative Banach algebras

$$\mathbb{A}\left(\|f_1(A)\|_{\mathcal{L}(\mathcal{H})}, \|f_2(B)\|_{\mathcal{L}(\mathcal{H})}\right) \text{ and } \mathbb{D}\left(f_1(A)^\sim, f_2(B)^\sim\right).$$

In order to investigate the modified Feynman integral in the setting of Feynman's operational calculus using the Taylor calculus, we proceed as follows. First, we construct a family of disentangling algebras $\mathbb{D}_1, \mathbb{D}_2, \dots$. Indeed, we define

$$\begin{aligned} &\mathbb{D}_1 := \mathbb{D}\left(f_1(B)^\sim, f_1(A)^\sim\right), \\ &\mathbb{D}_2 := \mathbb{D}\left(f_2(B)^\sim, f_2(B)^\sim, f_2(A)^\sim, f_2(A)^\sim\right), \\ &\mathbb{D}_3 := \mathbb{D}\left(f_3(B)^\sim, f_3(B)^\sim, f_3(B)^\sim, f_3(A)^\sim, f_3(A)^\sim, f_3(A)^\sim\right), \\ &\quad\vdots \\ &\mathbb{D}_k := \mathbb{D}\left(\overbrace{f_k(B)^\sim, f_k(B)^\sim, \dots, f_k(B)^\sim}^{k \text{ terms}}, \overbrace{f_k(A)^\sim, f_k(A)^\sim, \dots, f_k(A)^\sim}^{k \text{ terms}}\right) \\ &\quad\vdots \end{aligned} \tag{4.8}$$

For each $k \in \mathbb{N}$, the norm on $\mathbb{D}_k$ is

$$\begin{aligned} &\left\| g\left(f_k(B)^\sim, \dots, f_k(B)^\sim, f_k(A)^\sim, \dots, f_k(A)^\sim\right)\right\|_{\mathbb{D}_k} \\ &= \sum_{m_1,\dots,m_k,m_{k+1},\dots,m_{2k}=0}^{\infty} |a_{m_1,\dots,m_{2k}}| \, \|f_k(B)\|_{\mathcal{L}(\mathcal{H})}^{m_1+\dots+m_k} \, \|f_k(A)\|_{\mathcal{L}(\mathcal{H})}^{m_{k+1}+\dots+m_{2k}} \end{aligned} \tag{4.9}$$

where we have used the Taylor series for $g \in \mathbb{D}_k$ in the usual way (see (2.5)). With the family $\{\mathbb{D}_k\}_{k=1}^{\infty}$ in hand, we construct the direct sum Banach algebra (see [18], for instance)

$$\mathbb{D}_{\oplus} := \bigoplus_{k=1}^{\infty} \mathbb{D}_k \tag{4.10}$$

with norm

$$\|\{g_k\}_{k=1}^{\infty}\|_{\oplus} := \sup_{k\in\mathbb{N}} \|g_k\|_{\mathbb{D}_k}. \tag{4.11}$$

The reader will note that we have not yet associated any time-ordering measures to our operators. (There is no time-dependence in our operators, so we do not need the measures to define the disentangling algebras; see [15, Chapter 2].) Fix a $k \in \mathbb{N}$. Choose real numbers $a_1, b_1, c_1, d_1, \dots, a_k, b_k, c_k, d_k$ which satisfy

$$0 < a_1 < b_1 < c_1 < d_1 < a_2 < b_2 < c_2 < d_2 < \dots < a_k < b_k < c_k < d_k < T.$$

Next, choose continuous Borel probability measures μ_j on $[a_j, b_j]$, $j = 1, \dots, k$ and choose continuous Borel probability measures ν_j on $[c_j, d_j]$ for $j = 1, \dots, k$. (To be explicit, we assume that the support of each μ_j, $S(\mu_j)$, satisfies $S(\mu_j) \subseteq [a_j, b_j]$ and that the support of each ν_j, $S(\nu_j)$, satisfies $S(\nu_j) \subseteq [c_j, d_j]$ for $j = 1, \dots, n$.) We associate the measures μ_j, $j = 1, \dots, k$, to the k operators $f_k(B)$ and associate the measures ν_j, $j = 1, \dots, n$, to the

k operators $f_k(A)$. It is clear that we can carry out this construction for every $k \in \mathbb{N}$.

With the time-ordering measures in hand, we now define the disentangling map on $\mathbb{D}_\oplus$. First, recall that, for each $k \in \mathbb{N}$, we have the disentangling map

$$\mathcal{T}_{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_k} : \mathbb{D}_k \longrightarrow \mathcal{L}(\mathcal{H})$$

exactly as in Subsection 2.2. We use these disentangling maps to define

$$\mathcal{T}_\oplus : \mathbb{D}_\oplus \longrightarrow \bigoplus_{k=1}^{\infty} \mathcal{L}(\mathcal{H}) \tag{4.12}$$

by

$$\mathcal{T}_\oplus \left[\{g_k\}_{k=1}^{\infty} \right] := \{\mathcal{T}_{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_k} g_k\}_{k=1}^{\infty} . \tag{4.13}$$

It is clear that $\mathcal{T}_\oplus$ is linear and, since

$$\begin{aligned} \left\| \mathcal{T}_\oplus \left[\{g_k\}_{k=1}^{\infty} \right] \right\| &= \sup_{k \in \mathbb{N}} \left\| \mathcal{T}_{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_k} g_k \right\|_{\mathcal{L}(\mathcal{H})} \\ &\le \sup_{k \in \mathbb{N}} \|g_k\|_{\mathbb{D}_k} \\ &= \left\| \{g_k\}_{k=1}^{\infty} \right\|_\oplus , \end{aligned} \tag{4.14}$$

$\mathcal{T}_\oplus$ is a linear contraction. (Compare to [15], Chapters 2, 6 and 8.)

We are interested in the elements of $\mathbb{D}_k$ of the form

$$P^{1,\ldots,1,1,\ldots,1} \left(\overbrace{f_k(B)^\sim, \ldots, f_k(B)^\sim}^{k \text{ terms}}, \overbrace{f_k(A)^\sim, \ldots, f_k(A)^\sim}^{k \text{ terms}} \right). \tag{4.15}$$

The corresponding element of $\mathbb{D}_\oplus$ is

$$\Theta := \left\{ P^{1,\ldots,1,1,\ldots,1} \left(\overbrace{f_k(B)^\sim, \ldots, f_k(B)^\sim}^{k \text{ terms}}, \overbrace{f_k(A)^\sim, \ldots, f_k(A)^\sim}^{k \text{ terms}} \right) \right\}_{k=1}^{\infty} \tag{4.16}$$

and so

$$\mathcal{T}_\oplus(\Theta) = \left\{ P^{1,\ldots,1,1,\ldots,1}_{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_k} \left(\overbrace{f_k(B), \ldots, f_k(B)}^{k \text{ terms}}, \overbrace{f_k(A), \ldots, f_k(A)}^{k \text{ terms}} \right) \right\}_{k=1}^{\infty} . \tag{4.17}$$

Now, because of the way that the supports $S(\mu_j)$ and $S(\nu_j)$ are ordered, we can apply Corollary 3.3.3 of [15] to deduce that

$$\begin{aligned} &P^{1,\ldots,1,1,\ldots,1}_{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_k} \left(\overbrace{f_k(B), \ldots, f_k(B)}^{k \text{ terms}}, \overbrace{f_k(A), \ldots, f_k(A)}^{k \text{ terms}} \right) \\ &= \left[f_k(B) f_k(A) \right]^k \end{aligned} \tag{4.18}$$

$$= \left[\left(I + \frac{it}{k} B \right)^{-1} \left(I + \frac{it}{k} A \right)^{-1} \right]^k$$

We can now obtain the modified Feynman integral. Let $V : \mathbb{R}^d \to \mathbb{R}$ be Lebesgue measurable. The modified Feynman integral associated with the potential V, denoted by $\mathscr{F}_M^t(V)$, is defined as the strong operator limit

$$\mathscr{F}_M^t(V) = \lim_{k\to\infty} \left(\left[I + \frac{it}{k} H_0 \right]^{-1} \left[I + \frac{it}{k} V \right]^{-1} \right)^k \tag{4.19}$$

in $\mathcal{L}(L^2(\mathbb{R}^d))$ when this limit exists. (See [14, Definition 11.4.4].)

Following [14, Theorem 11.4.2 and Corollary 11.4.5], we assume that $V : \mathbb{R}^d \to \mathbb{R}$ is Lebesgue measurable with $V_+ \in L^1_{loc}(\mathbb{R}^d)$ and assume that V_- is relatively form bounded with respect to H_0 with relative bound less than 1. With our function f_k, we note that, since the spectrum of H_0 is $[0, \infty)$, $f_k \in \mathscr{G}(H_0)$ and because V is real-valued (though possibly unbounded), $f_k \in \mathscr{G}(V)$ (see Subsection 2.4) for all $k \in \mathbb{N}$ and $t \geq 0$. Then $\mathscr{F}_M^t(V)$ exists for all $t \in \mathbb{R}$ and

$$\begin{aligned} \mathscr{F}_M^t(V) &= e^{-it(H_0+V)} && (4.20) \\ &= \lim_{k\to\infty} P^{1,\ldots,1,1,\ldots,1}_{\mu_1,\ldots,\mu_k,\nu_1,\ldots,\nu_k} \Big(\overbrace{f_k(H_0), \ldots, f_k(H_0)}^{k \text{ terms}}, \overbrace{f_k(V), \ldots, f_k(V)}^{k \text{ terms}} \Big) \end{aligned}$$

where the (strong operator) limit is uniform in t on all bounded subsets of $\mathbb{R}$. We note that the hypothesis on V allows highly singular potentials – see Remark 11.4.6 of [14] and Example 11.4.7 of [14]. While it would be advantageous to be able to address this limit using an aspect of the stability theory for Feynman's operational calculus, the current state of the art of the stability theory does not allow this to be done.

4.3. Stability with respect to the functions f_j. In this subsection, we will investigate the stability (with respect to the functions f_j) of the operational calculus for unbounded operators obtained above using A. E. Taylor's ideas. As remarked on in Subsection 3.3, there are two varieties of stability to consider, with respect to the time-ordering measures and with respect to the operators (or operator-valued functions). (Actually, there is a third type of stability, joint stability, which is the stability of the operational calculus with respect to *both* the operators (operator-valued functions) and the measures. See Chapter 7 of [15].) Here, we will not consider the stability with respect to the time-ordering measures, as the paper [28] addresses this aspect of the stability theory for this setting.

By considering the stability of the operational calculus with respect to the functions f_j, we are essentially looking at the stability of the operational calculus with respect to the operators (since the operators change with the choice of the function from $\mathscr{G}(T)$.)

We will begin our discussion by considering an arbitrary unbounded, closed and densely defined operator T on X. (The ideas we use throughout this subsection are introduced and discussed in Subsection 2.4.) We next choose a $f \in \mathscr{G}(T)$. We obtain a corresponding Cauchy domain D_f with $\sigma(T) \subseteq D_f \subseteq \overline{D_f} \subseteq \Delta(f)$ with $\Delta(f)$ open in $\mathbb{C}$. (See Subsection 2.4.) Moreover, the finitely many components of D_f are bounded by finitely many rectifiable Jordan curves, no two of which intersect. (The closures of the components of D_f are disjoint, by definition; see Subsection 2.4.) Finally, D_f has at most one unbounded component and the number of components is fixed by the number of components of $\sigma(T)$.

Definition 4.1. Given $f \in \mathscr{G}(T)$ and its associated Cauchy domain D_f as in the previous paragraph, we say that a sequence $\{f_k\}_{k=1}^{\infty}$ from $\mathscr{G}(T)$ converges to $f \in \mathscr{G}(T)$ if

$$(1)\ \Delta(f_k) \supseteq \Delta(f), \text{ for all } k \in \mathbb{N} \tag{4.21}$$

and

$$(2)\ \{f_k\}_{k=1}^{\infty} \text{ converges uniformly on compact subsets of } \Delta(f). \tag{4.22}$$

Choose such a sequence $\{f_k\}_{k=1}^{\infty}$ from $\mathscr{G}(T)$. For each $k \in \mathbb{N}$, f_k has an associated Cauchy domain D_{f_k}. From the definition of Cauchy domain (see Subsection 2.4, above), each Jordan curve bounding D_{f_k} is homotopic to the corresponding Jordan curve of D_f. Consequently, for each $k \in \mathbb{N}$,

$$\frac{1}{2\pi i}\int_{\partial D_{f_k}} f_k(\xi)R(\xi;T)\,d\xi = \frac{1}{2\pi i}\int_{\partial D_f} f_k(\xi)R(\xi;T)\,d\xi. \tag{4.23}$$

(Each integral is, in fact, a finite sum of integrals over rectifiable Jordan arcs.) Now, denote by $C_1^f, \ldots, C_N^f$ the components of D_f and assume, without loss of generality, that C_1^f is the unbounded component. For each $k \in \mathbb{N}$, we may then write

$$\frac{1}{2\pi i}\int_{\partial D_f} f_k(\xi)R(\xi;T)\,d\xi = \sum_{j=1}^{N}\frac{1}{2\pi i}\int_{\partial C_j^f} f_k(\xi)R(\xi;T)\,d\xi. \tag{4.24}$$

Now, for each $j \in \{1, \ldots, N\}$,

$$\begin{aligned}
&\left\|\frac{1}{2\pi i}\int_{\partial C_j^f} f(\xi)R(\xi;T)\,d\xi - \frac{1}{2\pi i}\int_{\partial C_j^f} f_k(\xi)R(\xi;T)\,d\xi\right\|_{\mathcal{L}(X)} \\
&\quad\le \frac{1}{2\pi}\int_{\partial C_j^f} |f(\xi) - f_k(\xi)|\ \|R(\xi;T)\|_{\mathcal{L}(X)}d\xi \\
&\quad\le \frac{1}{2\pi}\sup\left\{|f(\xi) - f_k(\xi)| : \xi \in \partial C_j^f\right\}\cdot \\
&\qquad\quad \sup\left\{\|R(\xi;T)\|_{\mathcal{L}(X)} : \xi \in \partial C_j^f\right\}\ell\left(\partial C_j^f\right),
\end{aligned} \tag{4.25}$$

where $\ell(\Sigma)$ denotes the length of the rectifiable curve Σ. Next, since ∂C_j^f is a Jordan arc, it is compact for each $j = 1, \ldots, n$. So

$$\sup\left\{|f_k(\xi) - f(\xi)| : \xi \in C_j^f\right\} \xrightarrow[k\to\infty]{} 0.$$

We also know that the supremum of $\|R(\xi;T)\|_{\mathcal{L}(X)}$ is bounded on ∂C_j^f. Therefore, because the sum in (4.24) is finite,

$$\lim_{k\to\infty}\left\|\frac{1}{2\pi i}\int_{\partial D_f} f(\xi)R(\xi;T)\,d\xi - \frac{1}{2\pi i}\int_{\partial D_f} f_k(\xi)R(\xi;T)\,d\xi\right\|_{\mathcal{L}(X)} = 0. \tag{4.26}$$

We have therefore determined that the sequence

$$\left\{\frac{1}{2\pi i}\int_{\partial D_f} f_k(\xi)R(\xi;T)\,d\xi\right\}_{k=1}^{\infty} \tag{4.27}$$

of bounded linear operators converges in norm to the bounded linear operator

$$\frac{1}{2\pi i}\int_{\partial D_f} f(\xi)R(\xi;T)\,d\xi. \tag{4.28}$$

We now turn our attention back to the operators $A_1, \ldots, A_n$ and the functions $f_1, \ldots, f_n$ chosen from $\mathscr{G}(A_1), \ldots, \mathscr{G}(A_n)$, respectively. As we know, these functions determine bounded linear operators $f_1(A_1), \ldots, f_n(A_n)$. Associate the continuous Borel probability measure λ_j to $f_j(A_j)$ for each $j = 1, \ldots, n$. (We will use continuous time-ordering measures here mostly for convenience. The same stability result obtained at the end of this section can be stated and proved for the case where our time-ordering measures have a finitely supported discrete part, although combinatorial issues greatly complicate the arguments.) Fix, momentarily, $j \in \{1, \ldots, n\}$ and select a sequence $\{f_{j,k}\}_{k=1}^{\infty}$ from $\mathscr{G}(A_j)$ converging to f_j in the sense defined above. We have a corresponding family $\left\{D_{f_{j,k}}\right\}_{k=1}^{\infty}$, $j = 1, \ldots, n$, of Cauchy domains. We have observed above that

$$\frac{1}{2\pi i}\int_{\partial D_{f_{j,k}}} f_{j,k}(\xi)R(\xi;A_j)\,d\xi = \frac{1}{2\pi i}\int_{\partial D_{f_j}} f_{j,k}(\xi)R(\xi;A_j)\,d\xi. \tag{4.29}$$

It follows that

$$\begin{aligned}\|f_{j,k}(A_j)\|_{\mathcal{L}(X)} &\le |f_{j,k}(\infty)| + \frac{1}{2\pi}\int_{\partial D_{f_j}} |f_{j,k}(\xi)|\,\|R(\xi;A_j)\|_{\mathcal{L}(X)}\,d\xi \\ &\le |f_{j,k}(\infty)| + \frac{M_j}{2\pi}\int_{\partial D_{f_j}} |f_{j,k}(\xi)|\,d\xi,\end{aligned} \tag{4.30}$$

where

$$M_j := \sup\left\{\|R(\xi;A_j)\|_{\mathcal{L}(X)} : \xi \in \partial D_{f_j}\right\};$$

see (4.3) above. Based on the calculation just above, for each $k \in \mathbb{N}$ we construct the commutative Banach algebra

$$\mathbb{A}_k := \mathbb{A}\left(R_{1,k}, \ldots, R_{n,k}\right), \tag{4.31}$$

where

$$R_{j,k} := |f_{j,k}(\infty)| + \frac{M_j}{2\pi} \int_{\partial D_f} |f_{j,k}(\xi)|\, d\xi, \tag{4.32}$$

for $j = 1, \ldots, n$. We also define

$$\mathbb{A}_0 := \mathbb{A}\left(R_1, \ldots, R_n\right), \tag{4.33}$$

where, as above,

$$R_j = |f_j(\infty)| + \frac{M_j}{2\pi} \int_{\partial D_{f_j}} |f_j(\xi)|\, d\xi, \tag{4.34}$$

for $j = 1, \ldots, n$. We now define

$$\mathbb{A}_\oplus := \bigoplus_{k=0}^{\infty} \mathbb{A}_k. \tag{4.35}$$

Then $\mathbb{A}_\oplus$ is a commutative Banach algebra with norm

$$\| \{g_k\}_{k=0}^{\infty} \|_\oplus := \sup_{k \in \mathbb{N} \cup \{0\}} \|g_k\|_{\mathbb{A}_k}. \tag{4.36}$$

For each $k \in \mathbb{N}$ we take $\pi_k : \mathbb{A}_\oplus \longrightarrow \mathbb{A}_k$ to be the canonical projection. Given $\theta := \{g_k\}_{k=0}^{\infty} \in \mathbb{A}_\oplus$, the disentangling map

$$\mathcal{T}_{\lambda_1,\ldots,\lambda_n}^{\mathbb{A}_\oplus} : \mathbb{A}_\oplus \longrightarrow \bigoplus_{k=0}^{\infty} \mathcal{L}(X)$$

is defined by

$$\mathcal{T}_{\lambda_1,\ldots,\lambda_n}^{\mathbb{A}_\oplus} \theta := \left\{ (g_k)_{\lambda_1,\ldots,\lambda_n} \left(f_{1,k}(A_1), \ldots, f_{n,k}\left(A_n\right)\right) \right\}_{k=0}^{\infty}. \tag{4.37}$$

Remark 4.2. The reader may have noticed that we are working here with the algebras $\mathbb{A}_k$ of analytic functions. This is for convenience; the algebras $\mathbb{A}_k$ and the disentangling algebras $\mathbb{D}_k$ (if they were constructed here) are, as noted in Subsection 2.2, isometrically isomorphic.

It is straightforward to show that $\mathcal{T}_{\lambda_1,\ldots,\lambda_n}^{\mathbb{A}_\oplus}$ is a bounded linear operator (see [15, Chapters 7, 8], [24], [28], [29]).

Now, let $\theta := \{g_k\}_{k=0}^{\infty} \in \mathbb{A}_\oplus$ be such that $g_k \equiv g$ for all $k \in \mathbb{N} \cup \{0\}$. We consider

$$\mathcal{T}_{\lambda_1,\ldots,\lambda_n}^{\mathbb{A}_\oplus} \left(\{g_k\}_{k=0}^{\infty}\right) = \left\{ \mathcal{T}_{\lambda_1,\ldots,\lambda_n} g \left(f_{1,k}\left(A_1\right)^{\sim}, \ldots, f_{n,k}\left(A_n\right)^{\sim}\right) \right\}_{k=0}^{\infty} \tag{4.38}$$

and claim that

$$\left\| g_{\lambda_1,\ldots,\lambda_n} \left(f_{1,k}\left(A_1\right), \ldots, f_{n,k}\left(A_{n,k}\right)\right) - g_{\lambda_1,\ldots,\lambda_n} \left(f_1\left(A_1\right), \ldots, f_n\left(A_n\right)\right) \right\|_{\mathcal{L}(X)} \to 0 \tag{4.39}$$

as $k \to \infty$. To show that this is indeed the case, we start with

$$\left\| P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(f_{1,k}\left(A_1\right),\ldots,f_{n,k}\left(A_n\right)\right) - P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(f_1\left(A_1\right),\ldots,f_n\left(A_n\right)\right)\right\|_{\mathcal{L}(X)}$$
$$\leq \sum_{\pi\in S_m}\int_{\Delta_m(\pi)} \left\| C^k_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C^k_{\pi(1)}\left(s_{\pi(1)}\right) - \right. \tag{4.40}$$
$$\left. C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\right\|_{\mathcal{L}(X)} \left(\lambda_1^{m_1}\times\cdots\times\lambda_n^{m_n}\right)(ds_1,\ldots,ds_m).$$

(Note: We are making use, here, of the notation introduced in Subsection 2.1 and we use the notation $C^k_j(\cdot)$ to denote the particular operator $f_{j,k}(A_j)$. We have also applied Theorem 2.8 to write out the disentangling using continuous time-ordering measures.) We know that, in operator norm on $\mathcal{L}(X)$, $f_{j,k}\left(A_j\right) \longrightarrow f_j\left(A_j\right)$ as $k\to\infty$. Also, there is no time-dependence in our operators. A standard argument shows that

$$C^k_{\pi(m)}\cdots C^k_{\pi(1)} - C_{\pi(m)}\cdots C_{\pi(1)} \xrightarrow[k\to\infty]{} 0$$

in operator norm, for any $\pi\in S_m$. Therefore

$$\sum_{\pi\in S_m}\int_{\Delta_m(\pi)} \left\| C^k_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C^k_{\pi(1)}\left(s_{\pi(1)}\right)\right.$$
$$\left. - C_{\pi(m)}\left(s_{\pi(m)}\right)\cdots C_{\pi(1)}\left(s_{\pi(1)}\right)\right\|_{\mathcal{L}(X)} \left(\lambda_1^{m_1}\times\cdots\times\lambda_n^{m_n}\right)(ds_1,\ldots,ds_m)$$
$$= \sum_{\pi\in S_m}\left\| C^k_{\pi(m)}\cdots C^k_{\pi(1)} - C_{\pi(m)}\cdots C_{\pi(1)}\right\|_{\mathcal{L}(X)} \left(\lambda_1^{m_1}\times\cdots\times\lambda_n^{m_n}\right)(\Delta_m(\pi))$$
$$\xrightarrow[k\to\infty]{} 0$$

and so

$$\left\| P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(f_{1,k}\left(A_1\right),\ldots,f_{n,k}\left(A_n\right)\right)\right.$$
$$\left. - P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(f_1\left(A_1\right),\ldots,f_n\left(A_n\right)\right)\right\|_{\mathcal{L}(X)} \xrightarrow[k\to\infty]{} 0. \tag{4.41}$$

To verify (4.39), for arbitrary g (where $\{g_k\}_{k=0}^\infty \in \mathbb{A}_\oplus$, $g_k = g$ for all $k\in\mathbb{N}$), we apply the dominated convergence theorem. To see how this works, we first note that, given $\epsilon > 0$, there is a positive integer k_0 such that

$$\|\{g_k\}_{k=0}^\infty\|_{\mathbb{A}_\oplus} = \sup_{k\in\mathbb{N}\cup\{0\}} \|g\|_{\mathbb{A}_k} < \|g\|_{\mathbb{A}_{k_0}} + \epsilon. \tag{4.42}$$

Next, note that for any $k\in\mathbb{N}$,

$$\left\| P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(f_{1,k}\left(A_1\right),\ldots,f_{n,k}\left(A_n\right)\right) - P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(f_1\left(A_1\right),\ldots,f_n\left(A_n\right)\right)\right\|_{\mathcal{L}(X)}$$
$$\leq \left\|f_{1,k}\left(A_1\right)\right\|^{m_1}_{\mathcal{L}(X)}\cdots\left\|f_{n,k}\left(A_n\right)\right\|^{m_n}_{\mathcal{L}(X)} + \left\|f_1\left(A_1\right)\right\|^{m_n}_{\mathcal{L}(X)}\cdots\left\|f_n\left(A_n\right)\right\|^{m_n}_{\mathcal{L}(X)} \tag{4.43}$$
$$\leq R^{m_1}_{1,k}\cdots R^{m_n}_{n,k} + R^{m_1}_1\cdots R^{m_n}_n.$$

It follows that, writing

$$g(z_1, \ldots, z_n) = \sum_{m_1,\ldots,m_n=0}^{\infty} a^g_{m_1,\ldots,m_n} z_1^{m_1} \cdots z_n^{m_n},$$

we have

$$\begin{aligned}
&\|g_{\lambda_1,\ldots,\lambda_n}(f_{1,k}(A_1), \ldots, f_{n,k}(A_n)) - g_{\lambda_1,\ldots,\lambda_n}(f_1(A_1), \ldots, f_n(A_n))\|_{\mathcal{L}(X)} \\
&\le \sum_{m_1,\ldots,m_n=0}^{\infty} |a^g_{m_1,\ldots,m_n}| R_{1,k}^{m_1} \cdots R_{n,k}^{m_n} + \sum_{m_1,\ldots,m_n=0}^{\infty} |a^g_{m_1,\ldots,m_n}| R_1^{m_1} \cdots R_n^{m_n} \\
&< \sum_{m_1,\ldots,m_n=0}^{\infty} |a^g_{m_1,\ldots,m_n}| R_{1,k_0}^{m_1} \cdots R_{n,k_0}^{m_n} + \epsilon \qquad (4.44) \\
&\qquad + \sum_{m_1,\ldots,m_n=0}^{\infty} |a^g_{m_1,\ldots,m_n}| R_1^{m_1} \cdots R_n^{m_n} \\
&= \|g\|_{\mathbb{A}_{k_0}} + \epsilon + \|g\|_{\mathbb{A}_0}.
\end{aligned}$$

Based on this computation, we define $\Lambda : \mathbb{N}^n \cup \{(0, \ldots, 0)\} \longrightarrow \mathbb{R}$ by

$$\begin{aligned}
\Lambda(m_1, \ldots, m_n) := |a^g_{m_1,\ldots,m_n}| \left[R_{1,k_0}^{m_1} \cdots R_{n,k_0}^{m_n} + R_{1,k}^{m_1} \cdots R_{n,k}^{m_n}\right] \qquad (4.45) \\
+ \frac{\epsilon}{2^{m_1+\cdots+m_n+n}}.
\end{aligned}$$

The function Λ is clearly a summable dominating function for

$$\begin{aligned}
\sum_{m_1,\ldots,m_n=0}^{\infty} |a^g_{m_1,\ldots,m_n}| \Big\| P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}(f_{1,k}(A_1), \ldots, f_{n,k}(A_n)) - \\
P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}(f_1(A_1), \ldots, f_n(A_n)) \Big\|_{\mathcal{L}(X)}.
\end{aligned}$$

We may therefore apply the dominated convergence theorem (for Bochner integrals) as well as (4.41) to obtain

$$\begin{aligned}
&\lim_{k\to\infty} \|g_{\lambda_1,\ldots,\lambda_n}(f_{1,k}(A_1), \ldots, f_{n,k}(A_n)) \\
&\qquad - g_{\lambda_1,\ldots,\lambda_n}(f_1(A_1), \ldots, f_n(A_n))\|_{\mathcal{L}(X)} \\
&\le \lim_{k\to\infty} \sum_{m_1,\ldots,m_n=0}^{\infty} |a^g_{m_1,\ldots,m_n}| \Big\| P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}(f_{1,k}(A_1), \ldots, f_{n,k}(A_n)) \\
&\qquad - P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}(f_1(A_1), \ldots, f_n(A_n)) \Big\|_{\mathcal{L}(X)} \\
&= 0. \qquad (4.46)
\end{aligned}$$

We have therefore established the following theorem.

Theorem 4.3. *Let $A_1, \ldots, A_n$ be closed densely defined linear operators in $\mathscr{C}(X)$, X a Banach space. Choose, for each $j \in \{1, \ldots, n\}$, an $f_j \in$*

$\mathscr{G}(A_j)$ (*notation as above*). *Associate to each bounded operator* $f_j(A_j)$, $j = 1, \ldots, n$, *the continuous Borel probability measure* λ_j *on* $[0,T]$. *Furthermore, for each* $j \in \{1, \ldots, n\}$, *choose a sequence* $\{f_{j,k}\}_{k=1}^{\infty}$ *converging to* f_j *in the sense of* (4.21) *and* (4.22). *Construct the commutative Banach algebras* $\mathbb{A}_k$, $k \in \mathbb{N} \cup \{0\}$ *and* $\mathbb{A}_{\oplus}$ *as in* (4.31), (4.33) *and* (4.35). *Let* $\{g_k\}_{k=0}^{\infty} \in \mathbb{A}_{\oplus}$, $g_k \equiv g$, *for all* $k \in \mathbb{N} \cup \{0\}$. *Then*

$$\lim_{k\to\infty} \left\| \pi_k \left[\mathcal{T}^{\mathbb{A}_\oplus}_{\lambda_1,\ldots,\lambda_n} \left(\left\{ g_k \left(f_{1,k} \left(A_1 \right)^{\sim}, \ldots, f_{n,k} \left(A_n \right)^{\sim} \right) \right\} \right) \right] - \mathcal{T}_{\lambda_1,\ldots,\lambda_n} g \left(f_1 \left(A_1 \right)^{\sim}, \ldots, f_n \left(A_n \right)^{\sim} \right) \right\|_{\mathcal{L}(X)} = 0. \qquad (4.47)$$

Remark 4.4. We remind the reader that this theorem also holds when the time-ordering measures are allowed to have a nonzero, finitely supported, discrete part. Due to combinatorial issues, the disentanglings are much more complicated and so the argument above is more involved, though reasonably straightforward. See, for example, Chapter 8 of [15] for some stability arguments using time-ordering measures with nonzero discrete parts.

4.4. Comments on other stability properties. The stability theorem, Theorem 4.3, addresses the stability of the operational calculus with respect to the functions used in the Taylor calculus approach to the use of unbounded operators. It is also possible to investigate the stability of the operational calculus with respect to the time-ordering measures – see [29, 28, 23, 22, 24, 27, 25, 21, 16] as well as [13], [14] for stability in a different, but similar, setting.

Though we will not delve into the details of the stability theory of the operational calculus with respect to the time-ordering measures, we will take the time to sketch out the essential ideas. Given operators (or operator-valued functions) $A_1, \ldots, A_n$ and associated Borel (typically probability) time-ordering measures $\lambda_1, \ldots, \lambda_n$ on $[0,T]$, we construct the disentangling algebra $\mathbb{D}(A_1^{\sim}, \ldots, A_n^{\sim})$ and obtain the disentangling map $\mathcal{T}_{\lambda_1,\ldots,\lambda_n} : \mathbb{D}(A_1^{\sim}, \ldots, A_n^{\sim}) \longrightarrow \mathcal{L}(X)$. To investigate the stability of the operational calculus with respect to the time-ordering measures, we select sequences $\{\lambda_{j,k}\}_{k=1}^{\infty}$, $j = 1, \ldots, n$, of Borel (typically probability) measures on $[0,T]$ converging weakly to λ_j. (By definition, this means that $\int_{[0,T]} f\, d\lambda_{j,k} \longrightarrow \int_{[0,T]} f\, d\lambda_j$ for all bounded continuous functions f on $[0,T]$.) Since each n-tuple $(\lambda_{1,k}, \ldots, \lambda_{n,k})$ determines a operational calculus via the disentangling map $\mathcal{T}_{\lambda_{1,k},\ldots,\lambda_{n,k}}$, stability of FOC with respect to the time-ordering measures means that the family $\left\{\mathcal{T}_{\lambda_{1,k},\ldots,\lambda_{n,k}}\right\}_{k=1}^{\infty}$ of operational calculi has a limiting operational calculus; i.e.,

$$\lim_{k\to\infty} \mathcal{T}_{\lambda_{1,k},\ldots,\lambda_{n,k}} f = \mathcal{T}_{\lambda_1,\ldots,\lambda_n} f$$

for all $f \in \mathbb{D}$. (This is stated somewhat informally.)

We can also study the stability of the operational calculus with respect to both the operators (operator-valued functions) and the time-ordering measures. This is the so-called *joint stability* of the operational calculus. In

Subsection 3.2, sequences $\{f_{j,k}\}_{k=1}^{\infty}$, $j=1,\ldots,n$, give rise to $\{f_{j,k}(A_j)\}_{k=1}^{\infty}$, $j=1,\ldots,n$, of bounded operators. If we also select weakly convergent sequences $\{\lambda_{j,k}\}_{k=1}^{\infty}$ of measures, joint stability asks if

$$\lim_{k\to\infty}\mathcal{T}_{\lambda_{1,k},\ldots,\lambda_{n,k}}g\left(f_{1,k}(A_1),\ldots,f_{n,k}(A_n)\right)=\mathcal{T}_{\lambda_1,\ldots,\lambda_n}g\left(f_1(A_1),\ldots,f_n(A_n)\right).$$

Joint stability of the operational calculus is studied for the abstract approach to the operational calculus found in [15] and the results therein can be applied in this setting where the sequences of operators are $\{f_{j,k}(A_j)\}_{k=1}^{\infty}$, $j=1,\ldots,n$. It should be noted that the setting of [15] allows only bounded operators (with the exception of the generator of a (C_0) semigroup). In the next section, we study joint stability for the operational calculus when using analytic families of closed unbounded operators. Just as with the operational calculus in the setting of [15], we select sequences of operators as well as sequences of time-ordering measures. However, in the setting of this paper, the sequences of operators are sequences of closed and unbounded operators and so we use the idea of generalized convergence. This complicates the study of joint stability somewhat, giving the result a slightly different flavor than the stability theory in the "standard" setting of [15]. The discussion of joint stability for the operational calculus using analytic families found in Subsection 5.2, below, relies very much on the material found in Subsections 3.2–3.4, where the "heavy lifting" is carried out. Indeed, with the formalism introduced in these subsections, the discussion of joint stability is quite straightforward and brief.

5. FOC for unbounded operators using analytic families

In this section we turn to the operational calculus using analytic families of unbounded closed operators. (See Subsection 2.7 for necessary definitions.) To this end, let D_0 be an open neighborhood of $0\in\mathbb{C}$. Let $T_1(z),\ldots,T_n(z)$, $z\in D_0$, be analytic families of unbounded closed operators on the Banach space X. Furthermore, for each $j\in\{1,\ldots,n\}$ we will assume that, in the definition from Subsection 2.7, the Banach space Z equals X and so there are bounded analytic families $U_j(z),V_j(z)\in\mathcal{L}(X)$ with $U_j(z)$ injective onto $\mathcal{D}(T_j(z))$ and

$$T_j(z)U_j(z)=V_j(z). \tag{5.1}$$

It is the operator-valued functions $V_j(z)$ which will be used in the operational calculus. To this end, we choose $T>0$ such that $[0,T]\subset D_0\cap\mathbb{R}$. For each $j=1,\ldots,n$, associate Borel probability measures (time-ordering measures) λ_j on $[0,T]$ to the operator-valued function $V_j(\cdot):[0,T]\longrightarrow\mathcal{L}(X)$. (We assume, as always, that the discrete parts of the time-ordering measures, when present, are finitely supported.) As in Subsection 2.2, we construct the Banach algebra $\mathbb{A}(R_1,\ldots,R_n)$ with the radii R_j, $j=1,\ldots,n$, given by

$$R_j:=\int_{[0,T]}\|V_j(s)\|_{\mathcal{L}(X)}\lambda_j(ds), \tag{5.2}$$

which is assumed finite for each j. We also construct the disentangling algebra $\mathbb{D}(V_1(\cdot)^\sim, \ldots, V_n(\cdot)^\sim)$, again following Subsection 2.2. Given $g \in \mathbb{D}(V_1(\cdot)^\sim, \ldots, V_n(\cdot)^\sim)$, we write, in the usual way,

$$g\left(V_1(\cdot)^\sim, \ldots, V_n(\cdot)^\sim\right) = \sum_{m_1,\ldots,m_n=0}^{\infty} a^g_{m_1,\ldots,m_n} \left[V_1(\cdot)^\sim\right]^{m_1} \cdots \left[V_n(\cdot)^\sim\right]^{m_n}.$$

The disentangling map is defined exactly as in Subsection 2.2. Indeed,

$$\begin{aligned}
&\mathcal{T}_{\lambda_1,\ldots,\lambda_n} g\left(V_1(\cdot)^\sim, \ldots, V_n(\cdot)^\sim\right) \\
&= g_{\lambda_1,\ldots,\lambda_n}\left(V_1(\cdot), \ldots, V_n(\cdot)\right) \\
&= \sum_{m_1,\ldots,m_n=0}^{\infty} a^g_{m_1,\ldots,m_n} P^{m_1,\ldots,m_n}_{\lambda_1,\ldots,\lambda_n}\left(V_1(\cdot), \ldots, V_n(\cdot)\right)
\end{aligned} \tag{5.3}$$

exactly as in Subsection 2.2. Consequently, the operational calculus in this setting has all of the properties outlined in Subsection 2.2. The reader will note that, just as with the use of Taylor's calculus, using analytic families is ultimately very easy, at least in the definition of the operational calculus. This is due to the fact that we are "hiding" the unbounded operators $T(z)$ using the bounded operators $V(z)$. This leads us to the following comments.

While unbounded operators do not appear explicitly in the disentangled operator, they are present in the definition of the operator-valued functions (or, more carefully, the boundedly analytic families) $V_j(\cdot)$, $j = 1, \ldots, n$. The injective operator-valued functions $U_j(\cdot)$ serve to take a vector from X and map it to the domain of the operator $T_j(\cdot)$. To be a bit more precise, take $j_1, \ldots, j_m \in \{1, \ldots, n\}$. We may write, for $\phi \in X$,

$$V_{j_m}(s_m) \cdots V_{j_1}(s_1)\phi = \left[T_{j_m}(s_m)U_{j_m}(s_m)\right] \cdots \left[T_{j_1}(s_1)U_{j_1}(s_1)\right]\phi.$$

Since $U_{j_1}(s_1)\phi \in \mathcal{D}(T_{j_1}(s_1))$, it makes sense to apply $T_{j_1}(s_1)$. Similarly,

$$U_{j_2}(s_2)\left(T_{j_1}(s_1)U_{j_1}(s_1)\right) \in \mathcal{D}(T_{j_2}(s_2))$$

and so we can apply $T_{j_2}(s_2)$ to the vector $U_{j_2}(s_2)\left(T_{j_1}(s_1)U_{j_1}(s_1)\phi\right)$. This process continues through the entire operator product. So, even though the operator products in the disentangling series do not involve the unbounded operators alone, the structure of the analytic families allow the unbounded operators to appear in a natural way in the operator products present in the disentangling series – the disentangled monomials in (5.3).

5.1. Stability results for FOC using analytic families. We investigate here the stability of FOC when using analytic families of operators. Fix, for the moment, $j \in \{1, \ldots, n\}$. Select a sequence $\{T_{j,k}(z)\}_{k=1}^{\infty}$ of analytic families in $D_0 \subseteq \mathbb{C}$ (D_0 is as above). We take $Z = X$ in the definition of analytic family, Definition 2.32. Furthermore, we will assume that all of our analytic families $T_n(z)$, $T(z)$ satisfy Definition 2.34. Then there are, for each $k \in \mathbb{N}$, bounded analytic families $U_{j,k}(z), V_{j,k}(z) \in \mathcal{L}(X)$, with $U_{j,k}(z)$ injective onto $\mathcal{D}(T_{j,k}(z))$ and

$$T_{j,k}(z)U_{j,k}(z) = V_{j,k}(z). \tag{5.4}$$

Fix, along with our j, a $z \in D_0$. Then

$$\mathcal{G}(T_{j,k}(z)) = \{(U_{j,k}(z)\phi, V_{j,k}(z)\phi) : \phi \in X\} \tag{5.5}$$

and

$$\mathcal{G}(T_j(z)) = \{(U_j(z)\phi, V_j(z)\phi) : \phi \in X\}. \tag{5.6}$$

We assume that $T_{j,k}(z) \longrightarrow T_j(z)$ as $k \to \infty$ in the generalized sense of Subsection 2.5. Then

$$\hat{\delta}\left(\mathcal{G}(T_{j,k}(z)), \mathcal{G}(T_j(z))\right) \longrightarrow 0 \tag{5.7}$$

as $k \to \infty$. It follows at once that

$$\delta\left(\mathcal{G}(T_{j,k}(z), \mathcal{G}(T_j(z))\right) \xrightarrow[k\to\infty]{} 0 \tag{5.8}$$

and

$$\delta\left(\mathcal{G}(T_j(z)), \mathcal{G}(T_{j,k}(z))\right) \xrightarrow[k\to\infty]{} 0. \tag{5.9}$$

We will use (5.8). Define, for each $k \in \mathbb{N}$ and $z \in D_0$,

$$\delta_k(z) := \delta\left(\mathcal{G}(T_{j,k}(z)), \mathcal{G}(T_j(z))\right). \tag{5.10}$$

Then $\delta_k(z)$ is the smallest positive real number for which

$$\operatorname{dist}\left(u, \mathcal{G}(T_{j,k}(z))\right) \le \delta_k(z)\|u\|_{\mathcal{G}(T_j(z))} \tag{5.11}$$

for all $u \in \mathcal{G}(T_j(z))$ (see (2.25)). (Note that the graph norm for an operator $T \in \mathscr{C}(X, Y)$ is $\|(f, Tf)\|_{\mathcal{G}(T)} := \left(\|f\|_X^2 + \|Tf\|_Y^2\right)^{1/2}$. Indeed, for Banach spaces X and Y, the product $X \times Y$ becomes a Banach space under the norm $\|(x, y)\|_{X\times Y} = \left(\|x\|_X^2 + \|y\|_Y^2\right)^{1/2}$.) Fix $u_j(z) \in \mathcal{G}(T_j(z))$ and, for each $k \in \mathbb{N}$, choose $v_{j,k}(z) \in \mathcal{G}(T_{j,k}(z))$ with

$$\|u_j(z) - v_{j,k}(z)\|_{X\times X} < \operatorname{dist}\left(u(z), \mathcal{G}(T_{j,k}(z))\right) + \frac{1}{k} \le \delta_k(z)\|u(z)\|_{\mathcal{G}(T_j(z))} + \frac{1}{k}. \tag{5.12}$$

We obtain a sequence $\{v_{j,k}(z)\}_{k=1}^{\infty}$ in $X \times X$ for which

$$\|u_j(z) - v_{j,k}(z)\|_{X\times X} \xrightarrow[k\to\infty]{} 0, \tag{5.13}$$

because $\delta_k(z) \to 0$ as $k \to \infty$. Now, using (5.5) and (5.6), we may write

$$u_j(z) = (U_j(z)\phi_j(z), V_j(z)\phi_j(z)) \tag{5.14}$$

for some $\phi_j(z) \in X$ and

$$v_{j,k}(z) = (U_{j,k}(z)\psi_{j,k}(z), V_{j,k}(z)\psi_{j,k}(z)) \tag{5.15}$$

for some $\psi_{j,k}(z) \in X$. Hence

$$\|U_j(z)\phi_j(z) - U_{j,k}(z)\psi_{j,k}(z)\|_{X\times X} \xrightarrow[k\to\infty]{} 0 \tag{5.16}$$

and

$$\|V_j(z)\phi_j(z) - V_{j,k}(z)\psi_{j,k}(z)\|_{X\times X} \xrightarrow[k\to\infty]{} 0. \tag{5.17}$$

We remind the reader that these limits are for our fixed $z \in D_0$; i.e., these limits are pointwise limits.

We now address stability of the operational calculus for this setting. In order to do so, we need to switch our attention to the formalism of Feynman's operational calculus in arbitrary Banach algebras which was developed in Subsection 3.2. Therefore, we will start by assuming that X is a Banach algebra, and, to avoid unnecessary (in the view of the author) technicalities, we will assume that X is separable. (See Subsection 3.1, above.)

To continue the development of the stability theory in the current setting, we define, for each $j \in \{1,\dots,n\}$,

$$\begin{aligned} W_j : [0,T] &\longrightarrow X, \\ s &\longmapsto V_j(s)\phi_j(s), \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} W_{j,k} : [0,T] &\longrightarrow X, \\ s &\longmapsto V_{j,k}(s)\psi_{j,k}(s). \end{aligned} \tag{5.19}$$

Because our analytic families satisfy Definition 2.34, we now appeal to the discussion contained in Subsection 2.8 to conclude that the maps $W_j(\cdot)$ and $W_{j,k}(\cdot)$ are continuous on D_0 and so on $[0,T] \subseteq D_0 \cap \mathbb{C}$. Also, it follows from Equation (5.17) that

$$W_{j,k}(s) \xrightarrow[k\to\infty]{} W_j(s) \tag{5.20}$$

in X-norm for each $s \in [0,T]$ and $j = 1,\dots,n$.

We wish to apply the ideas of Theorem 8.5.13 of [15] concerning the stability of the operational calculus with respect to the operator-valued functions. In order to use the ideas of this theorem, we need our functions $W_j(\cdot)$ and $W_{j,k}(\cdot)$ to be λ_j-measurable in the sense discussed in Subsection 3.1. Indeed, because our functions $W_j(\cdot)$ and $W_{j,k}(\cdot)$ are continuous and since X is separable, it follows that our functions are all measurable in the appropriate sense. The analogue of Theorem 8.5.13 which we will use is stated as follows.

Theorem 5.1 ([15, Theorem 8.5.13 Analog]). *Let X be a separable Banach algebra. Let $W_j : [0,T] \to X$, $j = 1,\dots,n$, and associate to each $W_j(\cdot)$ the Borel probability measure λ_j on $[0,T]$ where $\lambda_j = \mu_j + \eta_j$, with μ_j a continuous Borel measure and with η_j a discrete finitely supported measure on $[0,T]$. Let $\{\tau_1,\dots,\tau_h\}$ with $0 < \tau_1 < \cdots < \tau_h < T$ be the union of the supports of the discrete measures $\eta_1,\dots,\eta_n$, and write*

$$\eta_j = \sum_{i=1}^{h} p_{ji}\delta_{\tau_i}$$

for each $j = 1, \ldots, n$. *We assume that* $W_j(\cdot)$ *is* λ_j*-measurable for each* $j = 1, \ldots, n$. *Furthermore, choose sequences* $\{W_{j,k}(\cdot)\}_{k=1}^{\infty}$, $j = 1, \ldots, n$, *of* λ_j*-measurable functions from* $[0,T]$ *to* X *such that, for each* $j = 1, \ldots, n$, $W_{j,k}(\cdot) \to W_j(\cdot)$ λ_j*-almost everywhere in* X*-norm as* $k \to \infty$. *Assume that, for each* $j = 1, \ldots, n$,

$$\sup\{\|W_{j,k}(s)\|_X : k \in \mathbb{N}, s \in [0,T]\} < \infty \tag{5.21}$$

and

$$\sup\{\|W_j(s)\|_X : s \in [0,T]\} < \infty. \tag{5.22}$$

Define real numbers r_j, $r_{j,k}$ *for* $j = 1, \ldots, n$ *and* $k \in \mathbb{N}$ *by*

$$r_j := \int_{[0,T]} \|W_j(s)\|_X \, \lambda_j(ds) \tag{5.23}$$

and

$$r_{j,k} := \int_{[0,T]} \|W_{j,k}(s)\|_X \, \lambda_j(ds). \tag{5.24}$$

Construct the commutative Banach algebras $\mathbb{A}(r_{1,k}, \ldots, r_{n,k})$, $\mathbb{A}(r_1, \ldots, r_n)$ *and the associated disentangling algebras*

$$\mathbb{D}_k := \mathbb{D}\left((W_{1,k}(\cdot), \lambda_1)^{\sim}, \ldots, (W_{n,k}(\cdot), \lambda_n)^{\sim}\right) \tag{5.25}$$

and

$$\mathbb{D}_0 := \mathbb{D}\left((W_1(\cdot), \lambda_1)^{\sim}, \ldots, (W_n(\cdot), \lambda_n)^{\sim}\right). \tag{5.26}$$

Finally, let

$$\mathbb{D}_{\oplus} := \bigoplus_{k \in \mathbb{N} \cup \{0\}} \mathbb{D}_k \tag{5.27}$$

be the direct-sum Banach algebra. Let π_k, $k \in \mathbb{N} \cup \{0\}$, *be the canonical projection of* $\mathbb{D}_{\oplus}$ *onto* $\mathbb{D}_k$.

Then, for any $\theta_f \in \mathbb{D}_{\oplus}$, $\theta_f = (f, f, \ldots)$, *we have*

$$\mathcal{T}_{\lambda_1,\ldots,\lambda_n}(\pi_k(\theta_f)) \to \mathcal{T}_{\lambda_1,\ldots,\lambda_n}(\pi_0(\theta_f)) \tag{5.28}$$

in norm on X *as* $k \to \infty$. *More transparently, we may write the above as*

$$\lim_{k\to\infty} \mathcal{T}_{\lambda_1,\ldots,\lambda_n} f\left(\tilde{W_{1,k}(\cdot)}, \ldots, \tilde{W_{n,k}(\cdot)}\right) = \mathcal{T}_{\lambda_1,\ldots,\lambda_n} f\left(W_1(\cdot), \ldots, W_n(\cdot)\right) \tag{5.29}$$

in norm on X.

The proof of this theorem proceeds in exactly the same way as in [15], working in the Banach algebra X instead of the Banach algebra $\mathcal{L}(X)$.

To use Theorem 5.1, we note that the functions $W_j(\cdot)$ and $W_{j,k}(\cdot)$ defined in (5.18) and (5.19), respectively are, continuous on $[0,T]$. Since X is separable, each $W_j(\cdot)$ is λ_j-measurable and each $W_{j,k}(\cdot)$ is λ_j-measurable. It has been observed in Equation (5.20) that $W_{j,k}(s) \to W_j(s)$ in X-norm for every $s \in [0,T]$ and each $j = 1, \ldots, n$. Furthermore, from the continuity of $W_j(\cdot)$ and $W_{j,k}(\cdot)$ on $[0,T]$, the norm-boundedness conditions (5.21) and

(5.22) are clearly satisfied. By a direct application of Theorem 5.1 with $\theta_f = (f, f, f, \dots) \in \mathbb{D}_{\oplus}$,

$$\lim_{k\to\infty} f_{\lambda_1,\dots,\lambda_n}\left(W_{1,k}(\cdot),\dots,W_{n,k}(\cdot)\right) = f_{\lambda_1,\dots,\lambda_n}\left(W_1(\cdot),\dots,W_n(\cdot)\right)$$

in norm on X. (Recall that $f_{\lambda_1,\dots,\lambda_n} =: \mathcal{T}_{\lambda_1,\dots,\lambda_n} f$.)

5.2. Joint stability of F.O.C. in the setting of analytic families. In this section, we address the stability of the operational calculus with respect to generalized convergence of sequences of analytic families of operators and weak convergence of sequences of time-ordering measures. Of course, in the previous subsection, we addressed the stability of the operational calculus in the setting of analytic families of closed operators and so this subsection adds weakly convergent sequences of time-ordering measures. We will use the notation introduced in Subsection 2.8 and in Subsection 5.1, just above.

To begin, we will follow the general setup detailed at the beginning of the current section. Hence, we let $D_0 \subseteq \mathbb{C}$ be an open neighborhood of $0 \in \mathbb{C}$. We also take $T_1(z), \dots, T_n(z)$ to be analytic families of unbounded closed operators on the separable Banach space X. For each $j = 1, \dots, n$, we will assume, in the definition of analytic family found in Subsection 2.7, that the Banach space Z equals X and so there are boundedly analytic functions $U_j(\cdot), V_j(\cdot)$ mapping D_0 into $\mathcal{L}(X)$ with $U_j(\cdot)$ injective onto the domain $\mathcal{D}(T_j(z))$ of $T_j(z)$, $z \in D_0$ and

$$T_j(z)U_j(z) = V_j(z),$$

for $z \in D_0$. We now, just as before, choose $T > 0$ such that $[0, T] \subseteq D_0 \cap \mathbb{R}$ and for each $j = 1, \dots, n$, associate to each $V_j(\cdot)$ a Borel probability measure λ_j on $[0, T]$. (As always, if a discrete part of a time-ordering measure is present in any of the λ_j, we assume it to be finitely supported.) Define, as before,

$$R_j := \int_{[0,T]} \|V_j(s)\|_{\mathcal{L}(X)}\, \lambda_j(ds)$$

which we assume finite for each $j = 1, \dots, n$. Construct the disentangling algebra $\mathbb{D}\left(V_1(\cdot)^{\sim}, \dots, V_n(\cdot)^{\sim}\right)$ and obtain, given $g \in \mathbb{D}\left(V_1(\cdot)^{\sim} \dots, V_n(\cdot)^{\sim}\right)$, the disentangled operator $g_{\lambda_1,\dots,\lambda_n}\left(V_1(\cdot), \dots, V_n(\cdot)\right)$ as in (3.1), above.

We now select, for each $j = 1, \dots, n$, a sequence $\{T_{j,k}(\cdot)\}_{k=1}^{\infty}$ of analytic families in D_0, D_0 being just as above. Also, as above, take the Banach space Z in the definition of analytic families to be equal to X. We will also assume that our analytic families $T_1(\cdot), \dots, T_n(\cdot)$ and $T_{1,k}(\cdot), \dots, T_{n,k}(\cdot)$ satisfy Definition 2.34; i.e., they are uniform on D_0. There are, for each $k \in \mathbb{N}$, boundedly analytic families $U_{j,k}(\cdot), V_{j,k}(\cdot) \in \mathcal{L}(X)$ with $U_{j,k}(\cdot)$ injective onto $\mathcal{D}(T_{j,k}(z))$ and

$$T_{j,k}(z)U_{j,k}(z) = V_{j,k}(z)$$

for each $z \in D_0$.

We now follow the exposition at the beginning of Subsection 5.1 to obtain the maps

$$\begin{aligned} W_j : [0,T] &\to X, \\ s &\mapsto V_j(s)\phi_j(s), \end{aligned}$$

and

$$\begin{aligned} W_{j,k} : [0,T] &\to X, \\ s &\mapsto V_{j,k}(s)\psi_{j,k}(s). \end{aligned}$$

(See equations (5.18) and (5.19).) By virtue of our assumed uniformity for the analytic families we appeal to the discussion of Subsection 2.8 to conclude that $W_j(\cdot)$ and $W_{j,k}(\cdot)$ are continuous and, from (5.17), $W_{j,k}(\cdot) \longrightarrow W_j(\cdot)$ pointwise in X-norm as $k \to \infty$.

Now that we have our sequences $\{W_{j,k}(\cdot)\}_{k=1}^{\infty}$ of continuous X-valued functions, we turn to our time-ordering measures. For each $j = 1, \ldots, n$, select a sequence $\{\lambda_{j,k}\}_{k=1}^{\infty}$ of Borel probability measures on $[0,T]$ such that each measure is allowed to have a discrete part with finite support (if present) and $\lambda_{j,k} \rightharpoonup \lambda_j$ as $k \to \infty$. For each $j = 1, \ldots, n$ we associate the Borel measures $\lambda_{j,k}$ and λ_j to the X-valued functions $W_{j,k}(\cdot)$, $W_j(\cdot)$, $k \in \mathbb{N}$.

To obtain our joint stability result, we let our continuous functions $W_j(\cdot)$, $j = 1, \ldots, n$, play the role of the functions $Y_j(\cdot)$, $j = 1, \ldots, n$, of Subsection 3.4. This being done, we can apply Theorem 3.16 to conclude that, for $f \in \mathbb{A}\left(M_1, \ldots, M_n\right)$ ($M_1, \ldots, M_n$ are defined just as in (3.57)),

$$\lim_{k\to\infty} f_{\lambda_{1,k},\ldots,\lambda_{n,k}}\left(W_{1,k}(\cdot), \ldots, W_{n,k}(\cdot)\right) = f_{\lambda_1,\ldots,\lambda_n}\left(W_1(\cdot), \ldots, W_n(\cdot)\right). \tag{5.30}$$

This is our joint stability result for the operational calculus in the setting of analytic families of operators. The reader will observe that, as has been the case throughout our investigations of the operational calculus using unbounded operators, the unbounded operators here are “hiding” in the functions $W_j(\cdot)$ and $W_{j,k}(\cdot)$ via the boundedly analytic functions $U_{j,k}(\cdot), U_j(\cdot), V_{j,k}(\cdot)$ and $V_j(\cdot)$. Moreover, since $W_j(s) = V_j(s)\phi_j(s)$ and $W_{j,k}(s) = V_{j,k}(s)\psi_{j,k}(s)$ and because the vectors $\phi_j(s)$ and $\psi_{j,k}(s)$ are most certainly not arbitrary (see (5.14) and (5.15), respectively), the joint stability result stated here is not as general as one may hope. Nevertheless, it is satisfying to find an analogue of joint stability in the general setting of closed unbounded operators.

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