New York Journal of Mathematics

New York J. Math. **28** (2022) 800–823.

Isolated points of the Zariski space

Dario Spirito

ABSTRACT. Let D be an integral domain and L be a field containing D. We study the isolated points of the Zariski space $\mathrm{Zar}(L|D)$, with respect to the constructible topology. In particular, we completely characterize when L (as a point) is isolated and, under the hypothesis that L is the quotient field of D, when a valuation domain of dimension 1 is isolated; as a consequence, we find all isolated points of $\mathrm{Zar}(D)$ when D is a Noetherian domain and, under the hypothesis that D and D' are Noetherian, local and countable, we characterize when $\mathrm{Zar}(D)$ and $\mathrm{Zar}(D')$ are homeomorphic. We also show that if V is a valuation domain and L is transcendental over V then the set of extensions of V to L has no isolated points.

CONTENTS

1. Introduction

Let D be an integral domain with quotient field K, and let L be a field containing K. The *Zariski space* of L over D, denoted by $\mathrm{Zar}(L|D)$, is the set of all valuation rings containing D and having quotient field L. O. Zariski introduced this set (under the name *abstract Riemann surface*) and endowed it with a natural topology (later called the *Zariski topology*) during its study of resolution of singularities; in particular, he used the compactness of the Zariski space to reduce the problem of gluing infinitely many projective models to the gluing of only

Received February 2, 2021.

2010 *Mathematics Subject Classification.* 13F30; 13A15; 13A18; 54D99.

Key words and phrases. Zariski space; constructible topology; Cantor space; isolated points; perfect spaces; extensions of valuations.

ISSN 1076-9803/2022

finitely many of them [30, 31]. Later on, it was showed that $\mathrm{Zar}(L|D)$ enjoys even deeper topological properties: in particular, it is a *spectral space*, meaning that there is always a ring R such that $\mathrm{Spec}(R)$ (endowed with the Zariski topology) is homeomorphic to $\mathrm{Zar}(L|D)$, and an example of such an R can be found using the Kronecker function ring construction [5, 6, 8]. Beyond being a very natural example of a spectral space "occurring in nature", the Zariski topology can also be used, for example, to study representation of integral domains as intersection of overrings [19, 20, 21], or in real and rigid algebraic geometry [15, 24].

As a spectral space, two other topologies can be constructed on $\mathrm{Zar}(L|D)$ starting from the Zariski topology: the *inverse* and the *constructible* (or *patch*) topology. Both of them give rise to spectral spaces (in particular, they are compact); furthermore, the constructible topology gains the property of being Hausdorff, and plays an important role in the topological characterization of spectral spaces (see for example Hochster's article [14]). The constructible topology can also be studied through ultrafilters [7], and this point of view allows to give many examples of spectral spaces, for example by finding them inside other spectral spaces (see [21, Example 2.2(1)] for some very general constructions, [27] for examples in the overring case, and [10, 9] for examples in the setting of semistar operations).

In this paper, we want to study the points of $\mathrm{Zar}(L|D)$ that are isolated, with respect to the constructible topology. Our starting point is a new interpretation of a result about the compactness of spaces in the form $\mathrm{Zar}(K|D) \setminus \{V\}$ [26, Theorem 3.6], where K is the quotient field of D: in particular, we show that if V is isolated in $\mathrm{Zar}(L|D)$, where L is a field containing V, then V is the integral closure of $D[x_1, \dots, x_n]_M$ for some $x_1, \dots, x_n \in L$, where M is a maximal ideal of $D[x_1, \dots, x_n]$ (Theorem 3.4). Through this result, we characterize when L is an isolated point of $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ (i.e., $\mathrm{Zar}(L|D)$ endowed with the constructible topology; Proposition 4.1) and, under the hypothesis that $L = K$ is the quotient field of D, when the one-dimensional valuation overrings are isolated (Theorem 5.2).

In Section 6, we study the isolated points of the constructible topology when D is a Noetherian domain and $L = K$ is its quotient field. Theorem 6.3 gives a complete characterization: $V \in \mathrm{Zar}(K|D) = \mathrm{Zar}(D)$ is isolated if and only if the center P of V on D has height at most 1 and P is contained in only finitely many prime ideals; in particular, this cannot happen if D is local and of dimension at least 3. In the countable case, we also give a complete characterization of when $\mathrm{Zar}(D)^{\mathrm{cons}} \simeq \mathrm{Zar}(D')^{\mathrm{cons}}$ under the hypothesis that D and D' are Noetherian and local (Theorem 6.11).

The last two sections of the paper explore the case of extension of valuations. Section 7 studies the case where D itself is a field: in particular, we show that if the transcendence degree of L over D is at least 2 then $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ has no isolated points, improving [3, Theorem 4.45]. In Section 8, we show that if V is a valuation domain that is not a field and $K(X)$ is the field of rational functions in one indeterminate, then the set of extensions of V to $K(X)$ has no isolated points (Theorem 8.2), and as a consequence we further extend [3, Theorem 4.45] to $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ when D is an arbitrary integral domain (Theorem 7.3 and Corollary 8.6).

2. Notation and preliminaries

Throughout the paper, all rings will be commutative, unitary and will have no zero-divisors (that is, they are integral domains). We usually denote by D such a domain and by K its quotient field; we use $\overline{D}$ to denote the integral closure of D in K.

2.1. Spectral spaces. A topological space X is *spectral* if it is homeomorphic to the prime spectrum of a ring, endowed with the Zariski topology; spectral spaces can also be characterized in a purely topological way (see [14] and [4]). Among their properties, spectral spaces are always compact and have a basis of open and compact sets. If $\Delta \subseteq X$, we denote by $\mathrm{Cl}(\Delta)$ the closure of Δ. The topology of X induces an order such that $x \leq y$ if and only if $y \in \mathrm{Cl}(x)$. If $Y \subseteq X$, the *closure under generization* of Y if the set $Y^{\mathrm{gen}} := \{x \in X \mid x \leq y$ for some $y \in Y\}$, where $\leq$ is the order induced by the topology, and Y is *closed by generizations* if $Y = Y^{\mathrm{gen}}$.

If X is a spectral space, the *inverse topology* on X is the coarsest topology such that the open and compact subsets of X are closed. We denote by X^{inv} the space X, endowed with the inverse topology. A subset $Y \subseteq X$ is closed in the inverse topology if and only if it is compact in the starting topology and closed by generizations; in particular, if Y is compact in the starting topology then its closure in the inverse topology is Y^{gen}.

If X is a spectral space, the *constructible topology* (or *patch topology*) on X is the coarsest topology such that the open and compact subsets of X are both open and closed. We denote by X^{cons} the space X, endowed with the constructible topology; if $Y \subseteq X$, we denote by Y^{cons} the subset Y considered with respect to the constructible topology, and by $\mathrm{Cl}^{\mathrm{cons}}(Y)$ the closure of Y in X^{cons}. If $Y = \mathrm{Cl}^{\mathrm{cons}}(Y)$, then Y is compact in the starting topology; conversely, if Y is closed in the starting topology or in the inverse topology, then it is closed also in the constructible topology.

Both X^{inv} and X^{cons} are spectral spaces, and in particular compact spaces; moreover, X^{cons} is Hausdorff and zero-dimensional.

A map $f : X \longrightarrow Y$ of spectral spaces is a *spectral map* if $f^{-1}(\Omega)$ is open and compact for every open and compact subset Ω of Y; in particular, a spectral map is continuous. If f is both spectral and closed, then it is also proper, and in particular $f^{-1}(\Omega)$ is compact for every compact subset Ω of Y [4, 5.3.7(i)]. If $f : X \longrightarrow Y$ is a spectral map, then it is spectral also when X and Y are both endowed with the inverse topology, and when they are both endowed with the constructible topology [4, Theorem 1.3.21]. In the latter case, f is also closed, since it is a continuous map between Hausdorff compact spaces.

2.2. Isolated points. If X is a topological space, a point $p \in X$ is *isolated* in X if $\{p\}$ is an open set. If X has no isolated points, then X is said to be *perfect*. The set of points that are not isolated in X is a closed set, called the *derived set* of X.

If $\Omega \subseteq X$ and $p \in \Omega$ is isolated in X, then p is also isolated in Ω; if Ω is open, then p is isolated in X if and only if p is isolated in Ω.

2.3. Valuation domains. A *valuation domain* is an integral domain V such that, for every $x \neq 0$ in the quotient field of V, at least one of x and x^{-1} is in V. Any valuation domain is local; we denote the maximal ideal of V by $\mathfrak{m}_V$. If L is a field containing the quotient field K of V, an *extension* of V to L is a valuation domain W having quotient field L such that $W \cap K = V$. We denote the set of extension of V to L by $\mathcal{E}(L|V)$; this set is always nonempty (see e.g. [12, Theorem 20.1]).

If D is an integral domain and L is a field containing D, the *Zariski space* (or *Zariski-Riemann space*) of L over D, denoted by $\mathrm{Zar}(L|D)$, is the set of all valuation domains containing D and having quotient field L. The Zariski space $\mathrm{Zar}(L|D)$ is always nonempty. When L is the quotient field of D, we denote $\mathrm{Zar}(L|D)$ simply by $\mathrm{Zar}(D)$, and we call its elements the *valuation overrings* of D.[1] If D' is the integral closure of D in L, then $\mathrm{Zar}(L|D) = \mathrm{Zar}(L|D')$; in particular, $\mathrm{Zar}(D) = \mathrm{Zar}(\overline{D})$. A valuation ring in $\mathrm{Zar}(L|D)$ is *minimal* if it is minimal with respect to containment.

The Zariski-Riemann space $\mathrm{Zar}(L|D)$ can be endowed with a natural topology, called the *Zariski topology*, which is the topology generated by the basic open sets

$$\mathcal{B}(x_1, \dots, x_n) := \{V \in \mathrm{Zar}(L|D) \mid x_1, \dots, x_n \in V\},$$

as $x_1, \dots, x_n$ range among the elements of L; we use the notation $\mathcal{B}^L(x_1, \dots, x_n)$ if we need to underline the field L. Under this topology, $\mathrm{Zar}(L|D)$ is a spectral space whose order is the opposite of the containment order [6, 5]; in particular, the minimal valuation rings in $\mathrm{Zar}(L|D)$ are maximal with respect to the order induced by the Zariski topology. As a spectral space, we can define the inverse and the constructible topology on $\mathrm{Zar}(L|D)$; a set $\Delta \subseteq \mathrm{Zar}(L|D)$ is closed with respect to the inverse topology if and only if it is compact with respect to the Zariski topology and $\Delta = \{W \in \mathrm{Zar}(L|D) | W \supseteq V \text{ for some } V \in \Delta\}$ [8, Remark 2.2 and Proposition 2.6].

Since $\mathcal{B}(z_1, \dots, z_n) = \mathcal{B}(z_1) \cap \cdots \cap \mathcal{B}(z_n)$ for every $z_1, \dots, z_n \in L$, a basis of the constructible topology of $\mathrm{Zar}(L|D)$ is the family of the sets in the form $\mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c$, as $x_1, \dots, x_n, y_1, \dots, y_m$ range in L. In particular, V is isolated in $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ if and only if

$$\begin{aligned}\{V\} &= \mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c = \\ &= \mathrm{Zar}(L|D[x_1, \dots, x_n]) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c\end{aligned}$$

for some $x_1, \dots, x_n, y_1, \dots, y_m \in L$.

[1]An *overring* of D is, more generally, a ring contained between D and its quotient field.

If $L' \subseteq L$ is a field extension and $D \subseteq L'$, we have a restriction map

$$\begin{aligned} \rho\colon \operatorname{Zar}(L|D) &\longrightarrow \operatorname{Zar}(L'|D), \\ V &\longmapsto V \cap L'. \end{aligned}$$

The map ρ is surjective due to Chevalley's extension theorem (see e.g. [1, Theorme 5.21] or [12, Theorem 19.5]), and is a spectral map since $\rho^{-1}(\mathcal{B}^{L'}(x)) = \mathcal{B}^{L}(x)$. Therefore, it is spectral and closed with respect to the constructible topology (on both sets). In particular, if $V \in \operatorname{Zar}(L'|D)$, then $\mathcal{E}(L|V) = \rho^{-1}(V)$; hence, $\mathcal{E}(L|V)$ is always closed in $\operatorname{Zar}(L|D)^{\text{cons}}$, and in particular it is compact both in the Zariski and the constructible topology.

Since, by definition, the spectrum $\operatorname{Spec}(D)$ is a spectral space (when endowed with the Zariski topology), we can define the inverse and the constructible topology also on $\operatorname{Spec}(D)$. For every ideal I of D, set $\mathcal{V}(I) := \{P \in \operatorname{Spec}(D) \mid I \subseteq P\}$ and $\mathcal{D}(I) := \operatorname{Spec}(D) \setminus \mathcal{V}(I)$: then, a basis of $\operatorname{Spec}(D)^{\text{cons}}$ is given by the intersections $\mathcal{D}(aD) \cap \mathcal{V}(I)$, as a ranges in D and I among the finitely generated ideals of D [4, Theorem 12.1.10(iv)].

For every field L, we can define a map

$$\begin{aligned} \gamma\colon \operatorname{Zar}(L|D) &\longrightarrow \operatorname{Spec}(D), \\ V &\longmapsto \mathfrak{m}_V \cap D, \end{aligned}$$

which is called the *center map*. When $\operatorname{Zar}(L|D)$ and $\operatorname{Spec}(D)$ are endowed with the Zariski topology, γ is spectral (in particular, continuous; see [32, Chapter VI, §17, Lemma 1] or [5, Theorem 4.1]), surjective (this follows, for example, from [1, Theorem 5.21] or [12, Theorem 19.6]) and closed [5, Theorem 2.5], so in particular it is proper. Therefore, γ is a spectral map also when $\operatorname{Zar}(L|D)$ and $\operatorname{Spec}(D)$ are endowed with their respective constructible topologies.

3. General results

We begin by establishing some general criteria to determine which valuation domains are isolated in $\operatorname{Zar}(D)$.

Let D be an integral domain: a prime ideal is called *essential* if D_P is a valuation domain, and D_P is said to be an *essential valuation overring* of D. We shall need the following weaker notion: we say that a prime ideal P of D is *almost essential* if there is a unique valuation overring of D having center P; equivalently, P is almost essential if and only if the integral closure of D_P is a valuation domain V. When this happens, we say that V is an *almost essential valuation overring* of D.

In the context of almost essential primes and valuation overrings, isolated valuation rings correspond to isolated prime ideals.

Proposition 3.1. *Let D be an integral domain, and let P be an almost essential prime ideal of D; let V be the valuation overring with center P. Then, V is isolated in* $\operatorname{Zar}(D)^{\text{cons}}$ *if and only if P is isolated in* $\operatorname{Spec}(D)^{\text{cons}}$.

Proof. Let $\gamma : \mathrm{Zar}(D) \longrightarrow \mathrm{Spec}(D)$ be the center map. If P is isolated in $\mathrm{Spec}(D)^{\mathrm{cons}}$, then $\{P\}$ is open and thus, as γ is continuous, $\{V\} = \gamma^{-1}(\{P\})$ is open in $\mathrm{Zar}(D)^{\mathrm{cons}}$, i.e., V is isolated. Conversely, if V is isolated then $\mathrm{Zar}(D) \setminus \{V\}$ is closed, with respect to the constructible topology, and thus $\gamma(\mathrm{Zar}(D) \setminus \{V\}) = \mathrm{Spec}(D) \setminus \{P\}$ is closed in $\mathrm{Spec}(D)^{\mathrm{cons}}$. Hence, $\{P\}$ is open and P is isolated in $\mathrm{Spec}(D)^{\mathrm{cons}}$, as claimed. □

Corollary 3.2. *Let D be a Prüfer domain, and let V be a valuation overring of D with center P. Then, V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$ if and only if P is isolated in $\mathrm{Spec}(D)^{\mathrm{cons}}$. In particular, $\mathrm{Zar}(D)^{\mathrm{cons}}$ is perfect if and only if $\mathrm{Spec}(D)^{\mathrm{cons}}$ is perfect.*

Proof. Since D is a Prüfer domain, every valuation overring is essential. The claim follows from Proposition 3.1. □

In general, almost essential valuation overrings are rare; for example, if D is Noetherian, no prime ideal of height 2 or more can be almost essential. For this reason, we need more general results; the first step is connecting isolated valuation rings with compactness.

Proposition 3.3. *Let X be a spectral space, and let x be a maximal element with respect to the order induced by the topology. Then, the following are equivalent:*

(i) x is isolated in X^{cons};
(ii) $X \setminus \{x\}$ is compact, with respect to the starting topology;
(iii) $X \setminus \{x\}$ is closed, with respect to the inverse topology.

Proof. Let $Y := X \setminus \{x\}$.

The equivalence of (ii) and (iii) follows from the fact that Y is closed by generizations.

If (i) holds, then $\{x\}$ is an open set in the constructible topology, and thus Y is closed; since X^{cons} is compact, it follows that Y is compact in the constructible topology and thus also in the Zariski topology (which is coarser). Thus, (ii) holds.

Conversely, if (iii) holds, then Y is closed also in the constructible topology; hence, $\{x\}$ is open and x is isolated. Thus, (i) holds. □

In particular, the previous proposition applies when $X = \mathrm{Zar}(L|D)$ and V is a minimal element of $\mathrm{Zar}(L|D)$, with respect to containment. In this case, the fact that $\mathrm{Zar}(L|D) \setminus \{V\}$ is compact has very strong consequences.

Theorem 3.4. *Let D be an integral domain and let $V \in \mathrm{Zar}(L|D)$. Then, the following are equivalent.*

(i) V is isolated in $\mathrm{Zar}(L|D)^{\mathrm{cons}}$;
(ii) there are $x_1, \dots, x_n \in L$ and a maximal ideal M of $D[x_1, \dots, x_n]$ such that V is the integral closure of $D[x_1, \dots, x_n]_M$ and M is isolated in

$$\mathrm{Spec}(D[x_1, \dots, x_n])^{\mathrm{cons}};$$

(iii) there are $x_1, \dots, x_n \in L$ *and a prime ideal* P *of* $D[x_1, \dots, x_n]$ *such that* V *is the integral closure of* $D[x_1, \dots, x_n]_P$ *and* P *is isolated in*

$$\mathrm{Spec}(D[x_1, \dots, x_n])^{\mathrm{cons}}.$$

Proof. Let X be an indeterminate over D, and let $R := D + XL[[X]]$. By the reasoning in the proof of [28, Proposition 3.3] (or by Lemma 4.2 below) the Zariski space $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ is homeomorphic to $(\mathrm{Zar}(R) \setminus \{L((X))\})^{\mathrm{cons}}$, which is open in $\mathrm{Zar}(R)^{\mathrm{cons}}$; in particular, a $W \in \mathrm{Zar}(L|D)$ is isolated with respect to the constructible topology if and only if $W + XL[[X]]$ is isolated in $\mathrm{Zar}(R)^{\mathrm{cons}}$. Therefore, without loss of generality we can suppose that L is the quotient field of D.

(i) $\Longrightarrow$ (iii) Since V is isolated, there are $x_1, \dots, x_k, y_1, \dots, y_m \in L$ such that $\{V\} = \mathrm{Zar}(D[x_1, \dots, x_k]) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c$. In particular, V is a minimal valuation overring of $D[x_1, \dots, x_k]$. By Proposition 3.3, $\mathrm{Zar}(D[x_1, \dots, x_k]) \setminus \{V\}$ is compact, with respect to the Zariski topology; therefore, by [26, Theorem 3.6], there are $x_{k+1}, \dots, x_n \in L$ such that V is the integral closure of

$$D[x_1, \dots, x_k][x_{k+1}, \dots, x_n]_M = D[x_1, \dots, x_n]_M$$

for some maximal ideal M of $D[x_1, \dots, x_n]$. Hence, M is almost essential in $D[x_1, \dots, x_n]$, and by Proposition 3.1, M is isolated in $\mathrm{Spec}(D[x_1, \dots, x_n])^{\mathrm{cons}}$. Thus (ii) holds.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i) The set $\mathrm{Zar}(D[x_1, \dots, x_n]) = \mathcal{B}(x_1, \dots, x_n)$ is open in the constructible topology, and thus V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$ if and only if it is isolated in $\mathrm{Zar}(D[x_1, \dots, x_n])^{\mathrm{cons}}$. By hypothesis, P is almost essential for $D[x_1, \dots, x_n]$, and thus by Proposition 3.1 the integral closure V of $D[x_1, \dots, x_n]_P$ is isolated, as claimed. □

4. Dimension 0

In this section, we study when the field L is isolated in $\mathrm{Zar}(L|D)^{\mathrm{cons}}$. If L is the quotient field of D, then L is an essential valuation overring of D, and thus one can reason through Proposition 3.1; however, it is possible to use a more general approach.

A domain D with quotient field K is said to be a *Goldman domain* (or a *G-domain*) if K is a finitely generated D-algebra, or equivalently if $K = D[u]$ for some $u \in K$.

Proposition 4.1. *Let* D *be an integral domain with quotient field* K*, and let* L *be a field extension of* K*. Then,* L *is isolated in* $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ *if and only if* D *is a Goldman domain and* $K \subseteq L$ *is an algebraic extension.*

Proof. Suppose first that the two conditions hold. Then, $K = D[u]$ for some $u \in K$; since $K \subseteq L$ is algebraic, it follows that $\mathcal{B}(u) = \mathrm{Zar}(L|K) = \{L\}$. Hence, L is isolated in $\mathrm{Zar}(L|D)^{\mathrm{cons}}$.

Conversely, suppose that L is isolated. By Theorem 3.4, there are $x_1, \dots, x_n \in L$ such that L is the integral closure of $D[x_1, \dots, x_n]_M$ for some maximal ideal

M; since M must have height 0, $F := D[x_1, \dots, x_n]$ must be a field such that $F \subseteq L$ is algebraic.

Suppose that F is transcendental over K: then, we can take a transcendence basis $y_1, \dots, y_k$ of F over K. By construction, F is algebraic over the quotient field of $D[y_1, \dots, y_k]$; since F is a field, it is a Goldman domain, and thus by [16, Theorem 22] so should be $D[y_1, \dots, y_k]$, against [16, Theorem 21]. Thus, F is algebraic over K. Applying again [16, Theorem 22] to the extension $D \subset F$, we see that D is a Goldman domain; furthermore, L is algebraic over F and thus over K. The claim is proved. □

The previous result can be used to give some necessary conditions for V to be isolated. We premise a lemma.

Lemma 4.2. *Let D be an integral domain, L be a field containing D, and let $W \in \mathrm{Zar}(L|D)$. Let $\pi : W \longrightarrow W/\mathfrak{m}_W$ be the quotient map. Then, the map*

$$\overline{\pi}\colon \{Z \in \mathrm{Zar}(L|D) \mid Z \subseteq W\} \longrightarrow \mathrm{Zar}(W/\mathfrak{m}_W|D/(\mathfrak{m}_W \cap D)),$$
$$Z \longmapsto \pi(Z)$$

is a homeomorphism, when both sets are endowed with either the Zariski or the constructible topology.

Proof. Let $Z \in \mathrm{Zar}(L|D)$: then, $\ker \pi = \mathfrak{m}_W \subseteq Z$ since Z and W are valuation domains with the same quotient field and $Z \subseteq W$. Hence, $\pi(Z) = Z/\mathfrak{m}_W$ is a valuation ring containing $D/(\mathfrak{m}_W \cap D)$; moreover, since W is a localization of Z, $W/\mathfrak{m}_W$ is a localization of $Z/\mathfrak{m}_W$ and thus $W/\mathfrak{m}_W$ is the quotient field of $\pi(Z)$. Hence, $\overline{\pi}$ is well-defined.

Moreover, if $Z' \in \mathrm{Zar}(W/\mathfrak{m}_W|D/(\mathfrak{m}_W \cap D))$, then $Z := \pi^{-1}(Z')$ is the pullback of Z' along the quotient $W \longrightarrow W/\mathfrak{m}_W$. Thus, Z is a valuation domain by [11, Proposition 1.1.8(1)], and its quotient field is L by [11, Lemma 1.1.4(10)]. Hence $\overline{\pi}$ is surjective. Furthermore, if $Z \in \mathrm{Zar}(L|D)$ and $Z \subseteq W$, then $\ker \pi \subseteq Z$ and thus $\pi^{-1}(\pi(Z)) = Z$; hence, $\overline{\pi}$ is bijective.

Let now $x \in W/\mathfrak{m}_W$. Then, $Z \in \overline{\pi}^{-1}(\mathcal{B}(x))$ if and only if $x \in \pi(Z)$. Since $\ker \pi \subseteq Z$, this happens if and only if Z contains all of $\pi^{-1}(x)$; thus, for every $y \in \pi^{-1}(x)$, we have $\overline{\pi}^{-1}(\mathcal{B}(x)) = \mathcal{B}(y)$, and likewise $\overline{\pi}(\mathcal{B}(x)) = \mathcal{B}(\pi(x))$ for every $x \in L$. Hence, $\overline{\pi}$ is continuous and open when both $\{Z \in \mathrm{Zar}(L|D) \mid Z \subseteq W\}$ and $\mathrm{Zar}(W/\mathfrak{m}_W|D/(\mathfrak{m}_W \cap D))$ are endowed with the Zariski topology, and thus it is a homeomorphism. It follows that it is also a homeomorphism when both sets are endowed with the constructible topology, as claimed. □

Proposition 4.3. *Let $V \in \mathrm{Zar}(D)$ be a valuation domain with center P on D. If V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$, then the field extension $D_P/PD_P \subseteq V/\mathfrak{m}_V$ is algebraic.*

Proof. Consider $\Delta := \{W \in \mathrm{Zar}(D) \mid W \subseteq V\}$. Since $\mathfrak{m}_V \cap D = P$, by Lemma 4.2, the quotient map $V \longrightarrow V/\mathfrak{m}_V$ induces a homeomorphism between Δ^{cons} and $\mathrm{Zar}(V/\mathfrak{m}_V|D/P)^{\mathrm{cons}}$, and thus $V/\mathfrak{m}$ is isolated in $\mathrm{Zar}(V/\mathfrak{m}_V|D/P)^{\mathrm{cons}}$. Let F be the quotient field of D/P: then, $F = (D/P)_{P/P} = D_P/PD_P$. By Proposition 4.1, $F \subseteq V/\mathfrak{m}_V$ must be algebraic, as claimed. □

Corollary 4.4. *Let D be an integral domain, let $\gamma : \mathrm{Zar}(D) \longrightarrow \mathrm{Spec}(D)$ be the center map and let $V \in \mathrm{Zar}(D)$. If V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$, then V is minimal in $\gamma^{-1}(\gamma(V))$.*

Proof. Let $P := \gamma(V)$. If V is not minimal, then $V/\mathfrak{m}_V$ is not minimal in $\mathrm{Zar}(V/\mathfrak{m}_V|D_P/PD_P)$; hence, the extension $D_P/PD_P \subseteq V/\mathfrak{m}_V$ cannot be algebraic, against Proposition 4.3. □

5. Dimension 1

We now analyze the case where the valuation ring V has (Krull) dimension 1; however, the methods we use only work when V is a valuation overring of D, i.e., only for the space $\mathrm{Zar}(D) = \mathrm{Zar}(K|D)$, where K is the quotient field of D. Unlike in the proof of Theorem 3.4, we cannot use [28, Proposition 3.3] to extend these results to arbitrary Zariski spaces $\mathrm{Zar}(L|D)$, because that construction changes the dimension of the valuation domains involved.

The idea of this section is to study the maximal ideals of the finitely generated algebras $D[x_1, \dots, x_n]$.

Proposition 5.1. *Let $(D, \mathfrak{m})$ be an integrally closed local domain, and let $T \neq D$ be a finitely generated D-algebra contained in the quotient field K of D. If $\mathfrak{m}T \neq T$, then no maximal ideal of T above $\mathfrak{m}$ has height 1.*

Proof. Let $T := D[x_1, \dots, x_n]$; we proceed by induction on n.

Suppose $n = 1$, and let $x := x_1$; then, $x \notin D$. If $x^{-1} \in D$, then $x \in \mathfrak{m}$, and thus $\mathfrak{m}T = T$, a contradiction. Hence, $x, x^{-1} \notin D$. By [25, Theorem 6], the ideal $\mathfrak{p} := \mathfrak{m}T$ is prime but not maximal; since every maximal ideal of T above $\mathfrak{m}$ must contain $\mathfrak{p}$, it follows that no such maximal ideal can have height 1.

Suppose that the claim holds up to $n - 1$; let $A := D[x_1, \dots, x_{n-1}]$, so that $T = A[x_n]$; without loss of generality, $A \neq D$ and $x_n \notin A$. Let M be a maximal ideal of T above $\mathfrak{m}$. If x_n is integral over A, then T is integral over A, and thus the height of M is equal to the height of $M \cap A$, which is not equal to 1 by induction.

Suppose that x_n is not integral over A. Let A' be the integral closure of A; then, $T \subseteq A'[x_n]$ is an integral extension, and since x_n is not integral over A it follows that $A' \subsetneq A'[x_n]$. Take a maximal ideal M' of $A'[x_n]$ above M. Let $N := M' \cap A'$; then, N is a nonzero prime ideal of A', and thus $A'' := (A')_N$ is a local integrally closed domain with maximal ideal $N(A')_N \neq (0)$. Then, the ring $A''[x_n]$ is the quotient ring of $A'[x_n]$ with respect to the multiplicatively closed set $A'[x_n] \setminus N$, the set $M'' := M'A''[x_n]$ is a maximal ideal, and $N(A')_N \subseteq M''$. Applying the case $n = 1$ to A'' and $A''[x_n]$, it follows that the height of M'' is not 1; since the height of M'' is the same of the height of M' and of M, it follows that the height of M is not 1, as claimed. □

Theorem 5.2. *Let D be an integral domain, and let $V \in \mathrm{Zar}(D)$ be a valuation overring of dimension 1. Then, V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$ if and only if V is a localization of $\overline{D}$ and its center on $\overline{D}$ is isolated in $\mathrm{Spec}(\overline{D})^{\mathrm{cons}}$.*

Proof. Since $\mathrm{Zar}(D) = \mathrm{Zar}(\overline{D})$, we can suppose without loss of generality that D is integrally closed.

If the two conditions hold, then V is isolated by Proposition 3.1.

Suppose that V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$. Let P be the center of V on D, and suppose that $V \neq D_P$. Since V is also isolated in $\mathrm{Zar}(D_P)^{\mathrm{cons}}$, by Theorem 3.4 there are $x_1, \dots, x_n \in K \setminus D_P$ such that V is the integral closure of $D_P[x_1, \dots, x_n]_M$, where M is a maximal ideal of $D_P[x_1, \dots, x_n]$. However, $\mathfrak{m}_V \cap D_P[x_1, \dots, x_n] = M$, and thus $M \cap D_P = PD_P$, so that $PD_P \cdot D_P[x_1, \dots, x_n] \neq D_P[x_1, \dots, x_n]$; by Proposition 5.1, M cannot have height 1. However, the dimension of the integral closure of $D_P[x_1, \dots, x_n]_M$ is exactly the height of M; hence, this contradicts the fact that V has dimension 1. Thus, $V = D_P$. The fact that P is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$ now follows from Proposition 3.1. □

Corollary 5.3. *Let D be an integral domain, and let $V \in \mathrm{Zar}(D)$ be a minimal valuation overring of D. If $\dim(V) = 1$ and V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$, then the center of V on D has height 1.*

Proof. The claim is a direct consequence of Theorem 5.2. □

Theorem 5.2 does not work when V has dimension 2 or more, as the next example shows.

Example 5.4. Let F be a field, take two independent indeterminates X and Y, and consider $D := F + XF(Y)[[X]]$, i.e., D is the ring of all power series with coefficients in $F(Y)$ such that the 0-degree coefficient belongs to F. Then, D is a one-dimensional local integrally closed domain (its maximal ideal is $XF(Y)[[X]]$), and its valuation overrings are its quotient field, $F(Y)[[X]]$ and the rings in the form $W+XF(Y)[[X]]$, where W belongs to $\mathrm{Zar}(F(Y)|F)\setminus\{F(Y)\}$, i.e., W is either $F[Y]_{(f)}$ for some irreducible polynomial $f \in F[Y]$ or $W = F[Y^{-1}]_{(Y^{-1})}$.

Each of these $W + XF(Y)[[X]]$ is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$, since each W is isolated in $\mathrm{Zar}(F(Y)|F)$ (this follows, for example, by applying Theorem 6.3 below to $F[Y]$ or to $F[Y^{-1}]$). However, since every $W + XF(Y)[[X]]$ has dimension 2, it can't be a localization of $D = \overline{D}$.

6. The Noetherian case

In this section, we want to characterize the isolated points of $\mathrm{Zar}(D)^{\mathrm{cons}}$ when D is a Noetherian domain. If D is integrally closed, this is a straightforward consequence of Theorem 5.2; to extend it to the non-integrally closed case, we need a few lemmas. (Note that the integral closure of a Noetherian domain is not necessarily Noetherian; see e.g. [18, Example 5, page 209].)

Lemma 6.1. *Let D be an integral domain. Let P be a prime ideal of D and let $\Delta \subseteq \mathrm{Spec}(D)$. If $P = \bigcap\{Q \mid Q \in \Delta\}$, then $P \in \mathrm{Cl}^{\mathrm{cons}}(\Delta)$.*

Proof. Let $\Omega = \mathcal{D}(aD) \cap \mathcal{V}(J)$ be a basic subset of $\mathrm{Spec}(D)^{\mathrm{cons}}$ containing P, where $a \in D$ and J is a finitely generated ideal. We claim that $\Omega \cap \Delta \neq \emptyset$.

Indeed, $\Delta \subseteq \mathcal{V}(J)$ since $J \subseteq P$ and $P \subseteq Q$ for every $Q \in \Delta$. Moreover, since $a \notin P$, there must be a $\overline{Q} \in \Delta$ such that $a \notin \overline{Q}$; thus, $\overline{Q} \in \mathcal{D}(aD) \cap \mathcal{V}(J) \cap \Delta = \Omega \cap \Delta$. In particular, $\Omega \cap \Delta \neq \emptyset$ and $P \in \mathrm{Cl}^{\mathrm{cons}}(\Delta)$. □

Lemma 6.2. *Let $A \subseteq B$ be an integral extension, and let $P \in \mathrm{Spec}(A)$, $Q \in \mathrm{Spec}(B)$ be such that $Q \cap A = P$. If $\bigcap\{P' \in \mathrm{Spec}(A) \mid P' \supsetneq P\} = P$, then $\bigcap\{Q' \in \mathrm{Spec}(B) \mid Q' \supsetneq Q\} = Q$.*

Proof. Let $I := \bigcap\{Q' \in \mathrm{Spec}(B) \mid Q' \supsetneq Q\}$, and suppose $I \neq Q$; then, $Q \subsetneq I$ and $\mathcal{V}(I) = \mathcal{V}(Q) \setminus \{Q\}$. Consider the canonical map of spectra $\phi : \mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$: then, ϕ is closed (with respect to the Zariski topology) [2, Chapter V, §2, Remark (2)], and thus $\phi(\mathcal{V}(I))$ is closed in $\mathrm{Spec}(A)$.

By the lying over and the going up theorems, every $P' \supsetneq P$ belongs to $\phi(\mathcal{V}(I))$, while $P \notin \phi(\mathcal{V}(I))$; hence, $\phi(\mathcal{V}(I)) = \mathcal{V}(P) \setminus \{P\}$. However, the condition $\bigcap\{P' \in \mathrm{Spec}(A) \mid P' \supsetneq P\} = P$ shows that $\mathcal{V}(P) \setminus \{P\}$ is not closed (its closure is $\mathcal{V}(P)$), a contradiction. Hence, $I = Q$, as claimed. □

Theorem 6.3. *Let D be a Noetherian domain, and let $V \in \mathrm{Zar}(D)$; let P be the center of V on D. Then, V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$ if and only if $h(P) \leq 1$ and $\mathcal{V}(P)$ is finite.*

Proof. Suppose first that V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$.

If $\dim(V) > 1$, then V is not Noetherian. By Theorem 3.4, V is the integral closure of $D[x_1, \dots, x_n]_M$, for some $x_1, \dots, x_n \in V$ and some maximal ideal M. However, $D[x_1, \dots, x_n]$ is Noetherian, and thus so is $D[x_1, \dots, x_n]_M$; hence, its integral closure is a Krull domain, which can't be a non-Noetherian valuation domain, a contradiction.

If $\dim(V) = 0$, then $V = K$. By Proposition 4.1, D must be a Goldman domain; by [16, Theorem 146], $\mathcal{V}(P)$ is finite.

If $\dim(V) = 1$, then by Theorem 5.2 V is the localization of $\overline{D}$ at a prime ideal of Q of height 1; hence, V is an essential prime ideal of $\overline{D}$ and thus Q is isolated in $\mathrm{Spec}(\overline{D})^{\mathrm{cons}}$ by Proposition 3.1.

Let $P := Q \cap D$. If $\mathcal{V}(P)$ is infinite, then P is the intersection of all the prime ideals properly containing it (since D/P is not a Goldman domain); by Lemma 6.2, the same property holds for Q, and thus by Lemma 6.1, Q is not isolated in $\mathrm{Spec}(D)^{\mathrm{cons}}$. This is a contradiction, and thus $\mathcal{V}(P)$ must be finite.

Conversely, suppose the two conditions hold and let $\mathcal{V}(P) = \{P, Q_1, \dots, Q_n\}$. For each i, let $y_i \in Q_i \setminus P$ and let $x_i := 1/y_i$: then, $A := D[x_1, \dots, x_n]$ is a Noetherian domain such that PA is a maximal ideal of A of height ≤ 1; moreover, since $\mathfrak{m}_V \cap D = P$, each x_i belongs to V, and thus $V \in \mathrm{Zar}(A)$ and $\mathfrak{m}_V \cap A = PA$.

The subspace $\mathrm{Zar}(A) = \mathcal{B}(x_1, \dots, x_n)$ is an open set of $\mathrm{Zar}(D)^{\mathrm{cons}}$: therefore, all isolated points of $\mathrm{Zar}(A)^{\mathrm{cons}}$ are also isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$.

If P has height 0, then $A = K = V$ and thus V is isolated. Suppose that $h(P) = 1$.

Since A is Noetherian, $\{PA\} = \mathcal{V}(PA)$ is an open subset of $\mathrm{Spec}(A)^{\mathrm{cons}}$; hence, $\gamma_A^{-1}(PA)$ is an open subset of $\mathrm{Zar}(A)^{\mathrm{cons}}$, where $\gamma_A : \mathrm{Zar}(A) \longrightarrow \mathrm{Spec}(A)$

is the center map relative to A. However, $\gamma_A^{-1}(PA)$ is the set of valuation overrings of $A_{PA} = D_P$ centered on $(PA)A_{PA} = PD_P$; since P has height 1, D_P has dimension 1, and thus $\gamma_A^{-1}(PA)$ is in bijective correspondence with the maximal ideals of the integral closure B of D_P, which is Noetherian by [16, Theorem 93]. The Jacobson radical of B is nonzero (since it contains P), and thus B has only finitely many maximal ideal; thus, $\gamma_A^{-1}(PA)$ is an open finite set of the Hausdorff space $\mathrm{Zar}(A)^{\mathrm{cons}}$, and so it is discrete. Since $V \in \gamma_A^{-1}(PA)$, we have that V is isolated in $\mathrm{Zar}(A)^{\mathrm{cons}}$ and thus in $\mathrm{Zar}(D)^{\mathrm{cons}}$, as claimed. □

Corollary 6.4. *Let $(D, \mathfrak{m})$ be a Noetherian local domain of dimension at least 3. Then, $\mathrm{Zar}(D)^{\mathrm{cons}}$ is perfect.*

Proof. Suppose V is isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$. By Theorem 6.3, its center P must have height 1 and $\mathcal{V}(P)$ must be finite. However, since P has height 1 and the maximal ideal $\mathfrak{m}$ of D has height at least 3, there is at least one prime ideal between P and $\mathfrak{m}$, and since D is Noetherian there must be infinitely many of them [16, Theorem 144], a contradiction. Hence, no V can be isolated, and $\mathrm{Zar}(D)^{\mathrm{cons}}$ is perfect. □

We now want to show that, when D is countable, there are few possible topological structures for $\mathrm{Zar}(D)^{\mathrm{cons}}$. The one-dimensional case is very easy.

Proposition 6.5. *Let $(D, \mathfrak{m})$ and $(D', \mathfrak{m}')$ be two Noetherian local domains of dimension 1. The following are equivalent:*

(i) $|\mathrm{Max}(\overline{D})| = |\mathrm{Max}(\overline{D'})|$;
(ii) $\mathrm{Zar}(D) \simeq \mathrm{Zar}(D')$;
(iii) $\mathrm{Zar}(D)^{\mathrm{cons}} \simeq \mathrm{Zar}(D')^{\mathrm{cons}}$.

Proof. Since D is Noetherian and one-dimensional, $\overline{D}$ is a principal ideal domain with finitely many maximal ideals; hence, $\mathrm{Zar}(D) = \mathrm{Zar}(\overline{D}) \simeq \mathrm{Spec}(\overline{D})$, and the homeomorphism holds both in the Zariski and in the constructible topology.

Hence, if $|\mathrm{Max}(\overline{D})| = |\mathrm{Max}(\overline{D'})|$ then $\mathrm{Spec}(\overline{D}) \simeq \mathrm{Spec}(\overline{D'})$ and thus $\mathrm{Zar}(D)$ and $\mathrm{Zar}(D')$ are homeomorphic in both the Zariski and the constructible topology. Conversely, if $\mathrm{Zar}(D) \simeq \mathrm{Zar}(D')$ (in any of the two topologies) then in particular they have the same cardinality, which is equal to $|\mathrm{Max}(\overline{D})| + 1 = |\mathrm{Max}(\overline{D'})| + 1$; thus, $|\mathrm{Max}(\overline{D})| = |\mathrm{Max}(\overline{D'})|$. The claim is proved. □

For larger dimension, we need to join the previous theorems with the topological characterization of the Cantor set. We isolate a lemma.

Lemma 6.6. *Let D be a countable domain. Then, $\mathrm{Zar}(D)^{\mathrm{cons}}$ is metrizable.*

Proof. The space $\mathrm{Zar}(D)^{\mathrm{cons}}$ is compact and Hausdorff, hence normal [29, Theorem 17.10] and, in particular, regular. Furthermore, the family of sets $\mathcal{B}(t)$ and $\mathcal{B}(t)^c$ (as t ranges in the quotient field of D) form a subbasis of $\mathrm{Zar}(D)^{\mathrm{cons}}$, and thus $\mathrm{Zar}(D)^{\mathrm{cons}}$ is second countable. By Urysohn's metrization theorem (see e.g. [29, Theorem 23.1]), $\mathrm{Zar}(D)^{\mathrm{cons}}$ is metrizable. □

Proposition 6.7. *Let* $(D, \mathfrak{m})$ *and* $(D', \mathfrak{m}')$ *be two countable Noetherian local domains of dimension at least* 3*. Then,* $\mathrm{Zar}(D)^{\mathrm{cons}} \simeq \mathrm{Zar}(D')^{\mathrm{cons}}$.

Proof. Both $\mathrm{Zar}(D)^{\mathrm{cons}}$ and $\mathrm{Zar}(D')^{\mathrm{cons}}$ are Boolean spaces, hence totally disconnected and compact; they are also perfect by Corollary 6.4 and metrizable by Lemma 6.6.

By [29, Theorem 30.3], any two spaces with these properties are homeomorphic; hence, $\mathrm{Zar}(D)^{\mathrm{cons}} \simeq \mathrm{Zar}(D')^{\mathrm{cons}}$. □

To study the case of dimension 2, we need two further lemmas.

Lemma 6.8. *Let* $(D, \mathfrak{m})$ *be a local Noetherian domain with* $\dim(D) > 1$*. If* D *is countable, then* D *has exactly countably many prime ideals of height* 1.

Proof. By [16, Theorem 144], there are infinitely many prime ideals between (0) and $\mathfrak{m}$, and thus D has infinitely many prime ideals of height 1.

Moreover, every prime ideal is generated by a finite set, and thus the number of prime ideals of height 1 is at most equal to the number of finite subsets of D. Since D is countable, so is the set of its finite subsets; the claim is proved. □

Lemma 6.9. *Let* $(D, \mathfrak{m})$ *be a local Noetherian domain of dimension* 2 *with quotient field* K*, and let* X *be the set of isolated points of* $\mathrm{Zar}(D)^{\mathrm{cons}}$*. Then:*

- *(a) a valuation overring of* D *belongs to* X *if and only if its center has height* 1*;*
- *(b)* X *is nonempty and compact, with respect to the Zariski topology;*
- *(c) if* D *is countable, then* X *is countable;*
- *(d)* $\mathrm{Cl}^{\mathrm{cons}}(X) = X \cup \{K\}$*;*
- *(e) the only isolated point of* $(\mathrm{Zar}(D) \setminus X)^{\mathrm{cons}}$ *is* K*;*
- *(f)* $\mathrm{Zar}(D) \setminus (X \cup \{K\})$ *is closed and perfect, with respect to the constructible topology.*

Proof. (a) Let $V \in \mathrm{Zar}(D)$. If V is isolated, then its center has height at most 1 by Theorem 6.3, but the height can't be 0 since $\mathcal{V}((0))$ is infinite. Conversely, if $P := \mathfrak{m}_V \cap D$ has height 1, then $\mathcal{V}(P) = \{P, \mathfrak{m}\}$ is finite, and thus $V \in X$, by Theorem 6.3.

(b) Let X_1 be the set of all height 1 prime ideals of D: by the previous point, $X = \gamma^{-1}(X_1)$. Since γ is surjective, and X_1 is nonempty, also X is nonempty. Furthermore, since D is a Noetherian ring, $\mathrm{Spec}(D)$ is a Noetherian space with respect to the Zariski topology (i.e., all its subsets are compact; see [4, Theorem 12.4.3] or [1, Chapter 6, Exercises 5–8]). Since γ is a spectral closed map, it is proper, and thus the counterimage of any compact subset of $\mathrm{Spec}(D)$ is compact; therefore, $X = \gamma^{-1}(X_1)$ is compact with respect to the Zariski topology, as claimed.

(c) By Lemma 6.8, X_1 is countable; furthermore, $\gamma^{-1}(P)$ is finite for every $P \in X_1$, since it is in bijective correspondence with the set of maximal ideals of the integral closure of D_P. Since $X = \gamma^{-1}(X_1)$, it follows that X is countable.

(d) Since X is compact, the set $X^{\mathrm{gen}} = \{W \in \mathrm{Zar}(D) \mid W \supseteq V \text{ for some } V \in X\}$ is closed in the inverse topology, and thus in the constructible topology; since every element of X is a one-dimensional valuation ring, furthermore, $X^{\mathrm{gen}} = X \cup \{K\}$. Hence, $\mathrm{Cl}^{\mathrm{cons}}(X) \subseteq X \cup \{K\}$.

If they are not equal, then $\mathrm{Cl}^{\mathrm{cons}}(X) = X$. However, X is infinite (since X_1 is infinite, by Lemma 6.8) and discrete (by definition, all its points are isolated) and thus it is not compact with respect to the constructible topology; this is a contradiction, since a closed set of a compact set is compact. Thus, $\mathrm{Cl}^{\mathrm{cons}}(X) = X \cup \{K\}$, as claimed.

(e) The set $\mathrm{Zar}(D) \setminus (X \cup \{K\})$ is open, with respect to the constructible topology (by part (d)), and its elements are not isolated in $\mathrm{Zar}(D)^{\mathrm{cons}}$; therefore, none of its elements can be isolated in $(\mathrm{Zar}(D) \setminus X)^{\mathrm{cons}}$. On the other hand, let $x \in \mathfrak{m}$, $x \neq 0$: then, $D[x^{-1}]$ is a Noetherian domain of dimension 1, and its maximal ideals are extensions of prime ideals of D of height 1. Therefore, if $V \in \mathrm{Zar}(D[x^{-1}])$ has dimension 1 then the center of V on D has height 1, and thus it is an isolated point of $\mathrm{Zar}(D)$, i.e., $\mathcal{B}(x^{-1}) = \mathrm{Zar}(D[x^{-1}]) \subseteq X \cup \{K\}$, and $\mathcal{B}(x^{-1}) \cap (\mathrm{Zar}(D) \setminus X) = \{K\}$. Since $\mathcal{B}(x^{-1})$ is open in $\mathrm{Zar}(D)^{\mathrm{cons}}$, it follows that K is isolated in $(\mathrm{Zar}(D) \setminus X)^{\mathrm{cons}}$.

(f) is a direct consequence of (e). □

Note that the set X of the previous proposition is *not* compact with respect to the constructible topology, as it is discrete and infinite.

Proposition 6.10. *Let* $(D, \mathfrak{m})$ *and* $(D', \mathfrak{m}')$ *be two countable Noetherian local domains of dimension* 2*. Then,* $\mathrm{Zar}(D)^{\mathrm{cons}} \simeq \mathrm{Zar}(D')^{\mathrm{cons}}$.

Proof. Denote by K, K' the quotient fields of D and D', respectively.

Let X be the set of isolated points of $\mathrm{Zar}(D)^{\mathrm{cons}}$, and let $C := \mathrm{Zar}(D) \setminus (X \cup \{K\})$: then, C is closed in $\mathrm{Zar}(D)^{\mathrm{cons}}$. Define in the same way X' and C' inside $\mathrm{Zar}(D')$; then, C' is closed.

As in the proof of Proposition 6.7, by Lemma 6.9(f) C^{cons} and $(C')^{\mathrm{cons}}$ are totally disconnected, perfect, compact and metrizable (with respect to the constructible topology), and thus they are homeomorphic. Let $\phi_C : C^{\mathrm{cons}} \longrightarrow (C')^{\mathrm{cons}}$ be a homeomorphism.

The set X is discrete and countable, and the unique nonisolated point of $X \cup \{K\}$ is K; since the same holds for X' and K', any bijection $X \longrightarrow X'$ extends to a homeomorphism $\phi_X : (X \cup \{K\})^{\mathrm{cons}} \longrightarrow (X' \cup \{K'\})^{\mathrm{cons}}$ by setting $\phi_X(K) = K'$. Define

$$\phi\colon \mathrm{Zar}(D)^{\mathrm{cons}} \longrightarrow \mathrm{Zar}(D')^{\mathrm{cons}},$$

$$V \longmapsto \begin{cases} \phi_C(V) & \text{if } V \in C, \\ \phi_X(V) & \text{if } V \in X \cup \{K\}. \end{cases}$$

By construction, ϕ is bijective, and ϕ is a homeomorphism when restricted to C and to $X \cup \{K\}$. Since these two sets are closed, by [29, Theorem 7.6] ϕ is a homeomorphism. In particular, $\mathrm{Zar}(D)^{\mathrm{cons}} \simeq \mathrm{Zar}(D')^{\mathrm{cons}}$. □

We summarize the previous results in the following theorem.

Theorem 6.11. *Let $(D, \mathfrak{m})$ and $(D', \mathfrak{m}')$ be two countable Noetherian local domains. Then,* $\mathrm{Zar}(D)^{\mathrm{cons}} \simeq \mathrm{Zar}(D')^{\mathrm{cons}}$ *if and only if one of the following conditions hold:*

(a) $\dim(D) = \dim(D') = 1$ *and* $|\operatorname{Max}(\overline{D})| = |\operatorname{Max}(\overline{D'})|$*;*
(b) $\dim(D) = \dim(D') = 2$*;*
(c) $\dim(D) \geq 3$ *and* $\dim(D') \geq 3$.

Proof. If D and D' satisfy one of the conditions, then $\mathrm{Zar}(D)^{\mathrm{cons}} \simeq \mathrm{Zar}(D')^{\mathrm{cons}}$ by, respectively, Proposition 6.5, Proposition 6.10 and Proposition 6.7.

Suppose now that $\mathrm{Zar}(D)^{\mathrm{cons}} \simeq \mathrm{Zar}(D')^{\mathrm{cons}}$.

If $\dim(D) = 1$, then $\mathrm{Zar}(D)$ is finite, and thus so must be $\mathrm{Zar}(D')$; hence, $\dim(D') = 1$, and $|\operatorname{Max}(\overline{D})| = |\operatorname{Max}(\overline{D'})|$ by Proposition 6.5.

Suppose $\dim(D), \dim(D') \geq 2$. By Corollary 6.4 and Lemma 6.8, $\mathrm{Zar}(D)^{\mathrm{cons}}$ has isolated points if and only if $\dim(D) = 2$; therefore, $\dim(D) = 2$ if and only if $\dim(D') = 2$, and $\dim(D) \geq 3$ if and only if $\dim(D') \geq 3$. The claim is proved. □

7. When D is a field

In this section we analyze the isolated points Zariski space $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ when $D = K$ is a field. Note that, if L is algebraic over K, then $\mathrm{Zar}(L|K)$ is just a point (L itself); thus, the only interesting case is when $\operatorname{trdeg}(L/K) \geq 1$.

We start by connecting the isolated points of $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ and of $\mathrm{Zar}(L'|D)^{\mathrm{cons}}$, where $L' \subseteq L$ is an algebraic extension.

Proposition 7.1. *Let V be a valuation domain, and $L' \subseteq L$ be an algebraic extension such that $V \subseteq L'$. Let $\rho : \mathrm{Zar}(L|V) \longrightarrow \mathrm{Zar}(L'|V)$ be the restriction map, and let $\mathcal{X} \subseteq \mathrm{Zar}(L|V)$ be a subset such that $\rho^{-1}(\rho(\mathcal{X})) = \mathcal{X}$. Then, the following hold.*

(a) If W is isolated in $\mathcal{X}^{\mathrm{cons}}$, then $\rho(W)$ is isolated in $\rho(\mathcal{X})^{\mathrm{cons}}$.
(b) If $\rho(\mathcal{X})$ is perfect and $|\rho(\mathcal{X})| > 1$, then $\mathcal{X}$ is perfect.

In particular, the previous statements apply to $\mathcal{X} = \mathrm{Zar}(L|V)$ and $\mathcal{X} = \mathcal{E}(L|V)$.

Proof. (a) Let W be an isolated point of $\mathcal{X}^{\mathrm{cons}}$, and let $W' := W \cap L' = \rho(W)$.

Suppose first that L is finite and normal over L'. Let G be the group of L'-automorphisms of L: then, every $\sigma \in G$ is continuous when seen as a map from $\mathrm{Zar}(L|V)^{\mathrm{cons}}$ to itself. Moreover, $\rho(\sigma(Z)) = \rho(Z)$ for every $Z \in \mathrm{Zar}(L|V)$, and thus σ restricts to a self-homeomorphism of $\mathcal{X}$.

Since G acts transitively on $\rho^{-1}(W')$ (see e.g. [12, Corollary 20.2]) and $W \in \rho^{-1}(W')$ is isolated, all points of $\rho^{-1}(W')$ are isolated in $\mathcal{X}$; hence, $\rho^{-1}(W')$ is open in $\mathcal{X}^{\mathrm{cons}}$. Since $\rho : \mathrm{Zar}(L|V) \longrightarrow \mathrm{Zar}(L'|V)$ is a closed map (with respect to the constructible topology), it is also closed when seen as a map $\mathcal{X} \longrightarrow \rho(\mathcal{X})$; therefore, $\rho(\mathcal{X} \setminus \rho^{-1}(W')) = \rho(\mathcal{X}) \setminus \{W'\}$ is closed in $\rho(\mathcal{X})$, with respect to the constructible topology, and thus W' is an isolated point of $\rho(\mathcal{X})^{\mathrm{cons}}$, as claimed.

Suppose now that L is finite over L', and let F be the normal closure of L'. Let $\rho_0 : \mathrm{Zar}(F|V) \longrightarrow \mathrm{Zar}(L|V)$ be the restriction map. Since W is isolated in $\mathcal{X}$, the set $\rho_0^{-1}(W)$ is open in $\rho_0^{-1}(\mathcal{X})^{\text{cons}}$; moreover, $\rho_0^{-1}(W)$ is finite since $[F : L] < \infty$. Therefore, $\rho_0^{-1}(W)$ is a discrete subspace of $\rho_0^{-1}(\mathcal{X})^{\text{cons}}$, and in particular each $Z \in \rho_0^{-1}(W)$ is isolated. Applying the previous part of the proof to the extension $L' \subseteq F$ and to any such Z, we obtain that $Z \cap L' = W \cap L' = \rho(W)$ is isolated, as claimed.

Suppose now that $L' \subseteq L$ is arbitrary. Since W is isolated in $\mathcal{X}$, there are $x_1, \dots, x_n, y_1, \dots, y_m \in L$ such that $\{W\} = \mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c \cap \mathcal{X}$. Let $F := L'(x_1, \dots, x_n, y_1, \dots, y_m)$: then, $W \cap F$ is isolated in $\{Z \cap F \mid Z \in \mathcal{X}\}$. Since $[F : L'] < \infty$, we can apply the previous part of the proof, obtaining that $W \cap F \cap L' = W \cap L' = \rho(W)$ is isolated in $\rho(\mathcal{X})^{\text{cons}}$, as claimed.

(b) Suppose that $\mathcal{X}$ is not perfect: then, there is a $W \in \mathcal{X}$ that is isolated. By the previous part of the proof, it would follow that $W \cap L'$ is isolated in $\rho(X)^{\text{cons}}$. Since $\rho(\mathcal{X})$ has more than one point, this is impossible, and so $\mathcal{X}$ is perfect.

The "in particular" statement follows from the fact that $\text{Zar}(L|V)$ and $\mathcal{E}(L|V)$ satisfy the hypothesis on $\mathcal{X}$. ☐

Corollary 7.2. *Let V be a valuation domain and $L' \subseteq L$ be an algebraic extension; suppose that $V \subseteq L'$ and that L' is transcendental over the quotient field of V. If $\text{Zar}(L'|V)^{\text{cons}}$ (respectively, $\mathcal{E}(L'|V)^{\text{cons}}$) is perfect, then $\text{Zar}(L|V)^{\text{cons}}$ (resp., $\mathcal{E}(L|V)^{\text{cons}}$) is perfect.*

Proof. It is enough to apply Proposition 7.1(b) to $\mathcal{X} = \text{Zar}(L|V)$ or $\mathcal{X} = \mathcal{E}(L|V)$, using the hypothesis that L' is transcendental over the quotient field of V to guarantee that $|\text{Zar}(L'|V)| > 1$ and $|\mathcal{E}(L'|V)| > 1$. ☐

The following result completely settles the problem of finding the isolated points when $\text{trdeg}(L/K) \geq 2$, generalizing [3, Theorem 4.45] and solving the authors' Conjecture A (in an even more general formulation). Note that the first case in the proof is exactly [3, Theorem 4.45], but we give a new proof of it using Theorem 6.3.

Theorem 7.3. *Let $K \subseteq L$ be a field extension with $\text{trdeg}(L/K) \geq 2$. Then, $\text{Zar}(L|K)^{\text{cons}}$ is perfect.*

Proof. Suppose first that $L = K(x_1, \dots, x_n)$ is a finitely generated purely transcendental extension of K, with transcendence basis $x_1, \dots, x_n$. Suppose there exists an isolated point W of $\text{Zar}(L|K)^{\text{cons}}$. By Proposition 4.1, $W \neq L$.

For each i, at least one of x_i and x_i^{-1} belongs to W; let it be t_i. Then, $W \in \text{Zar}(K[t_1, \dots, t_n])$, and so W is isolated in $\text{Zar}(K[t_1, \dots, t_n])^{\text{cons}}$. Let P be the center of W on $K[t_1, \dots, t_n]$; since $K[t_1, \dots, t_n]$ is Noetherian, by Theorem 6.3 P has height 1 and $\mathcal{V}(P)$ is finite.

Since $K[t_1, \dots, t_n]$ is isomorphic to a polynomial ring, every maximal ideal of $K[t_1, \dots, t_n]$ has height $n > 1$ [16, Section 3.2, Exercise 3], and thus P is not maximal. However, $K[t_1, \dots, t_n]$ is an Hilbert ring, and thus every non-maximal prime ideal is the intersection of the maximal ideals containing it [16, Theorem

147]; in particular, this happens for P, and thus $\mathcal{V}(P)$ must be infinite. This is a contradiction, and so $\mathrm{Zar}(L|K)^{\mathrm{cons}}$ is perfect.

Suppose now that L has finite transcendence degree over K, let $x_1, \dots, x_n$ be a transcendence basis of L and let $L' := K(x_1, \dots, x_n)$. By the previous part of the proof, $\mathrm{Zar}(L'|K)^{\mathrm{cons}}$ is perfect; since $L' \subseteq L$ is algebraic, by Corollary 7.2 also $\mathrm{Zar}(L|K)^{\mathrm{cons}}$ is perfect.

Take now any extension L of K, and suppose that W is an isolated point of $\mathrm{Zar}(L|K)^{\mathrm{cons}}$. Then, there are $x_1, \dots, x_n, y_1, \dots, y_m \in L$ such that $\{W\} = \mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c$. Take two elements $a, b \in L$ that are algebraically independent over K, and let $L' := K(a, b, x_1, \dots, x_n, y_1, \dots, y_m)$: then, $2 \leq \mathrm{trdeg}(L'/K) < \infty$. Set $V := W \cap L'$: then, $\{V\} = \mathcal{B}^{L'}(x_1, \dots, x_n) \cap \mathcal{B}^{L'}(y_1)^c \cap \cdots \cap \mathcal{B}^{L'}(y_m)^c$, and thus V is isolated in $\mathrm{Zar}(L'|V)^{\mathrm{cons}}$. However, by the previous part of the proof, $\mathrm{Zar}(L'|V)^{\mathrm{cons}}$ is perfect, a contradiction. Hence, $\mathrm{Zar}(L|K)^{\mathrm{cons}}$ is perfect. □

When the transcendence degree of L over K is 1, the picture is very different, because it may even happen that all elements of $\mathrm{Zar}(L|K)^{\mathrm{cons}}$ (except L itself) are isolated. Compare the next results with [26, Corollary 5.5(a)] and [28, Proposition 4.2].

Proposition 7.4. *Let K be a field. Then all points of $\mathrm{Zar}(K(X)|K)$, except $K(X)$, are isolated with respect to the constructible topology.*

Proof. The points of $\mathrm{Zar}(K(X)|K)$ are $K(X)$, $K[X^{-1}]_{(X^{-1})}$, and the rings $K[X]_{(f(X))}$, where $f(X)$ is an irreducible polynomial of $K[X]$. The first one is not isolated by Proposition 4.1; on the other hand, $\{K[X]_{(f(X))}\} = \mathcal{B}(f(X)^{-1})^c$ and $\{K[X^{-1}]_{(X^{-1})}\} = \mathcal{B}(X)^c$, and thus these domains are isolated, as claimed. □

Lemma 7.5. *Let D be an integral domain with quotient field K, and let $L' \subseteq L$ be two extensions of K. Let $V \in \mathrm{Zar}(L'|D)$. If V is isolated in $\mathrm{Zar}(L'|D)^{\mathrm{cons}}$ and $\mathcal{E}(L|V)$ is finite, then every $W \in \mathcal{E}(L|V)$ is isolated in $\mathrm{Zar}(L|D)^{\mathrm{cons}}$.*

Proof. Let $\rho : \mathrm{Zar}(L|D) \longrightarrow \mathrm{Zar}(L'|D)$ be the restriction map. Then, $\mathcal{E}(L|V) = \rho^{-1}(V)$ is open in $\mathrm{Zar}(L'|D)^{\mathrm{cons}}$ since V is isolated. Moreover, $\mathcal{E}(L|V)$ is finite by hypothesis, and, since the constructible topology is Hausdorff, all its points are isolated in $\mathrm{Zar}(L|D)^{\mathrm{cons}}$. □

Proposition 7.6. *Let K be a field and let L be an extension of K with $\mathrm{trdeg}(L/K) = 1$. Let $V \in \mathrm{Zar}(L|K)$, $V \neq L$. Then the following are equivalent:*

(i) V is isolated in $\mathrm{Zar}(L|K)^{\mathrm{cons}}$;
(ii) there exists a finitely generated extension L' of K such that $L' \subseteq L$ and $\mathcal{E}(L|V \cap L') = \{V\}$;
(iii) there exists a finitely generated extension L' of K such that $L' \subseteq L$ and $\mathcal{E}(L|V \cap L')$ is finite.

Proof. (i) $\Longrightarrow$ (ii) Since V is isolated, we have

$$\{V\} = \Omega := \mathcal{B}(x_1, \dots, x_n) \cap \mathcal{B}(y_1)^c \cap \cdots \cap \mathcal{B}(y_m)^c$$

for some $x_1, \dots, x_n, y_1, \dots, y_m \in L$. Let $L' = K(x_1, \dots, x_n, y_1, \dots, y_m)$. Then every extension of $V \cap L'$ to L belongs to Ω, and thus it is equal to V. Hence, L' is the required field.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i) Since $\mathcal{E}(L|V \cap L')$ is finite, $L' \subseteq L$ must be algebraic and so $K \subseteq L'$ is transcendental; take any $X \in L'$ that is transcendental over K. Since $K \subseteq L'$ is finitely generated, $K(X) \subseteq L'$ must be a finite extension.

Since $V \neq L$, we have $V \cap K(X) \neq K(X)$; by Proposition 7.4, $V \cap K(X)$ is isolated in $\mathrm{Zar}(K(X)|K)$. Moreover, since $K(X) \subseteq L'$ is a finite extension, $\mathcal{E}(L'|V \cap K(X))$ is finite; by Lemma 7.5, all points of $\mathcal{E}(L'|V \cap K(X))$ are isolated in $\mathrm{Zar}(L'|K)^{\mathrm{cons}}$, and in particular this happens for $V \cap L'$. We can now apply Lemma 7.5 to $V \cap L'$ and L, obtaining that all elements of $\mathcal{E}(L|V \cap L')$ (in particular, V) are isolated in $\mathrm{Zar}(L|K)^{\mathrm{cons}}$. □

Proposition 7.7. *Let K be a field and let L be an extension of K with* $\mathrm{trdeg}(L/K) = 1$. *Let $\mathcal{X} := \mathrm{Zar}(L|K) \setminus \{L\}$. Then, the following are equivalent:*

(i) all points of $\mathcal{X}$ are isolated in $\mathrm{Zar}(L|K)^{\mathrm{cons}}$*;*

(ii) for every $X \in L$, transcendental over K, the set $\mathcal{E}(L|V)$ is finite for every $V \in \mathrm{Zar}(K(X)|K)$;

(iii) there is an $X \in L$, transcendental over K, such that the set $\mathcal{E}(L|V)$ is finite for every $V \in \mathrm{Zar}(K(X)|K)$.

Proof. (i) $\Longrightarrow$ (ii) Take any $X \in L$ that is transcendental over K, and let $V \in \mathrm{Zar}(K(X)|K)$. The space $\mathcal{E}(L|V)$ is closed in $\mathrm{Zar}(L|V)^{\mathrm{cons}}$, and thus it is compact. Since all its points are isolated, it is also discrete; hence, $\mathcal{E}(L|V)$ is finite.

(ii) $\Longrightarrow$ (iii) is obvious.

(iii) $\Longrightarrow$ (i) Apply Proposition 7.6, (iii) $\Longrightarrow$ (i) with $L' = K(X)$ to each $V \in \mathcal{X}$. □

Corollary 7.8. *Let K be a field and let L be a finitely generated extension of K such that* $\mathrm{trdeg}(L/K) = 1$. *Then, all points of* $\mathrm{Zar}(L|K) \setminus \{L\}$ *are isolated in* $\mathrm{Zar}(L|K)^{\mathrm{cons}}$.

Proof. It is enough to apply Proposition 7.7. □

Remark 7.9. Let $K \subseteq L$ be a transcendental extension of degree 1, and let $V \in \mathrm{Zar}(L|K)$. Let $X \in L$ be transcendental over K. By Proposition 7.6, if $\mathcal{E}(L|V \cap K(X))$ is finite, then V is isolated in $\mathrm{Zar}(L|K)^{\mathrm{cons}}$; however, unlike in Proposition 7.7, the converse does not hold, i.e., $\mathcal{E}(L|V \cap K(X))$ may be infinite even if V is isolated.

For example, let $W = K[X]_{(X)}$ (or more generally, we can take any $W \in \mathrm{Zar}(K(X)|K)$, $W \neq K(X)$). Since W is a discrete valuation ring, using [17] (see also [13, Section 3]), it is possible to construct a chain $K(X) \subset F_0 \subset F_1 \subset \cdots$ of extensions of $K(X)$ such that:

- the extensions $K(X) \subset F_0$ and $F_i \subset F_{i+1}$ are finite, for each $i > 0$;
- W has two extensions to F_0, say W_1 and W_2;
- W_1 has only one extension to F_i, for each $i > 0$;

- if W' is an extension of W_2 to F_i, then W' has more than one extension to F_{i+1}.

Let $L := \bigcup_{i\geq 0} F_i$. Then, W_1 has a unique extension V to L, while W_2 has infinitely many extensions; in particular, the set $\mathcal{X}$ of extensions of W to L is infinite. Let $y \in W_1 \setminus W_2$: then, $\mathcal{B}(y) \cap \mathcal{X} = \{V\}$, and thus $\mathcal{B}(y) \cap \mathcal{B}(X^{-1})^c = \{V\}$. Hence, V is isolated in $\mathrm{Zar}(L|K)^{\mathrm{cons}}$, despite $V \cap K(X) = W$ having infinitely many extensions to L.

The reason why the proof of Proposition 7.7 fails in this context is that we are not requiring the *other* extensions of W to L to be isolated.

8. Extensions of valuations

In this section, we extend the results of the previous section from the case where $D = K$ is a field to the case where $D = V$ is a valuation domain. In particular, we want to study the set $\mathcal{E}(L|V)$ of extensions of V to L.

The most important case is when $L = K(X)$ is the field of rational functions. If V is a valuation domain with quotient field K and $s \in K$, we set

$$V_s := \{\phi \in K(X) \mid \phi(s) \in V\}.$$

Then, V_s is an extension of V to $K(X)$, and it is possible to analyze quite thoroughly its algebraic properties (see for example [22, Proposition 2.2] for a description when V has dimension 1).

The following lemma is a partial generalization of [22, Theorem 3.2], of which we follow the proof.

Lemma 8.1. *Let V be a valuation domain with quotient field K, and let U be an extension of V to the algebraic closure $\overline{K}$. Let $s, t \in \overline{K}$. Then, $U_s \cap K(X) = U_t \cap K(X)$ if and only if s and t are conjugated over K.*

Proof. If s, t are conjugated, there is a K-automorphism σ of $\overline{K}$ sending s to t. Setting $\widetilde{\sigma}(\sum_i a_i X^i) := \sum_i \sigma(a_i) X^i$, we can extend σ to a $K(X)$-automorphism $\widetilde{\sigma}$ of $\overline{K}(X)$ such that $\widetilde{\sigma}(\phi)(t) = \sigma(\phi(s))$ for every $\phi \in \overline{K}(X)$; in particular, if $\phi \in K(X)$ then $\widetilde{\sigma}(\phi) = \phi$ and thus $\phi(s) \in V$ if and only if $\phi(t) \in V$, i.e., $\phi \in U_s \cap K(X)$ if and only if $\phi \in U_t \cap K(X)$. Therefore, $U_s \cap K(X) = U_t \cap K(X)$.

Conversely, suppose that s and t are not conjugate, and let $p(X)$ be the minimal polynomial of s over K: then, $p(t) \neq 0$, and thus there is a $c \in K$, $c \neq 0$, such that $v(c) > u(p(t))$ (where v and u are, respectively, the valuations with respect to V and U and $u|_K = v$). Then, $q(X) := \frac{p(X)}{c} \in K(X)$ belongs to U_s (since $q(s) = 0 \in V$) but not to U_t (since $u(q(t)) = u(p(t)) - v(c) < 0$). Hence, $U_s \cap K(X) \neq U_t \cap K(X)$, as claimed. □

Theorem 8.2. *If V is a valuation domain that is not a field, then $\mathcal{E}(K(X)|V)^{\mathrm{cons}}$ is perfect.*

Proof. Suppose first that K is algebraically closed. By [23, Theorem 7.2], for all extensions W of V to $K(X)$ there is a sequence $E = \{s_\nu\}_{\nu\in\Lambda}$ (where Λ is a well-ordered set without maximum) such that

$$W = V_E = \{\phi \in K(X) \mid \phi(s_\nu) \in V \text{ for all large } \nu\}$$

and $W \neq V_{s_\nu}$ for every ν. In particular, the elements $\phi(s_\nu)$ are either eventually in V or eventually out of V (by [23, Proposition 3.2]; see also the proof of Theorem 3.4 therein). Take $\psi \in K(X)$: then, if $W \in \mathcal{B}(\psi)$ then it must be $\psi(s_\nu) \in V$ eventually, and thus $\mathcal{B}(\psi)$ contains V_{s_ν} for all large ν; on the other hand, if $W \in \mathcal{B}(\psi)^c$ then $\psi(s_\nu) \notin V$ eventually, and thus $\mathcal{B}(\psi)^c$ contains V_{s_ν} for all large ν.

Now let

$$\Omega := \mathcal{B}(\psi_1, \dots, \psi_n) \cap \mathcal{B}(\phi_1)^c \cap \cdots \cap \mathcal{B}(\phi_m)^c \cap \mathcal{E}(K(X)|V)$$

be a basic open set of $\mathcal{E}(K(X)|V)^{\mathrm{cons}}$ containing W. For every i, there is an index N_i such that $\psi_i(s_\nu) \in V$ for all $\nu \geq N_i$; likewise, for every j there is a M_j such that $\phi_j(s_\nu) \notin V$ for all $\nu \geq M_j$. Therefore, for every

$$\nu \geq \max\{N_1, \dots, N_n, M_1, \dots, M_m\},$$

we have $V_{s_\nu} \in \Omega$. Hence W belongs to the closure of $\{V_{s_\nu}\}_{\nu \in \Lambda} \subseteq \mathcal{E}(K(X)|V)$, with respect to the constructible topology. It follows that W is not isolated in $\mathcal{E}(K(X)|V)^{\mathrm{cons}}$ and, since W was arbitrary, $\mathcal{E}(K(X)|V)^{\mathrm{cons}}$ is perfect.

Suppose now that K is any field. Let $W \in \mathcal{E}(K(X)|V)$, and suppose that W is isolated in $\mathcal{E}(K(X)|V)^{\mathrm{cons}}$. Let $\rho : \mathcal{E}(\overline{K}(X)|V) \longrightarrow \mathcal{E}(K(X)|V)$ be the restriction map. Since W is isolated and ρ is continuous, $\rho^{-1}(W)$ is open. Let $W' \in \rho^{-1}(W)$ and let $U := W' \cap \overline{K}$: then, U is an extension of V to $\overline{K}$.

By the previous part of the proof, for every open neighborhood Ω of W' there is an $s \in \overline{K}$ such that $U_s \neq W'$ and $U_s \in \Omega$; since $\rho^{-1}(W)$ is open, it follows that for every such Ω there is a $U_s \in \rho^{-1}(W)$ with these properties. Therefore, the set $\Delta := \{U_s \in \rho^{-1}(W) \mid s \in \overline{K}\}$ is dense in $\rho^{-1}(W)$. Since $U_s \cap K(X) = U_t \cap K(X) = W$ for every $U_s, U_t \in \Delta$, by Lemma 8.1 Δ is finite; since $\mathcal{E}(K(X)|V)^{\mathrm{cons}}$ is Hausdorff, it follows that $\Delta = \rho^{-1}(W)$, and in particular $\rho^{-1}(W)$ is finite. Hence, all its points are isolated. However, this contradicts the fact that $\mathcal{E}(\overline{K}(X)|V)^{\mathrm{cons}}$ is perfect; thus, also $\mathcal{E}(K(X)|V)^{\mathrm{cons}}$ must be perfect. □

The theorem above allows to determine the isolated points of $\mathrm{Zar}(K(X)|D)^{\mathrm{cons}}$ for every integral domain D.

Proposition 8.3. *Let D be an integral domain that is not a field, and let J be the intersection of the nonzero prime ideals of D.*

- *(i) If $J = (0)$, then $\mathrm{Zar}(K(X)|D)^{\mathrm{cons}}$ is perfect.*
- *(ii) If $J \neq (0)$, then the only isolated points of $\mathrm{Zar}(K(X)|D)^{\mathrm{cons}}$ are $K[X]_{(f(X))}$ (where $f(X)$ is an irreducible polynomial of $K[X]$) and $K[X]_{(X^{-1})}$.*

Proof. Let $W \in \mathrm{Zar}(K(X)|D)$. If $W \cap K \neq K$, then $\mathcal{E}(K(X)|W \cap K)$ is perfect (when endowed with the constructible topology) by Theorem 8.2. Since W belongs to this set, it is not isolated in $\mathrm{Zar}(K(X)|D)^{\mathrm{cons}}$.

Suppose that $W \cap K = K$. If $W = K(X)$, then W is not isolated by Proposition 4.1, since $K(X)$ is not algebraic over K. Thus let $W \neq K(X)$.

Suppose that $J = (0)$, and suppose that W is isolated in $\mathrm{Zar}(K(X)|D)^{\mathrm{cons}}$. Since $K \subseteq W$, we have $\mathfrak{m}_W \cap D = (0)$; by Lemma 4.2, the quotient map of W onto its residue field induces a homeomorphism between the spaces $\Delta := \{Z \in \mathrm{Zar}(K(X)|D) \mid Z \subseteq W\}$ and $\mathrm{Zar}(W/\mathfrak{m}_W|D)$, where W is sent to $W/\mathfrak{m}_W$. Since W is isolated in $\mathrm{Zar}(K(X)|D)^{\mathrm{cons}}$, it is also isolated in Δ^{cons}, and thus $W/\mathfrak{m}_W$ must be an isolated point of $\mathrm{Zar}(W/\mathfrak{m}_W|D)^{\mathrm{cons}}$. By Proposition 4.1, D must be a Goldman domain, against the hypothesis $J = (0)$. Therefore, W is not isolated and $\mathrm{Zar}(K(X)|D)^{\mathrm{cons}}$ is perfect.

Suppose now that $J \neq (0)$, and let $j \in J$, $j \neq 0$. Then, $D[j^{-1}] = K$, and thus $\mathcal{B}(j^{-1}) = \mathcal{E}(K(X)|K) = \mathrm{Zar}(K(X)|K)$ is a clopen subset of $\mathrm{Zar}(K(X)|D)^{\mathrm{cons}}$; in particular, $W \in \mathcal{E}(K(X)|K)$ is isolated in $\mathrm{Zar}(K(X)|D)^{\mathrm{cons}}$ if and only if it is isolated in $\mathrm{Zar}(K(X)|K)^{\mathrm{cons}}$. The claim now follows from Proposition 7.4. □

To conclude the paper, we extend Theorem 7.3 to valuation domains.

Theorem 8.4. *Let V be a valuation domain with quotient field K, and let L be a field extension of K such that* $\mathrm{trdeg}(L/K) \geq 2$*. Then, $\mathcal{E}(L|V)^{\mathrm{cons}}$ and $\mathrm{Zar}(L|V)^{\mathrm{cons}}$ are perfect.*

Proof. We first show that $\mathcal{E}(L|V)^{\mathrm{cons}}$ is perfect: suppose that is not, and let W be an isolated point.

Suppose that $L = K(x, z_2, \ldots)$ is purely transcendental over K, where $x, z_2, \ldots$, is a transcendence basis. Take an $m \in \mathfrak{m}_V \subseteq \mathfrak{m}_W$: then, at least one of mx and x^{-1} belongs to $\mathfrak{m}_W$. Let z_1 be that element. Then, $z_1, z_2, \ldots$ is also a transcendence basis of L.

Let $L' := K(z_1, z_3, \ldots,)$ be the extension of K obtained adjoining all the element of this basis except z_2. Then, $z_1^{-1} \in L' \setminus W$, and thus $W \cap L' \neq L'$; since, by construction, $L \simeq L'(X)$, by Theorem 8.2 $\mathcal{E}(L|W \cap L')^{\mathrm{cons}}$ is perfect. Since $\mathcal{E}(L|W \cap L') \subseteq \mathcal{E}(L|W)$, all the elements of $\mathcal{E}(L|W \cap L')$ (in particular, W) are not isolated in $\mathcal{E}(L|V)^{\mathrm{cons}}$. This is a contradiction, and thus $\mathcal{E}(L|V)^{\mathrm{cons}}$ is perfect.

Suppose now that L is arbitrary: then, we can find a purely transcendental extension L' of K such that $L' \subseteq L$ is algebraic. By the previous part of the proof, $\mathcal{E}(L'|V)^{\mathrm{cons}}$ is perfect; by Corollary 7.2, also $\mathcal{E}(L|V)^{\mathrm{cons}}$ is perfect. Therefore, $\mathcal{E}(L|V)^{\mathrm{cons}}$ is always perfect.

Finally, $\mathrm{Zar}(L|V)$ is the union of $\mathcal{E}(L|V_0)$, as V_0 ranges among the valuation overrings of V; since each of these is perfect with respect to the constructible topology (by the previous part of the proof), then also $\mathrm{Zar}(L|V)^{\mathrm{cons}}$ is perfect, as claimed. □

Corollary 8.5. *Let V be a valuation domain with quotient field K, suppose $V \neq K$, and let L be a transcendental field extension of K. Then, $\mathcal{E}(L|V)^{\mathrm{cons}}$ is perfect.*

Proof. If $\mathrm{trdeg}(L/K) \geq 2$ the claim follows from Theorem 8.4. If $\mathrm{trdeg}(L/K) = 1$, let $X \in L$ be transcendental over K. By Theorem 8.2, $\mathcal{E}(K(X)|V)^{\mathrm{cons}}$ is perfect; by Corollary 7.2, also $\mathcal{E}(L|V)^{\mathrm{cons}}$ is perfect. □

Corollary 8.6. *Let D be an integral domain, and let L be a transcendental extension of the quotient field K of D. If* $\mathrm{trdeg}(L/K) \geq 2$*, then* $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ *is perfect.*

Proof. Any $W \in \mathrm{Zar}(L|D)$ belongs to $\mathcal{E}(L|V)$ for some $V \in \mathrm{Zar}(D)$. By Theorem 8.4, all $\mathcal{E}(L|V)^{\mathrm{cons}}$ are perfect and thus no W is isolated. Hence, $\mathrm{Zar}(L|D)^{\mathrm{cons}}$ is perfect. □

Acknowledgments

I would like to thank the referee for pointing out several problems in the first version of the paper and for his or her patience in pointing them out.

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(Dario Spirito) DIPARTIMENTO DI MATEMATICA "TULLIO LEVI-CIVITA", UNIVERSITÀ DEGLI STUDI DI PADOVA, PADOVA, ITALY; CURRENT ADDRESS: DIPARTIMENTO DI SCIENZE MATEMATICHE, INFORMATICHE E FISICHE, UNIVERSITÀ DEGLI STUDI DI UDINE, UDINE, ITALY
spirito@math.unipd.it; dario.spirito@uniud.it

This paper is available via http://nyjm.albany.edu/j/2022/28-32.html.