

Geometric classification of isomorphism of unital graph C^* -algebras

Sara E. Arklint, Søren Eilers and Efren Ruiz

ABSTRACT. We geometrically describe the relation induced on a set of graphs by isomorphism of their associated graph C^* -algebras as the smallest equivalence relation generated by five types of moves. The graphs studied have finitely many vertices and finitely or countably infinitely many edges, corresponding to unital and separable C^* -algebras.

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1. Introduction

The graph C^* -algebra construction associates to any graph G with finitely many vertices a unital C^* -algebra $C^*(G)$ with properties reflecting the geometry of the graph in various ways. The map $G \mapsto C^*(G)$ is far from injective, and hence one is naturally led to ask for a characterization of when two graphs G and H give the same graph C^* -algebra in the sense that $C^*(G)$ is $*$ -isomorphic to $C^*(H)$. We provide such a characterization in the paper at hand, much in the way that Reidemeister moves determine homotopy of knots, by defining a short list of fundamental “moves” (O), (I+), (R+), (C+), (P+) on graphs that do not change the associated C^* -algebra, and then establishing that whenever $C^*(G) \cong C^*(H)$, then there is a finite list of moves transforming G into H . The argument is constructive, but the path through moves may be very long even for simple examples.

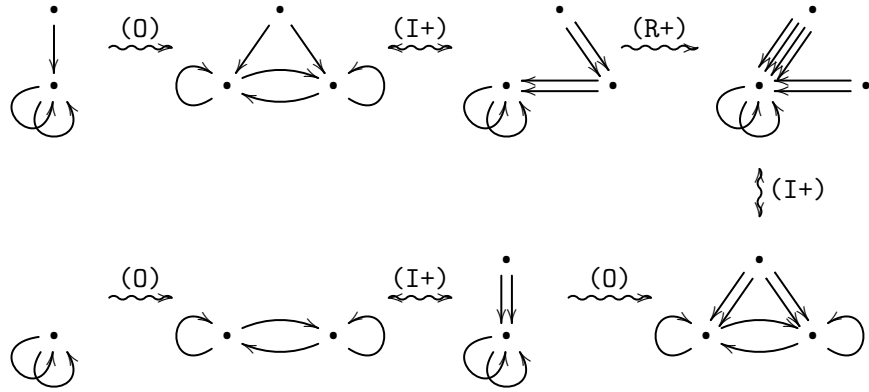
For instance, the graphs G and H with adjacency matrices [2] and $\begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$ are easily seen to define the Cuntz algebra \mathcal{O}_2 and the 2×2 -matrices $M_2(\mathcal{O}_2)$ over it,

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respectively. But it is well known that $\mathcal{O}_2 \cong M_2(\mathcal{O}_2)$, and our methods establish the path



which we do not know how to shorten.

A similar, but simpler, characterization of the coarser equivalence relation on such graphs induced by asking when G and H define *stably* isomorphic graph C^* -algebras was given in [ERRS21]. In fact, as is often the case, the classification problem for $*$ -isomorphism and for stable isomorphism among unital graph C^* -algebras were completed in tandem, in this case because the authors of [ERRS21] were able to extract the exact result as a corollary to the stabilized one. This established among other things that isomorphism amongst $C^*(G)$ and $C^*(H)$ was decidable by K -theory, but since the moves used to describe stable isomorphism do not respect $*$ -isomorphism in any natural way, one needs to carefully revise the list of moves in order to give the desired geometric description of “on the nose” isomorphism, whilst retaining the necessary flexibility. That is the problem we resolve here.

Our strategy of proof is by now standard, an elaboration of the original approach by Franks [Fra84] to classify irreducible shifts of finite type up to flow equivalence which draws significantly on previous refinements by Boyle and Huang ([BH03],[Boy02]) and by two of the authors with Restorff and Sørensen [ERRS21]. The key idea is to transform the question into one in algebra by proving that fundamental matrix operations such as row and column addition to adjacency matrices defined by the graphs under study are generated by a succession of legal moves. Our tool for doing this is an *antenna calculus* developed for this purpose which is used to represent the information remembered by the exact $*$ -isomorphism class (but forgotten after stabilization) by means of simple auxiliary configurations in such graphs, or – equivalently, as we shall see – as a vector complementing the adjacency matrix.

The work presented here was initiated during the 2016 Institut Mittag-Leffler focus program “Classification of operator algebras: complexity, rigidity, and dynamics” where we proved that $*$ -isomorphism among unital graph C^* -algebras was generated by a list of specific moves, and we presented our results at the

program’s workshop “Classification and discrete structures”. This work contained the development of the (R+) move which is essential here, but one of the other moves on our list was so arithmetic in nature that we couldn’t really defend calling our result geometric, and hence we refrained from publishing the result.

Recently the second and third authors have initiated in [ER] a systematic study of a refined collection of moves with the property that relevant subclasses of these moves generate isomorphisms of the graph C^* -algebras which respect additional structure such as diagonals and the canonical gauge action, and we were led to the definition of a refined type of insplitting – the (I+) move – which not only induces $*$ -isomorphism as opposed to the original version’s stable isomorphism, but also respects the additional structure mentioned above. To our immense satisfaction we have been able to show that the arithmetic move abandoned by the authors can be induced by the (I+) move along with the other honestly geometric moves already on our list, and hence we are now able to present a list of natural moves which all induce $*$ -isomorphism, and prove that any $*$ -isomorphism is induced by these moves by appealing to the argument we developed several years ago.

The bulk of the paper is devoted to the first step of Franks’ approach: To show that the elementary matrix operations may be implemented by geometric moves. This is particularly tricky in the situation studied here, but the most challenging technical difficulty is the same in all such problems: To ensure that the matrices visited as one tries to implement the given data are non-negative in an appropriate sense allowing them to make sense as adjacency matrices for intermediate graphs as well. For this, thankfully, we may appeal to the complicated analysis in [ERRS21] with rather minor adjustments.

In the interest of brevity, we relegate the proof that the improved moves indeed respect the exact isomorphism class of the C^* -algebras to the companion paper [ER], but we will describe them with some care below.

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2. Preliminaries

2.1. Notation and conventions. We use the definition of graph C^* -algebras in [FLR00] in which sinks and infinite emitters are singular vertices, and always consider $C^*(E)$ as a universal C^* -algebra generated by Cuntz-Krieger families $\{s_e, p_v\}$ with e ranging over edges and v ranging over vertices in E .

Unless stated otherwise, graphs E, F will always be considered as having finitely many vertices and finitely or countably infinitely many edges. We generally follow notation from [ERRS21, ERRS18], but will deviate slightly from these papers when describing graphs by matrices as explained below.

2.2. Legal moves. We briefly introduce the five types of moves we are considering. See [ER] for a full discussion.

In- and out-splitting works as in symbolic dynamics by distributing the incoming (resp. outgoing) edges to new vertices according to a given partition, and duplicating the outgoing (resp. incoming) ones. Note, however, the lack of symmetry below: Out-splitting may take place everywhere, but the partition cannot contain empty sets. In-splitting is restricted to regular vertices, but empty sets are allowed.

Definition 2.1 (Move (0): Outsplitt at a non-sink). Let $E = (E^0, E^1, r, s)$ be a graph, and let $w \in E^0$ be a vertex that is not a sink. Partition $s^{-1}(w)$ as a disjoint union of a finite number of nonempty sets

$$s^{-1}(w) = \mathcal{E}_1 \sqcup \mathcal{E}_2 \sqcup \cdots \sqcup \mathcal{E}_n$$

with the property that at most one of the \mathcal{E}_i is infinite. Let E_O denote the graph (E_O^0, E_O^1, r_O, s_O) defined by

$$\begin{aligned} E_O^0 &:= \{v^1 \mid v \in E^0 \text{ and } v \neq w\} \cup \{w^1, \dots, w^n\} \\ E_O^1 &:= \{e^1 \mid e \in E^1 \text{ and } r(e) \neq w\} \cup \{e^1, \dots, e^n \mid e \in E^1 \text{ and } r(e) = w\} \\ r_{E_O}(e^i) &:= \begin{cases} r(e)^1 & \text{if } e \in E^1 \text{ and } r(e) \neq w \\ w^i & \text{if } e \in E^1 \text{ and } r(e) = w \end{cases} \\ s_{E_O}(e^i) &:= \begin{cases} s(e)^1 & \text{if } e \in E^1 \text{ and } s(e) \neq w \\ s(e)^j & \text{if } e \in E^1 \text{ and } s(e) = w \text{ with } e \in \mathcal{E}_j. \end{cases} \end{aligned}$$

We say E_O is formed by performing move (0) to E .

Definition 2.2 (Move (I-): Insplitting). Let $E = (E^0, E^1, r_E, s_E)$ be a graph and let $w \in E^0$ be a regular vertex. Partition $r^{-1}(w)$ as a finite disjoint union of (possibly empty) subsets,

$$r^{-1}(w) = \mathcal{E}_1 \sqcup \mathcal{E}_2 \sqcup \cdots \sqcup \mathcal{E}_n.$$

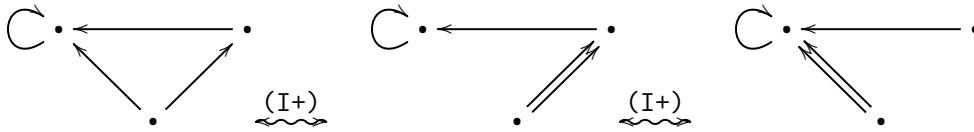
Let $E_I = (E_I^0, E_I^1, r_{E_I}, s_{E_I})$ be the graph defined by

$$\begin{aligned}
 E_I^0 &= \{v^1 : v \in E^0 \setminus \{w\}\} \cup \{w^1, w^2, \dots, w^n\} \\
 E_I^1 &= \{e^1 : e \in E^1, s_E(e) \neq w\} \cup \{e^1, e^2, \dots, e^n : e \in E, s_E(e) = w\} \\
 s_{E_I}(e^i) &= \begin{cases} s_E(e)^1 & \text{if } e \in E^1, s_E(e) \neq w \\ w^i & \text{if } e \in E^1, s_E(e) = w \end{cases} \\
 r_{E_I}(e^i) &= \begin{cases} r_E(e)^1 & \text{if } e \in E^1, r_E(e) \neq w \\ w^j & \text{if } e \in E^1, r_E(e) = w, e \in \mathcal{E}_j \end{cases}
 \end{aligned}$$

We say E_I is formed by performing move (I-) to E .

Definition 2.3 (Move (I+): Unital Insplitting). The graphs E and F are said to be *move (I+) equivalent* if there exists a graph G and a regular vertex $w \in G^0$ such that E and F are both the result of an (I-) move applied to G via a partition of $r_G^{-1}(w)$ using n sets.

Note that we do **not** consider the (I-) move further in this paper — it leaves $C^*(E) \otimes \mathbb{K}$ invariant, but not $C^*(E)$. It is convenient to think of an (I+) move as the result of redistributing the past of vertices having the same future. For instance we have



since all graphs may be obtained by an (I-) move applied to $C \cdot \Leftarrow \cdot$ with two sets in the partition.

Definition 2.4 (Move (R+): Unital Reduction). Let E be a graph and let w be a regular vertex which does not support a loop. Let E_{R+} be the graph defined by

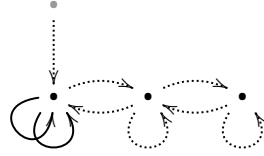
$$\begin{aligned}
 E_{R+}^0 &= (E^0 \setminus \{w\}) \sqcup \{\tilde{w}\} \\
 E_{R+}^1 &= (E^1 \setminus (r_E^{-1}(w) \cup s_E^{-1}(w))) \sqcup \{[ef] : e \in r_E^{-1}(w), f \in s_E^{-1}(w)\} \\
 &\quad \sqcup \{\tilde{f} : f \in s_E^{-1}(w)\}
 \end{aligned}$$

where the source and range maps of E_{R+} extend those of E , and satisfy

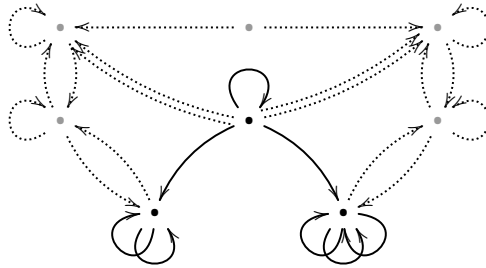
$$s_{E_{R+}}([ef]) = s_E(e), \quad s_{E_{R+}}(\tilde{f}) = \tilde{w}, \quad r_{E_{R+}}([ef]) = r_E(f), \quad r_{E_{R+}}(\tilde{f}) = r_E(f).$$

The (R+) move is best thought of as the result of removing a vertex and replacing all two-step paths through it by direct paths. The outgoing edges from the deleted vertex are preserved as edges from a source.

The (C+) and (P+) moves are defined by gluing on small graphs to the existing one under very precisely given conditions. Examples are



and



where the new parts of the graphs are indicated with gray dots and dotted arrows. Since they will not play a very central role in the arguments in the present paper, we refer to [ER] for a full discussion.

Throughout the paper we say “moves of type (X)” when we refer to a collection of such moves and their inverses. This applies in particular to the collection of moves of type (O), (I+), and (R+). In fact the remaining two types are in an appropriate sense their own inverses.

Theorem 2.5 ([ER]). *When F is obtained from E by one of the moves*

$$(O), (I+), (R+), (C+), (P+)$$

then $C^(E) \simeq C^*(F)$.*

2.3. Antenna calculus. Since regular sources are irrelevant from the point of view of stable isomorphism of unital graph C^* -algebras, we have disregarded them in our previous work [ERRS18] and [ERRS21], but since we are now aiming for exact $*$ -isomorphism, we need tools to keep track of them. For this purpose we introduce carefully selected notation which constitutes what we term an *antenna calculus*.

Our starting point is the observation that whenever a graph with finitely many vertices contains two or more regular sources, they may be collected to one by an (O) move in reverse, resulting in a graph with at most one regular source which represents the same C^* -algebra. In consequence, the salient information of the regular sources in any such graph is the number of edges that any other vertex receives from that source.

Assuming that there is at most one regular source, we enumerate the remaining vertices by $1, \dots, n$ and refer to a vertex by its number as \boxed{i} . Placing a regular

source first, the adjacency matrix has the form

$$\begin{bmatrix} 0 & c_1 & \cdots & c_n \\ 0 & a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ 0 & a_{n1} & \cdots & a_{nn} \end{bmatrix} \tag{2.1}$$

and we denote the submatrices with entries denoted a_{ij} and c_i by A and C , respectively, thinking of C as a row vector. In case there are no regular sources, we let C denote the zero vector and think of this setting as representing the graph with adjacency matrix A . Letting all $c_i = 0$ in (2.1) would give a very different system, so it is essential to deviate from the generic construction here. We think of the regular source as being “deleted” when $C = \underline{0}$.

In most cases we work instead of A and C with the pair (D, B) with B a matrix with the same dimensions as A , and D a column vector with the same number of entries as C , given by

$$b_{ij} = a_{ji} - \delta_{i,j} \quad (\text{Kronecker } \delta) \tag{2.2}$$

$$d_i = c_i + 1. \tag{2.3}$$

We will use round parentheses on B and D to set them aside from A and C given by bracketed matrices. It is clear that this contains the same information, and it will be obvious from Theorems 3.1 and 3.2, as well as from comparison with K -theory, why this notation is eminently adjusted to our needs. For the latter, note that when we indicate by B^\bullet and B° the matrices obtained by collecting, respectively, the columns corresponding to regular and singular vertices in the graph described, we have

Lemma 2.6. *When the graph E is represented by the pair (D, B) , we have*

$$(K_0(C^*(E)), [1_{C^*(E)}]) = (\text{cok } B^\bullet, D + \text{im } B^\bullet).$$

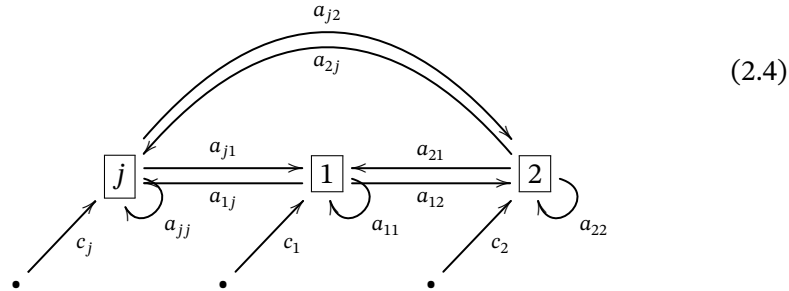
Proof. The result is standard (see e.g. [Tom03]) when there is no regular source and hence $D = \underline{1}$. If not, and the regular vertices are i_1, \dots, i_k , the K_0 -group is given by

$$\text{cok} \begin{pmatrix} -1 & 0 & \cdots & 0 \\ c_1 & b_{1i_1} & \cdots & b_{1i_k} \\ \vdots & \vdots & & \vdots \\ c_n & b_{ni_1} & \cdots & b_{ni_k} \end{pmatrix}$$

with the class of the unit represented by $\underline{1}$. This is isomorphic to the given data $(\text{cok } B^\bullet, D + \text{im } B^\bullet)$. □

The subdivision of data into D and B is theoretically and arithmetically convenient, but impractical for visualization purposes. Hence, whenever we are

depicting graphs, we will use the model



where $\boxed{1}$ and $\boxed{2}$ are different vertices which are not regular sources, and where \boxed{j} with $j > 2$ is an arbitrary vertex which is not a regular source. In this setting we outsplit to distribute the edges emanating from regular sources to a “shadow source”, and think of the edges enumerated by the c_i as *antennae* attached to an original graph.

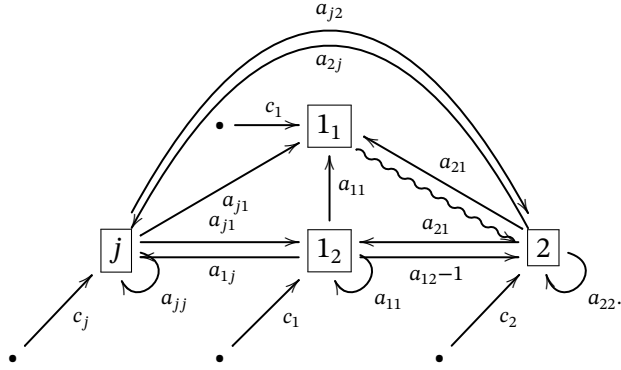
The entries a_{ij} are in the full range $0 \leq a_{ij} \leq \infty$. Note that entries c_i must be finite, and we use the convention that if some c_1, c_2 or c_j is zero, there is no corresponding source.

3. Elementary matrix operations

In this section we show how elementary matrix operations are induced on the (D, B) pair by our moves (0) , $(I+)$ and $(R+)$ applied to the graphs they represent. We follow the strategy of imposing any condition on the configuration necessary to establish the claims easily. In the ensuing sections we then proceed to remove many of these conditions.

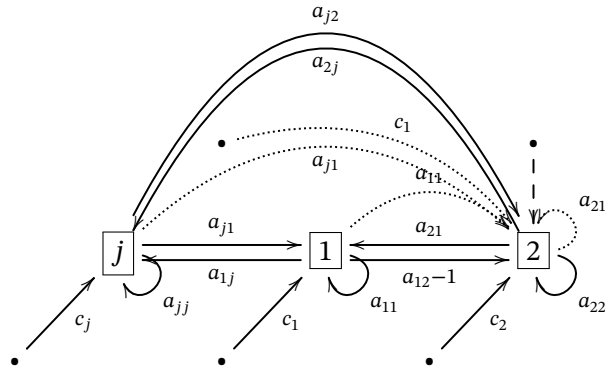
3.1. Outsplitting gives row operations. In this subsection we study how outsplitting translates to row operations on the A or B matrices which also influence the C and D vectors in a systematic way.

We start at (2.4) and assume there is at least one edge from $\boxed{1}$ to $\boxed{2}$, and $\boxed{1}$ emits at least one other edge (to any other vertex or to $\boxed{1}$ or $\boxed{2}$). Then we can outsplit at $\boxed{1}$ with one set in the partition being the selected edge from $\boxed{1}$ to $\boxed{2}$ and another containing the rest, and (0) gives us

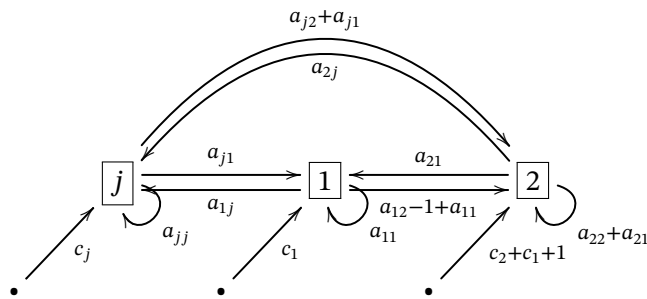


The squiggly arrow is alone in the sense that there is nothing else from $\boxed{1_1}$ to $\boxed{2}$, and we know that $\boxed{1_2}$ is not a sink.

Now we perform (R_+) to $\boxed{1_1}$ which we know is regular and does not support a loop, and the situation becomes



Here all the dotted edges are induced by paths that used to go via $\boxed{1_1}$, and the slashed edge is the extra introduced by (R_+) . We can collect sources and redraw this as



and then we get:

Proposition 3.1. *Given a pair (D, B) describing the graph E . When $b_{21} > 0$ and $\sum_{j=1}^n b_{j1} > 0$, we can go from E to the graph described by the pair (D', B') given as*

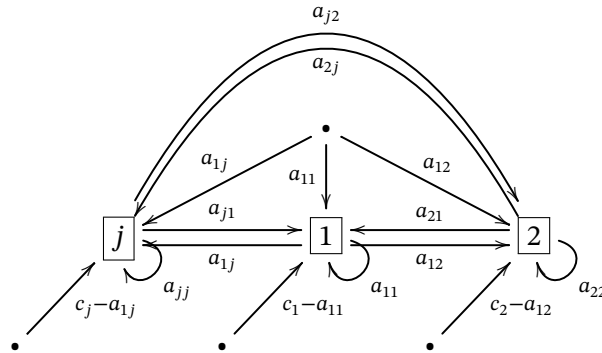
$$D' = \begin{pmatrix} d_1 \\ d_2 + d_1 \\ d_3 \\ \vdots \\ d_n \end{pmatrix} \quad B' = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} + b_{11} & b_{22} + b_{12} & b_{23} + b_{13} & \dots & b_{2n} + b_{1n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix}$$

by moves of type (O) and (R+).

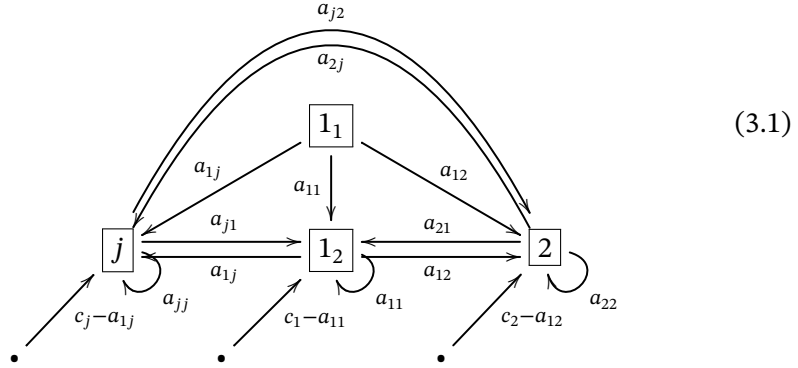
Recall b_{11} might be negative, so the condition $\sum_{j=1}^n b_{j1} > 0$ is not automatic from $b_{21} > 0$.

Proof. Follow the recipe given above, and substitute by (2.2) and (2.3) in the conclusion. □

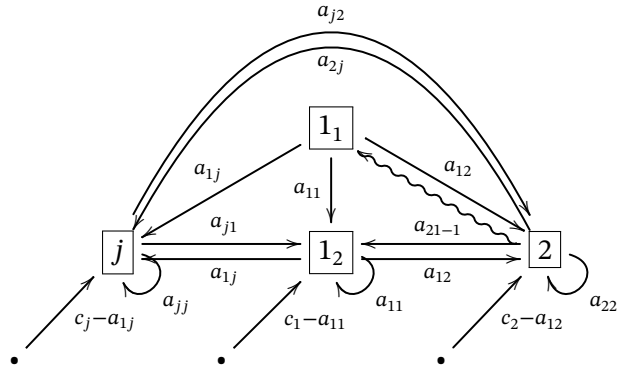
3.2. Insplitting gives column operations. We now pass to column operations on a given pair (D, B) . Let $\boxed{1}$ and $\boxed{2}$ be different vertices (not regular sources) with $\boxed{1}$ regular. We start with the setup as in (2.4) where this time we need to assert that there is at least one edge from $\boxed{2}$ to $\boxed{1}$, and further that there are at least as many antennae to any \boxed{j} as there are edges from $\boxed{1}$ to \boxed{j} for all j (including $j \in \{1, 2\}$). As above \boxed{j} and $\boxed{2}$ could be singular. The condition that $c_j \geq a_{1j}$ will allow us to redraw as



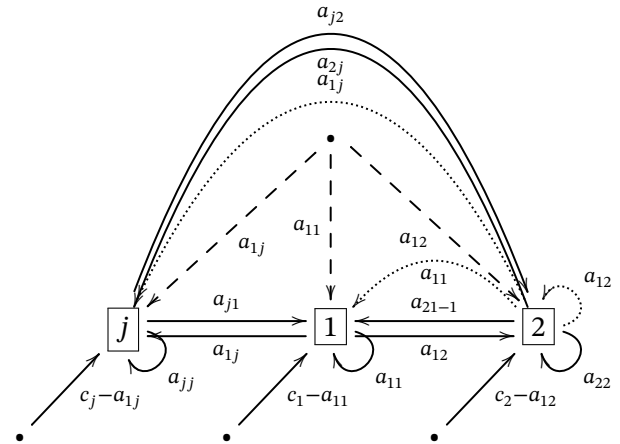
(the source in the middle is never a sink, and if any number $c_j - a_{1j}$ is zero, the source is deleted). Renaming the middle vertices



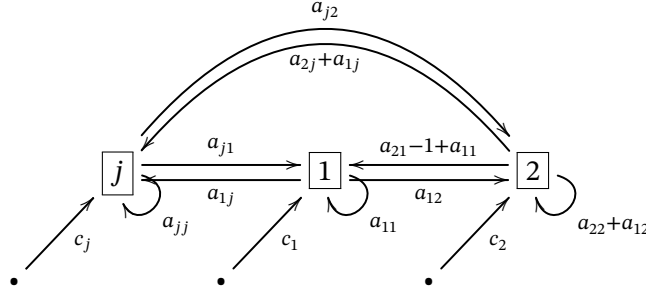
we see that $\boxed{1_1}$ and $\boxed{1_2}$ emit identically and hence we can use move (I+) (see comment just after Definition 2.3). We move one edge that used to go from $\boxed{2}$ to $\boxed{1_2}$ so that it now goes to $\boxed{1_1}$, and obtain



There is no loop on $\boxed{1_1}$, and it is regular, so we can use (R+) and get



which can be redrawn as



after appropriate moves of type (0). By (2.2) and (2.3), we get:

Proposition 3.2. *Given a pair (D, B) describing the graph E. When $b_{12} > 0$, $d_j \geq b_{j1} + 1$ for $j > 1$, and $d_1 \geq b_{11} + 2$, we can go from E to the graph described by the pair (D', B') given by*

$$D' = D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix} \quad B' = \begin{pmatrix} b_{11} & b_{12} + b_{11} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} + b_{21} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} + b_{31} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} + b_{n1} & b_{n3} & \dots & b_{nn} \end{pmatrix}$$

by moves of type (0), (I+), and (R+).

3.3. Insplitting gives column addition to antennae. In this section we show how to increase the size of the D vector in a pair (D, B) without changing B.

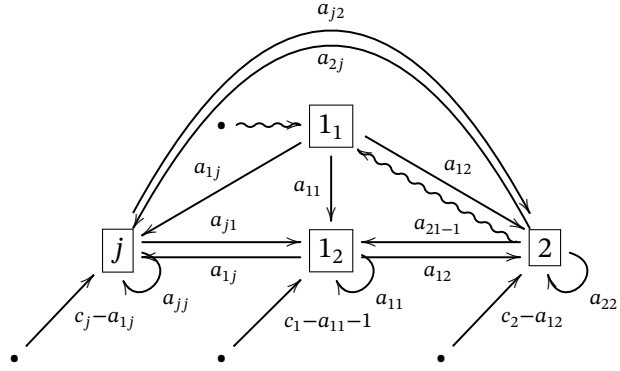
Proposition 3.3. *Given a pair (D, B) describing the graph E. When $b_{12} > 0$, $d_j \geq b_{j1} + 1$ for $j > 1$, and $d_1 \geq b_{11} + 3$, we can go from E to the graph described by (D', B') given by*

$$D' = \begin{pmatrix} d_1 + b_{11} \\ d_2 + b_{21} \\ d_3 + b_{31} \\ \vdots \\ d_n + b_{n1} \end{pmatrix} \quad B' = B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix}$$

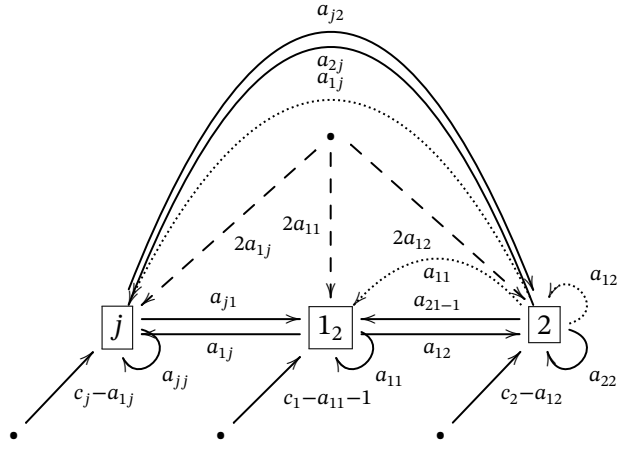
by moves of type (0), (I+), and (R+).

Proof. We proceed as in the previous section, but require further that $c_1 > a_{11}$. Then when we get to the stage (3.1) the middle source in the bottom supports at least one edge and hence has not been deleted. We use (I+) as before, but as we may, we also move one edge from the shadow source of $\boxed{1}$ over to $\boxed{1_1}$, so

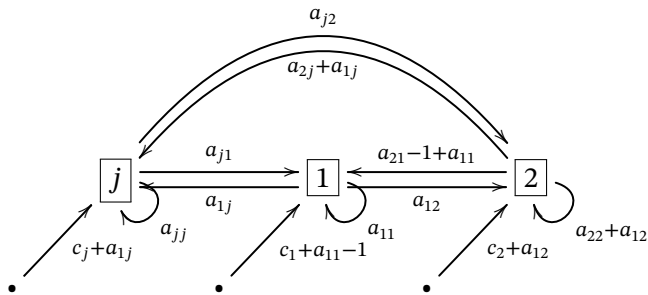
we get



Now as we perform (R_+) there will have been indirect paths from the new source to everything else receiving from $\boxed{1}$, the net effect being that the number of antennae arising from the (R_+) move is twice the number of edges originally emitted from $\boxed{1}$, resulting in



in which sources can be collected to form



This graph is represented by (D', B') given by

$$D' = \begin{pmatrix} d_1 + b_{11} \\ d_2 + b_{21} \\ d_3 + b_{31} \\ \vdots \\ d_n + b_{n1} \end{pmatrix} \quad B' = \begin{pmatrix} b_{11} & b_{12} + b_{11} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} + b_{21} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} + b_{31} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} + b_{n1} & b_{n3} & \dots & b_{nn} \end{pmatrix}$$

which is exactly the result of applying moves of type (0), (I+), and (R+) as in Proposition 3.2 to the graph described by (D', B) , noting that the requirements are met because of our assumptions. This proves the claim. \square

4. General matrix operations

In this section we generalize the elementary matrix operations of the previous section to much more general settings under rather modest conditions on the graphs studied. We also discuss in this context the possibility of performing the operations in reverse, as row or column subtractions rather than additions.

We generalize row addition before specifying conditions, as it will be convenient to prove that the conditions are always obtainable after moves of type (0), (I+) or (R+).

4.1. Improved row addition. The next result is a useful variation of Proposition 3.1, allowing the row addition to be performed at general entries \boxed{i} and \boxed{j} , provided that the former supports a loop and that there is a path connecting them in the appropriate sense.

Proposition 4.1. *Given a pair (D, B) describing the graph E . When two different vertices \boxed{i} and \boxed{j} are so that \boxed{i} supports a loop, and there is a path from \boxed{i} to \boxed{j} , then we can go from E to the graph described by the pair (D', B') given*

$$D' = \begin{pmatrix} \vdots \\ d_{j-1} \\ d_j + d_i \\ d_{j+1} \\ \vdots \end{pmatrix} \quad B' = \begin{pmatrix} \vdots & \vdots & \vdots & \dots & \vdots \\ b_{j-1,1} & b_{j-1,2} & b_{j-1,3} & \dots & b_{j-1,n} \\ b_{j1} + b_{i1} & b_{j2} + b_{i2} & b_{j3} + b_{i3} & \dots & b_{jn} + b_{in} \\ b_{j+1,1} & b_{j+1,2} & b_{j+1,3} & \dots & b_{j+1,n} \\ \vdots & \vdots & \vdots & \dots & \vdots \end{pmatrix}$$

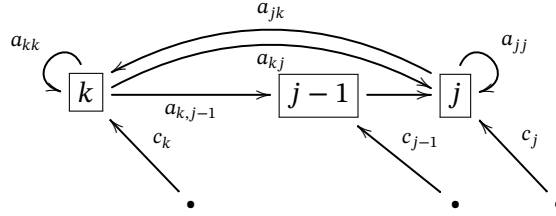
by moves of type (0) and (R+).

Proof. We may assume that $i = 1$ and that there is a minimal path from $\boxed{1}$ to \boxed{j} passing through $\boxed{2}, \dots, \boxed{j-1}$ in order. Hence, $a_{\ell, \ell+1} = b_{\ell+1, \ell} > 0$. Our argument depends on whether the intermediate $1 < \ell < j$ fall in the set

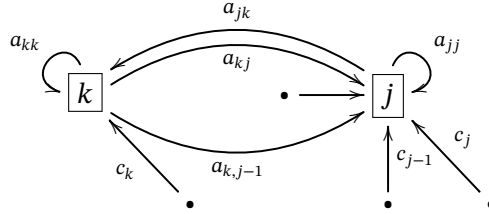
$$S = \left\{ \ell \in \{1, \dots, j-1\} \mid \sum_{i=1}^n b_{i\ell} = 0 \right\}$$

or not. Importantly, $1 \notin S$ by our assumption that $b_{11} \geq 0$ and $b_{21} > 0$.

We first consider $j - 1$, noting that $b_{j,j-1} > 0$. If $j - 1 \in S$ we know that $\boxed{j - 1}$ emits exactly one edge, namely to \boxed{j} , and hence the setup is



(generic $k \in \{1, \dots, n\} \setminus \{j - 1, j\}$) which becomes



after an (R_+) move. This is represented by the pair

$$\left(\begin{pmatrix} d_1 \\ \vdots \\ d_{j-2} \\ d_{j-1} + d_j \\ d_{j+1} \\ \vdots \end{pmatrix}, \begin{pmatrix} b_{11} & \cdots & b_{1,j-2} & b_{1j} & \cdots \\ \vdots & & \vdots & \vdots & \\ b_{j-2,1} & \cdots & b_{j-2,j-2} & b_{j-2,j} & \cdots \\ b_{j1} + b_{j-1,1} & \cdots & \mathbf{b_{j,j-2} + b_{j-1,j-2}} & b_{jj} + b_{j-1,j} & \cdots \\ b_{j+1,1} & \cdots & b_{j+1,j-2} & b_{j+1,j} & \cdots \\ \vdots & & \vdots & \vdots & \end{pmatrix} \right)$$

where the $(j - 1)$ st column has also been deleted. Note that the boldfaced entry is nonzero. When $j - 1 \notin S$, we note that Proposition 3.1 applies, and arrive at

$$\left(\begin{pmatrix} d_1 \\ \vdots \\ d_{j-2} \\ d_{j-1} \\ d_{j-1} + d_j \\ \vdots \end{pmatrix}, \begin{pmatrix} b_{11} & \cdots & b_{1,j-2} & b_{1,j-1} & b_{1j} & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \\ b_{j-2,1} & \cdots & b_{j-2,j-2} & b_{j-2,j-1} & b_{j-2,j} & \cdots \\ b_{j-1,1} & \cdots & b_{j-1,j-2} & b_{j-1,j-1} & b_{j-1,j} & \cdots \\ b_{j1} + b_{j-1,1} & \cdots & \mathbf{b_{j,j-2} + b_{j-1,j-2}} & b_{j,j-1} + b_{j-1,j-1} & b_{jj} + b_{j-1,j} & \cdots \\ \vdots & & \vdots & \vdots & \vdots & \end{pmatrix} \right)$$

In either case, we may use our knowledge that $b_{j-1,j-2} > 0$ to see that we may now use one of these operations to add row $j-2$ to row j (because of the nonzero entry at the boldfaced entries) and we can continue this way until we reach the pair (D'', B'') obtained by replacing the j th row in (D, B) by

$$\left(\left(\sum_{\ell=1}^j d_\ell \right), \left(\sum_{\ell=1}^j b_{\ell 1} \quad \cdots \quad \sum_{\ell=1}^j b_{\ell, j-2} \quad \sum_{\ell=1}^j b_{\ell, j-1} \quad \sum_{\ell=1}^j b_{\ell, j} \quad \cdots \right) \right)$$

and then deleting all rows and columns corresponding to entries in S . We note that by performing only the $j - 2$ first such steps, but starting from (D', B') , we also get to (D'', B'') , proving the claim. \square

4.2. Augmented canonical form. Here we describe the notion of *augmented canonical form* and explain how it may be algorithmically arranged. The notion is the direct extension of *canonical form* from [ERRS18] and [ERRS21] to the antenna calculus setting. We will use the convention that singular vertices are visualized using “ \circ ” while regular vertices are visualized using “ \bullet ”.

We recall the notational convention that \boxed{i} is never a regular source. We denote by $\gamma(i)$ the *component* associated to a vertex \boxed{i} as the largest set of vertices in $\{1, \dots, n\}$ so that $i \in \gamma(i)$ and so that whenever $j, k \in \gamma(i)$ are different, there is a path from \boxed{j} to \boxed{k} (and back by symmetry). We denote by $|\gamma(i)|$ the number of elements, divided

$$|\gamma(i)| = |\gamma(i)|^\bullet + |\gamma(i)|^\circ \quad (4.1)$$

into regular and singular vertices if necessary. Note that when $|\gamma(i)| > 1$, there is always a nontrivial path from \boxed{i} back to itself, but that this is not always the case when $|\gamma(i)| = 1$. We denote the set of components by Γ_E and note that it is pre-ordered because we may say that $\gamma(i) \leq \gamma(j)$ when there is a path from \boxed{j} to \boxed{i} (or when $\gamma(i) = \gamma(j)$).

By successively picking elements of Γ_E that are maximal amongst those not yet chosen, and permuting the enumeration of vertices correspondingly, we may assume A is given with the vertices ordered so that each component corresponds to a segment $i, \dots, i+k$, and so that it is upper block triangular. Working with the corresponding lower block triangular B , and denoting the block corresponding to rows from $\gamma(i)$ and columns from $\gamma(j)$ by $B_{\gamma(i), \gamma(j)}$, we write $B_{\gamma(i)}$ for the diagonal blocks $B_{\gamma(i), \gamma(i)}$. If Γ is a finite partially ordered set and $\underline{n} = (n_\gamma)_{\gamma \in \Gamma}$ is a vector of positive integers, we write $\mathcal{M}_\Gamma(\underline{n})$ for the integer matrices X in $\mathcal{M}_{\|\underline{n}\|_1}$ such that $X_{\gamma, \gamma'} = 0$ unless $\gamma \leq \gamma'$ for $\gamma, \gamma' \in \Gamma$. It is then clear that $B \in \mathcal{M}_{\Gamma_E}(\underline{n})$ for $\underline{n} = (|\gamma(i)|)_{\gamma(i) \in \Gamma_E}$. See [ERRS18, Section 4.1] for more details.

Definition 4.2. Let E be a given graph. We say that E is in *augmented canonical form* if

- (I) every regular vertex of E which is not a source supports a loop;
- (II) whenever there is a path from \boxed{i} to \boxed{j} , there is an edge from \boxed{i} to \boxed{j} ;
- (III) whenever there is a path from \boxed{i} to \boxed{j} , and \boxed{i} is an infinite emitter, there are infinitely many edges from \boxed{i} to \boxed{j} ;
- (IV) If there are two different paths from \boxed{i} back to itself (neither visiting \boxed{i} along the way), then
 - (a) \boxed{i} supports two loops,
 - (b) $|\gamma(i)|^\bullet \geq 3$, and,
 - (c) the Smith form of $B_{\gamma(i)}^\bullet$ has at least two ones in its diagonal;
- (V) there is at most one regular source in E .

If E has no regular sources, we just say that E is in canonical form.

The Smith form for a rectangular matrix works exactly as in the square case, see [ERRS21, Remark 8.1]

Since all vertices in a graph in augmented canonical form either supports a loop or is singular, the set of components in such a graph coincides with the union of the collection of all maximal strongly connected sets, and all singular vertices not supporting a path back to themselves, just like in [ERRS18, Definition 3.9]. The source, if it exists, is the only vertex not contained in a component.

We note from the outset that there is a trichotomy among components in a graph that is in augmented canonical form. If one (hence all) of the vertices in the component has more than one path back to itself, then there are more than a prescribed number of regular vertices in the component by (IV), all vertices are directly connected by (II), and every vertex supports two loops by (IV) again. If this is not the case, there is only one vertex in the component because of (I). If this vertex is regular, it supports exactly one loop, and if it is singular, it has no path back to itself and hence there are no edges in the component. We also get from (III) that any infinite emitter emits with infinite multiplicity to any vertex it reaches by a path.

It is worth translating these conditions to the pair (D, B) , and we see first that they only involve B . The diagonal blocks correspond to the components themselves, and we see that they come in three flavors: One type is “large” and contains only positive entries by (II) and (IV) (they are even ∞ in all columns corresponding to singular vertices by (II)), and the other two types are “small” and must be one of

$$(0) \quad (-1).$$

All off-diagonal blocks that are allowed to take nonzero entries by the condition defining $\mathcal{M}_{\Gamma_E}(n)$ are in fact positive everywhere because of (II), even ∞ on all singular columns by (III).

Proposition 4.3. *Any graph E may be transformed algorithmically into a graph E' in augmented canonical form by moves of type (0) , $(I+)$, and $(R+)$ as follows:*

- STEP 0 *Use an (0) move to ensure that there is at most one regular source.*
- STEP 1 *Use (0) moves to ensure that if \boxed{i} is an infinite emitter, then it emits either infinitely many or no edges to each vertex.*
- STEP 2 *Use $(R+)$ moves to ensure that each regular vertex not supporting a loop is a source.*
- STEP 3 *Use (0) moves to ensure that no component has only one vertex but two or more edges, so that the properties of STEP 1 and 2 are preserved.*
- STEP 4 *Use row addition to ensure that any component with two or more vertices has at least one vertex supporting a loop.*
- STEP 5 *Use row addition to ensure that all entries in the $B_{\gamma(i)}$ -block for any component $\gamma(i)$ with $|\gamma(i)| > 1$ are strictly positive.*
- STEP 6 *Use an (0) move to increase the size of any component not satisfying (IV) by one, so that the properties of STEP 1 and 2 are preserved. Go back to*

- STEP 5 *if the $B_{\gamma(i)}$ -block of the outsplit graph is not strictly positive for any $\gamma(i)$ with $|\gamma(i)| > 1$.*
- STEP 7 *Use row addition to ensure that $b_{ij} > 0$ whenever $\gamma(i) \neq \gamma(j)$, $\gamma(i) \leq \gamma(j)$ and $|\gamma(j)| > 1$.*
- STEP 8 *Use row addition to ensure that $b_{ij} > 0$ whenever $\gamma(i) \neq \gamma(j)$, $\gamma(i) \leq \gamma(j)$ and $B_{\gamma(i),\gamma(j)} \neq 0$.*
- STEP 9 *If different i, j, k exist with $b_{ij}, b_{jk} > 0$ and $b_{ik} = 0$, use row addition to arrange that $b_{ik} > 0$. Then return to STEP 8.*

Proof. We have explained the workings of STEP 0 in Section 2.3. STEP 1 is obtained by outsplitting, placing all edges parallel to infinitely many edges in one set of the partition, and the rest in another. STEP 2 is straightforward, but new sources will arise and must be collected into one as in STEP 0. We note that the properties arranged in STEP 0, 1 and 2 will not be affected in later steps, since care is taken when applying subsequent (0) moves and since row additions will never change these properties.

To see that STEP 3 is possible, note that we can replace $[n]$ by $[\begin{smallmatrix} 1 & 1 \\ n-1 & n-1 \end{smallmatrix}]$ and $[\infty]$ by $[\begin{smallmatrix} 1 & 1 \\ \infty & \infty \end{smallmatrix}]$. In the latter case, we must assign all edges leaving the component to the second vertex to preserve STEP 1.

There is only something to do in STEP 4 when a component $\gamma(i)$ has exclusively singular vertices, and two or more of these. By STEP 1, there is then an infinity of parallel edges from \boxed{j} to \boxed{i} with $i \neq j \in \gamma(i)$, and Proposition 3.1 applies to add row i to row j , thus obtaining an infinity of loops at \boxed{i} .

In STEP 5, we first ensure that all vertices in such a component supports a loop, using STEP 4 and subsequently adding rows with vertices already having a loop by means of Proposition 4.1. When all vertices support loops, we can sum all rows into the last row and see that this produces exclusively positive entries. This may then be added to all other rows by 4.1 again.

In STEP 6, we aim to obtain (IV)(b)(c) by increasing the number of regular vertices in the relevant components; this will automatically induce ones in the diagonal of the Smith forms. Since every vertex in such a component now supports at least two loops, we can outsplit any vertex without affecting the conditions obtained in STEP 0, 1 and 2.

In STEP 7, Proposition 4.1 again applies because any vertex in the emitting component supports a loop, and gives positive entries in the off-diagonal component since the diagonal entry in the added row is positive (because it in fact supports two loops). In STEP 8, we may assume by STEP 7 that there is only one column in the block, so the given nonzero entry can be used to make all entries positive by row operations among the vertices in $\gamma(i)$. Proposition 4.1 applies when $|\gamma(i)| > 1$, and when $|\gamma(i)| = 1$ there is nothing to do.

In STEP 9, we see by the previous steps that \boxed{i} , \boxed{j} and \boxed{k} lie in three different components, and that $|\gamma(k)| = 1$ so that the block $B_{\gamma(i),\gamma(k)}$ has only one column. We may apply Proposition 3.1 because there is an edge from \boxed{j} to \boxed{k} ,

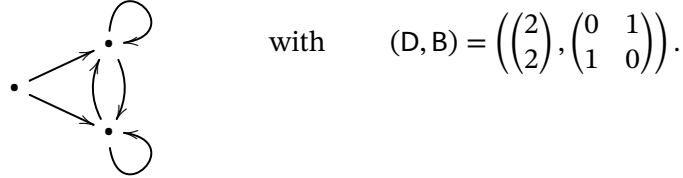
and because j emits at least one other edge. Indeed, it will either support two loops, one loop, or be an infinite emitter depending on which element in the trichotomy it belongs to.

The algorithm clearly terminates, and it follows that (I)–(V) are satisfied in the resulting graph. Here, (I) is a consequence of STEP 2, and STEP 9 ensures that (II) holds. Because we have maintained the property of STEP 1 throughout, this entails (III) as well, and (IV)(a) is a consequence of the positivity obtained in STEP 5. (IV)(b)(c) are obtained in STEP 6. \square

Example 4.4. The graph



satisfies all conditions for being in augmented canonical form except for the size requirements in (IV). Our algorithm would attend to this in STEP 3 with an outsplitting, leading to



In STEP 5, two row additions would obtain positivity, for instance as

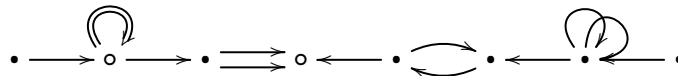
$$\left(\begin{pmatrix} 6 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \right)$$

and then another outsplitting in STEP 6 followed by row additions in STEP 5 could lead to the graph given by

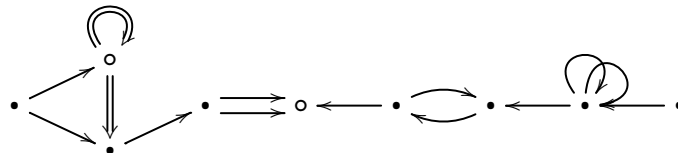
$$\left(\begin{pmatrix} 6 \\ 10 \\ 10 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \right)$$

It is easy to see that the Smith form of the latter B is the identity matrix, and hence meets all requirements in (IV).

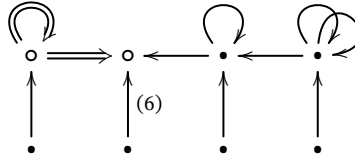
Example 4.5. The graph



fails all conditions (I)–(V) of Definition 4.2. Postponing STEP 0 for ease of visualization, STEP 1 gives



whereafter three (R+) moves mandated by STEP 2 leads to



after collection of sources into antenna form. Ordering components as specified earlier, we get the pair

$$\left(\begin{pmatrix} 2 \\ 2 \\ 2 \\ 7 \end{pmatrix}, \begin{pmatrix} \infty & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ \infty & 0 & 1 & 0 \end{pmatrix} \right)$$

which satisfies all conditions (I), (III), (V) for being in augmented canonical form. As in Example 4.4, a number of subsequent outsplittings and row additions in STEP 3,5 and 6 can lead to

$$\left(\begin{pmatrix} 2 \\ 4 \\ 3 \\ 2 \\ 6 \\ 10 \\ 10 \\ 7 \\ 2 \end{pmatrix}, \begin{pmatrix} \infty & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ \infty & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \infty & 2 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ \infty & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 3 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{0} & \mathbf{0} & 1 & 1 & 0 \\ \infty & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 0 \end{pmatrix} \right)$$

which satisfies (IV). All that remains to do is to increase the entries at the zeroes indicated in bold to meet (II). This is easily obtained in STEP 6 by row additions.

Remark 4.6. Although we do not require graphs to be in augmented canonical form before performing row additions, it is relevant for performing row *subtractions*. Indeed, it is obvious that when we can go from (D, B) to (D', B') by a row addition implemented by moves of type (O), (I+), and (R+), we can go from (D', B') to (D, B) by such moves as well. Starting from (D', B') , we just need to ensure that such an operation does not introduce entries inconsistent with the way we represent graphs: d_i must be at least one, b_{ij} must be nonnegative for $i \neq j$, and $b_{ii} \geq -1$.

Computing (D', B') from (D, B) is problematic in columns with infinite emitters, since $\infty - \infty$ is undefined, but it follows from (II) that any row addition in the presence of augmented canonical form does not alter such a column. Hence, it makes sense to use the convention $\infty - \infty = \infty$ in this case.

Remark 4.7. Graphs in augmented canonical form have not been considered before, but if one deletes the antennae, or – equivalently – considers only the B part of the data, we recover the canonical form of [ERRS21, Definition 7.3]. The

condition (I) studied here implies the conditions defining $\mathfrak{M}_p^{\circ\circ}(\mathbf{m} \times \mathbf{n}, \mathbb{Z})$ and $\mathfrak{M}_p^{\circ\circ\circ}(\mathbf{m} \times \mathbf{n}, \mathbb{Z})$ (see [ERRS18, Definition 4.15] for the definition of $\mathfrak{M}_p^{\circ\circ}(\mathbf{m} \times \mathbf{n}, \mathbb{Z})$ and $\mathfrak{M}_p^{\circ\circ\circ}(\mathbf{m} \times \mathbf{n}, \mathbb{Z})$) as well as (2) and the first half of (1) in the definition of canonical form in [ERRS21, Definition 7.3]. Our (II) similarly gives $\mathfrak{M}_p^{\circ}(\mathbf{m} \times \mathbf{n}, \mathbb{Z})$ and the second half of (1), and our (III) gives (4) of canonical form in [ERRS21, Definition 7.3]. (IV) gives the remaining conditions (3) and (5).

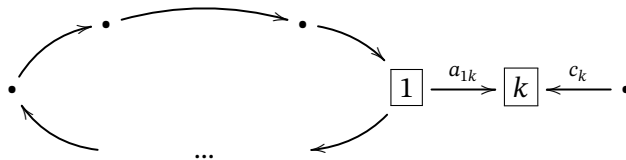
4.3. Increasing antenna counts. The following result is of key technical importance for us. We will use it to increase the number of antennae to suit our needs, in particular when generalizing the column operation from Proposition 3.2 to a much more general version. Employing an assumption of augmented standard form allows us to show that we may increase antenna counts like this in any such setting.

Theorem 4.8. *Let E be a graph in augmented canonical form represented by (D, B) . For any j with \boxed{j} regular, we can go to the graph described by the pair (D', B') given by*

$$D' = \begin{pmatrix} d_1 + b_{1j} \\ d_2 + b_{2j} \\ d_3 + b_{3j} \\ \vdots \\ d_n + b_{nj} \end{pmatrix} \quad B' = B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix}$$

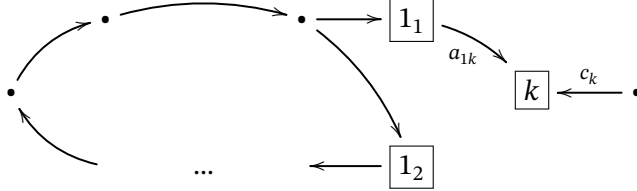
by moves of type (0), (I+), and (R+).

Proof. We assume without loss of generality that $j = 1$ but note for later use that Proposition 3.3 allows us to add any regular column, say i , to D provided $d_j \geq b_{ji} + 1$ for $j \neq i$, and $d_i \geq b_{ii} + 3$, when there is some $j \neq i$ so that $b_{ij} > 0$. CASE 1: Suppose $b_{11} = b_{12} = \dots = b_{1n} = 0$. Then $\boxed{1}$ supports a single loop and besides this loop receives only from a regular source. When also $b_{21} = \dots = b_{n1} = 0$ there is nothing to prove, so we assume that $\boxed{1}$ emits to at least two vertices. By repeated use of (R+) in reverse, we pass to the graph

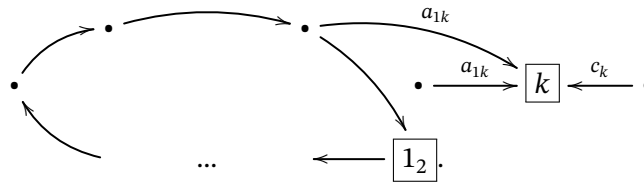


with the loop of length $d_1 = 1 + c_1$, having the important property that there is exactly one incoming edge to $\boxed{1}$. We now use (0) at $\boxed{1}$ (using here that it

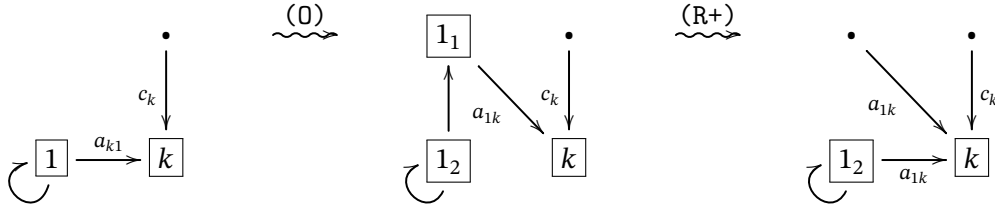
emits more than one edge) and get



and then with (R+) at $\boxed{1_1}$ we get



(some a_{1k} may be zero, but not all). Shortening the loop again with (R+) moves, we arrive at the desired situation. This works also when $c_1 = 0$ but takes the form



CASE 2: Suppose $b_{11} = 0$ but that the first row does not vanish. We may assume that $b_{12} > 0$. By (II) and (IV) of our assumption of augmented canonical form, $\boxed{1}$ is alone in its component, so we conclude that $b_{21} = 0$. Since $b_{12} \geq 1$, we may apply Proposition 3.1 and add row 2 to row 1 twice and get to

$$\left(\begin{pmatrix} d_1 + 2d_2 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix}, \begin{pmatrix} 0 & b_{12} + 2b_{22} & b_{13} + 2b_{23} & \dots & b_{1k} + 2b_{2k} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix} \right)$$

Since $d_1, d_2 > 0$ we can choose $M_j \geq 0$ so that

$$d_j + M_j(d_1 + 2d_2) \geq b_{j1} + 1 \tag{4.2}$$

for all j with $b_{j1} > 0$. For j with $b_{j1} = 0$, set $M_j = 0$. For all j , add row 1 to row j , M_j times to get to

$$\left(\begin{pmatrix} d_1 + 2d_2 \\ d_2 + M_2(d_1 + 2d_2) \\ d_3 + M_3(d_1 + 2d_2) \\ \vdots \\ d_n + M_n(d_1 + 2d_2) \end{pmatrix}, \begin{pmatrix} 0 & b_{12} + 2b_{22} & b_{13} + 2b_{23} & \dots & b_{1n} + 2b_{2n} \\ b_{21} & b_{22} + M_2(b_{12} + 2b_{22}) & b_{23} + M_2(b_{13} + 2b_{23}) & \dots & b_{2n} + M_2(b_{1n} + 2b_{2n}) \\ b_{31} & b_{32} + M_3(b_{12} + 2b_{22}) & b_{33} + M_3(b_{13} + 2b_{23}) & \dots & b_{3k} + M_3(b_{1n} + 2b_{2n}) \\ \vdots & \vdots & \vdots & \dots & \vdots \\ b_{n1} & b_{n2} + M_n(b_{12} + 2b_{22}) & b_{n3} + M_n(b_{13} + 2b_{23}) & \dots & b_{nk} + M_n(b_{1n} + 2b_{2n}) \end{pmatrix} \right)$$

and note that since $b_{12} > 0$ and (4.2) hold, Proposition 3.3 applies to take us to

$$\left(\begin{pmatrix} d_1 + 2d_2 \\ d_2 + M_2(d_1 + 2d_2) + b_{21} \\ d_3 + M_3(d_1 + 2d_2) + b_{31} \\ \vdots \\ d_n + M_n(d_1 + 2d_2) + b_{n1} \end{pmatrix}, \begin{pmatrix} 0 & b_{12} + 2b_{22} & b_{13} + 2b_{23} & \dots & b_{1n} + 2b_{2n} \\ b_{21} & b_{22} + M_2(b_{12} + 2b_{22}) & b_{23} + M_2(b_{13} + 2b_{23}) & \dots & b_{2k} + M_2(b_{1n} + 2b_{2n}) \\ b_{31} & b_{32} + M_3(b_{12} + 2b_{22}) & b_{33} + M_3(b_{13} + 2b_{23}) & \dots & b_{3k} + M_3(b_{1n} + 2b_{2n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} + M_n(b_{12} + 2b_{22}) & b_{n3} + M_n(b_{13} + 2b_{23}) & \dots & b_{nk} + M_n(b_{1n} + 2b_{2n}) \end{pmatrix} \right)$$

and then for all j , subtracting row 1 from row j M_j times, we get to

$$\left(\begin{pmatrix} d_1 + 2d_2 \\ d_2 + b_{21} \\ d_3 + b_{31} \\ \vdots \\ d_n + b_{n1} \end{pmatrix}, \begin{pmatrix} 0 & b_{12} + 2b_{22} & b_{13} + 2b_{23} & \dots & b_{1n} + 2b_{2n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix} \right)$$

Finally, recall that $b_{21} = 0$, thus subtracting row 2 from row 1 twice, we get to

$$\left(\begin{pmatrix} d_1 + 0 \\ d_2 + b_{21} \\ d_3 + b_{31} \\ \vdots \\ d_n + b_{n1} \end{pmatrix}, \begin{pmatrix} 0 & b_{12} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix} \right)$$

by a succession of moves of type (0), (I+), and (R+).

CASE 3: Since E is in augmented canonical form, the remaining case has $b_{11} > 0$, and we may assume that $b_{12} > 0$ and $b_{21} > 0$ for some regular [2]. Outsplitting [1] using a single loop on [1] in one set of the partition, and the rest of the outgoing edges in the other, we get to

$$\left(\begin{pmatrix} d_1 \\ d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix}, \begin{pmatrix} 0 & b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ 1 & b_{11} - 1 & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ 0 & b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix} \right)$$

by an (0) move.

We claim that we can get to

$$\left(\begin{pmatrix} d_1 \\ d_1 + N \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix}, \begin{pmatrix} 0 & b_{11} & b_{12} & b_{13} & \dots & b_{1n} \\ 1 & b_{11} - 1 & b_{12} & b_{13} & \dots & b_{1n} \\ 0 & b_{21} & b_{22} & b_{23} & \dots & b_{2n} \\ 0 & b_{31} & b_{32} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & b_{n1} & b_{n2} & b_{n3} & \dots & b_{nn} \end{pmatrix} \right)$$

by moves of type (I+), (0) and (R+) for all $N \geq 1$.

Since the $(2, 1)$ -entry of the above matrix is 1, we may add row 1 to row 2 twice by Proposition 3.1 and get to

$$\left(\begin{pmatrix} d_1 \\ 3d_1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 1 & 3b_{11} - 1 & 3b_{12} & 3b_{13} & \cdots & 3b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \right)$$

where we skip all unaltered lines to conserve space. Adding row 2 to row 1 (applying Proposition 3.1 since the $(1, 2)$ -entry of the above matrix is $b_{11} > 0$), we get to

$$\left(\begin{pmatrix} 4d_1 \\ 3d_1 \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 & 4b_{11} - 1 & 4b_{12} & 4b_{13} & \cdots & 4b_{1n} \\ 1 & 3b_{11} - 1 & 3b_{12} & 3b_{13} & \cdots & 3b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \right)$$

By Proposition 3.3 which applies because the two first entries in the vector dominate appropriately, and because the $(1, 2)$ -entry in the matrix is not zero, we get to:

$$\left(\begin{pmatrix} 4d_1 + N \\ 3d_1 + N \\ \vdots \end{pmatrix}, \begin{pmatrix} 1 & 4b_{11} - 1 & 4b_{12} & 4b_{13} & \cdots & 4b_{1n} \\ 1 & 3b_{11} - 1 & 3b_{12} & 3b_{13} & \cdots & 3b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \right)$$

and subtracting row 2 from row 1, we get to

$$\left(\begin{pmatrix} d_1 \\ 3d_1 + N \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 1 & 3b_{11} - 1 & 3b_{12} & 3b_{13} & \cdots & 3b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \right).$$

Subtracting row 1 from row 2 twice, we get to

$$\left(\begin{pmatrix} d_1 \\ d_1 + N \\ \vdots \end{pmatrix}, \begin{pmatrix} 0 & b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 1 & b_{11} - 1 & b_{12} & b_{13} & \cdots & b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \right) \tag{4.3}$$

by moves of type (0), (I+), and (R+), as claimed.

Choose $N \in \mathbb{N}$ such that

$$d_j + d_1 + N + 1 \geq b_{j1} + b_{11} - 1 \tag{4.4}$$

for all $j \geq 1$ with $b_{j1} > 0$ (recall that $b_{j1} < \infty$ since $\boxed{1}$ is regular). We now set $\Delta_j = 1$ when $b_{j1} > 0$ and $\Delta_j = 0$ otherwise.

Adding row 2 to row 1 to (4.3), as well as adding row 2 to row j for all j with $b_{j1} > 0$, we get to

$$\left(\begin{pmatrix} 2d_1 + N \\ d_1 + N \\ d_2 + \Delta_2(d_1 + N) \\ \vdots \\ d_n + \Delta_n(d_1 + N) \end{pmatrix}, \begin{pmatrix} 1 & 2b_{11} - 1 & 2b_{12} & 2b_{13} & \cdots & 2b_{1n} \\ 1 & b_{11} - 1 & b_{12} & b_{13} & \cdots & b_{1n} \\ \Delta_2 & b_{21} + \Delta_2(b_{11} - 1) & b_{22} + \Delta_2 b_{12} & b_{23} + \Delta_2 b_{13} & \cdots & b_{2n} + \Delta_2 b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_n & b_{n1} + \Delta_n(b_{11} - 1) & b_{n2} + \Delta_n b_{12} & b_{n3} + \Delta_n b_{13} & \cdots & b_{nn} + \Delta_n b_{1n} \end{pmatrix} \right).$$

Applying Proposition 3.3, which applies because of (4.4), we now get to

$$\left(\begin{pmatrix} 2d_1 + N + 2b_{11} - 1 \\ d_1 + N + b_{11} - 1 \\ d_2 + b_{21} + \Delta_2(d_1 + N + b_{11} - 1) \\ \vdots \\ d_n + b_{n1} + \Delta_n(d_1 + N + b_{11} - 1) \end{pmatrix}, \begin{pmatrix} 1 & 2b_{11} - 1 & 2b_{12} & 2b_{13} & \cdots & 2b_{1n} \\ 1 & b_{11} - 1 & b_{12} & b_{13} & \cdots & b_{1n} \\ \Delta_2 & b_{21} + \Delta_2(b_{11} - 1) & b_{22} + \Delta_2 b_{12} & b_{23} + \Delta_2 b_{13} & \cdots & b_{2n} + \Delta_2 b_{1n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \Delta_n & b_{n1} + \Delta_n(b_{11} - 1) & b_{n2} + \Delta_n b_{12} & b_{n3} + \Delta_n b_{13} & \cdots & b_{nn} + \Delta_n b_{1n} \end{pmatrix} \right).$$

Subtracting row 2 from row 1 and row 2 from row j for all j with $b_{j1} > 0$, we arrive at

$$\left(\begin{pmatrix} d_1 + b_{11} \\ d_1 + N + b_{11} - 1 \\ d_2 + b_{21} \\ \vdots \\ d_n + b_{n1} \end{pmatrix}, \begin{pmatrix} 0 & b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 1 & b_{11} - 1 & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix} \right)$$

and applying Proposition 3.3 in reverse $N - 1$ times to the first column, we get to

$$\left(\begin{pmatrix} d_1 + b_{11} \\ d_1 + b_{11} \\ d_2 + b_{21} \\ \vdots \\ d_n + b_{n1} \end{pmatrix}, \begin{pmatrix} 0 & b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ 1 & b_{11} - 1 & b_{12} & b_{13} & \cdots & b_{1n} \\ 0 & b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix} \right)$$

by moves of type (0), (I+) and (R+).

And finally, we reach

$$\left(\begin{pmatrix} d_1 + b_{11} \\ d_2 + b_{21} \\ \vdots \\ d_n + b_{n1} \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{n1} & b_{n2} & b_{n3} & \cdots & b_{nn} \end{pmatrix} \right)$$

by an (0) move in reverse. \square

Proposition 4.9. Consider matrices (D, B) describing the graph E in augmented canonical form. Suppose \boxed{i} and \boxed{j} are different regular vertices so that there is a path from \boxed{i} to \boxed{j} . Then we can go from E to the graph described by the pair (D', B') given by

$$D' = D = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \\ \vdots \\ d_n \end{pmatrix} \quad B' = \begin{pmatrix} \cdots & b_{1,j-1} & b_{1j} + b_{1i} & b_{1,j+1} & \cdots \\ \cdots & b_{2,j-1} & b_{2j} + b_{2i} & b_{2,j+1} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdots & b_{n,j-1} & b_{nj} + b_{ni} & b_{n,j+1} & \cdots \end{pmatrix}$$

by moves of type (0), (I+), and (R+).

Proof. We may assume $i = 1$ and $j = 2$, and because the graph is in augmented canonical form, there is an edge from $\boxed{1}$ to $\boxed{2}$. To apply Proposition 3.2 we use Theorem 4.8 four times to pass to the vector

$$\begin{pmatrix} d_1 + 2(b_{11} + b_{12}) \\ d_2 + 2(b_{21} + b_{22}) \\ d_3 + 2(b_{31} + b_{32}) \\ \vdots \\ d_n + 2(b_{n1} + b_{n2}) \end{pmatrix}$$

so that the conditions are met to get to

$$\left(\begin{pmatrix} d_1 + 2(b_{11} + b_{12}) \\ d_2 + 2(b_{21} + b_{22}) \\ d_3 + 2(b_{31} + b_{32}) \\ \vdots \\ d_n + 2(b_{n1} + b_{n2}) \end{pmatrix}, \begin{pmatrix} b_{11} & b_{12} + b_{11} & b_{13} & \dots & b_{1n} \\ b_{21} & b_{22} + b_{21} & b_{23} & \dots & b_{2n} \\ b_{31} & b_{32} + b_{31} & b_{33} & \dots & b_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} + b_{n1} & b_{n3} & \dots & b_{nn} \end{pmatrix} \right).$$

We then apply Theorem 4.8 in reverse two times to reach the conclusion in Proposition 3.2 irrespective of the original d_i . \square

5. Conclusion

Definition 5.1. We say that a pair of graphs (E, F) are in *augmented standard form* if both are in augmented canonical form, and if there is an isomorphism of partially ordered sets $\psi : \Gamma_E \rightarrow \Gamma_F$ so that

$$|\psi(\gamma(i))|^* = |\gamma(i)|^* \quad |\psi(\gamma(i))|^\circ = |\gamma(i)|^\circ$$

for all $\gamma(i) \in \Gamma_E$.

We recall the notation $|\cdot|^*$ and $|\cdot|^\circ$ from (4.1). We usually identify $\Gamma = \Gamma_E = \Gamma_F$ when Definition 5.1 applies.

Lemma 5.2. *Let E and F be directed graphs with finitely many vertices. When $C^*(E) \simeq C^*(F)$, then we can replace E by E' and F by F' by moves of type (0), (I+), and (R+), so that the pair (E', F') is in augmented standard form.*

Proof. By Proposition 4.3, there are graphs E', F' in augmented standard form such that $C^*(E) \simeq C^*(E')$ and $C^*(F) \simeq C^*(F')$. In particular, $C^*(E') \simeq C^*(F')$, and so their ideal lattices are also isomorphic. Since Γ_E and Γ_F are reflected in the ideal structure of the C^* -algebras, the $*$ -isomorphism implements an order isomorphism in a way that the corresponding components define gauge simple C^* -algebras that are mutually isomorphic, cf. [ERRS18, Lemmas 3.16 & 4.17]. Thus the types in the trichotomy as well as the number of singular vertices are the same. Arguing as in STEP 6 of Proposition 4.3 we may component-wise increase the number of regular vertices on either side to match them up, and run the algorithm to the end from there to reestablish augmented canonical form. \square

Definition 5.3. Assume that (E, F) are in augmented standard form with $\Gamma = \Gamma_E = \Gamma_F$ and set $\underline{n} = (|\gamma|)_{\gamma \in \Gamma}$ and $\underline{m} = (|\gamma|^*)_{\gamma \in \Gamma}$. We say that the graphs are GL_Γ -equivalent if there exist invertible $U \in \mathcal{M}_\Gamma(\underline{n})$ and $V \in \mathcal{M}_\Gamma(\underline{m})$ so that

$$UB_E^* = B_F^*V.$$

If further all diagonal blocks in U and V can be chosen with determinant 1, we say that the graphs are SL_Γ -equivalent.

If U may be chosen so that $UD_E - D_F \in \text{im } B_F^*$ we say that E and F are GL_Γ^+ - or SL_Γ^+ -equivalent.

[ERRS21]	Add	Subtract	Here	Add	Subtract
Row	$E_{ij}B$	$E_{ij}^{-1}B$	Row	$(E_{ij}D, E_{ij}B)$	$(E_{ij}^{-1}(Bz + D), E_{ij}^{-1}B)$
Column	BE_{ij}	BE_{ij}^{-1}	Column	(D, BE_{ij})	(D, BE_{ij}^{-1})

FIGURE 1. Left: Legal operations in [ERRS21]. Right: Legal operations here.

Let E_{ij} be the matrix which differs from the identity matrix only by a one in entry ij . We have established (in a way to be made precise in the ensuing proof) that the very special SL_{Γ}^+ -equivalences implemented by $(U, V) = (E_{ij}, I)$ or $(U, V) = (I, E_{ij})$ are given by moves. This we generalize as follows:

Theorem 5.4. *Let E and F be graphs with finitely many vertices so that the pair (E, F) is in augmented standard form and are SL_{Γ}^+ -equivalent. Then E may be transformed to F by moves of the type (0) , $(I+)$ and $(R+)$.*

Proof. Our aim is to go from (D_E, B_E) to (D_F, B_F) by row and column operations, visiting graphs specified by $(D^{(k)}, B^{(k)})$ in augmented canonical form along the way, with $(D^{(0)}, B^{(0)}) = (D_E, B_E)$ and reaching (D_F, B_F) at the end. This requires in particular that

$$d_i^{(k)} \geq 1 \quad b_{ij}^{(k)} \geq 0 \quad b_{ii}^{(k)} \geq -1$$

everywhere.

That this is possible for the B matrices is exactly proved in [ERRS21, Theorem 9.10]. More precisely, since the graphs represented by B_E and B_F are in standard form and SL_{Γ} -equivalent, a sequence of “legal” row and column operations is specified to obtain $B^{(k+1)}$ from $B^{(k)}$, going from $B^{(0)} = B_E$ to $B^{(M)} = B_F$. It is always legal to add row i to row j when $\gamma(i) \geq \gamma(j)$, but to perform the corresponding row subtraction we further need to ensure that $B^{(k+1)}$ remains in augmented standard form with $B^{(k)}$ (roughly speaking by not taking too much away). The same applies to column operations, but these are further restricted to the realm of regular vertices.

With E_{ij} the operation matrix introduced above, it is clear that the operations described are given as in the table to the left in Figure 1. Note also that the legality conditions imply that $E_{ij} \in \mathcal{M}_{\Gamma}(n)$ throughout. We use the dangerous convention that $\infty - \infty = \infty$ in row subtractions, cf. Remark 4.6.

The operation matrices implement U and V in the sense that the product of matrices acting from the left is U and the product of the matrices acting from the right becomes V after deletion of row and columns corresponding to singular vertices. We denote by $U^{(k)}$ the matrix obtained by multiplying all operator matrices applied from the left to reach step k , with $U^{(k+1)} = U^{(k)}$ whenever the operation is performed on the right.

It follows directly from Propositions 4.1 and 4.9 that we can extend three of these operations to pairs representing graphs in augmented canonical form, but

row subtraction requires care, since we may only meaningfully subtract row i from row j when $d_j > d_i$. However, as a consequence of the fact that every matrix $B^{(k)}$ is in augmented standard form with its predecessor, one may check that the procedure given in [ERRS21, Section 9] never makes a row subtraction of row i from row j unless there is a regular $\boxed{\ell}$ so that $b_{i\ell} < b_{j\ell}$. Hence, we may use Theorem 4.8 to pass to a pair where $d_i < d_j$ before effectuating the operation. This is indicated to the right in Figure 1. Here \mathbf{z} is a multiple of the basis vector \mathbf{e}_ℓ ; in particular it vanishes on all i corresponding to singular \boxed{i} .

We define $(D^{(k)}, B^{(k)})$ by these operations, and claim that

$$D^{(k)} = U^{(k)}D^{(0)} + B^{(k)}\mathbf{x}^{(k)} \tag{5.1}$$

with $\mathbf{x}^{(k)}$ a vector which vanishes on all singular entries. Indeed for row subtractions we have

$$\begin{aligned} D^{(k+1)} &= E_{ij}^{-1}(B^{(k)}\mathbf{z} + D^{(k)}) \\ &= B^{(k+1)}\mathbf{z} + E_{ij}^{-1}U^{(k)}D^{(0)} + E_{ij}^{-1}B^{(k)}\mathbf{x}^{(k)} \\ &= U^{(k+1)}D^{(0)} + B^{(k+1)}\mathbf{x}^{(k+1)} \end{aligned}$$

with $\mathbf{x}^{(k+1)} = \mathbf{z} + \mathbf{x}^{(k)}$, and the same with $\mathbf{z} = \mathbf{0}$ for row additions. For column operations we have $D^{(k+1)} = D^{(k)}$, but we must set $\mathbf{x}^{(k+1)}$ to either $E_{ij}^{-1}\mathbf{x}^{(k)}$ or $E_{ij}\mathbf{x}^{(k)}$ as appropriate.

We now know that we can go from $(D_E, B_E) = (D^{(0)}, B^{(0)})$ to $(D^{(M)}, B^{(M)})$ using moves (0), (I+), and (R+), and we know that $B^{(M)} = B_F$ as desired. We have by (5.1) that $D^{(M)}$ and $UD^{(0)}$ define the same element in $\text{cok } B^{(M)}$, and by our assumption we know this is also the same element as the one defined by D_F . In other words, we can write

$$D^{(M)} - D_F = B_F\mathbf{y}$$

with $y_i = 0$ for singular i . Redistributing according to signs we get

$$D^{(M)} + \sum_{i=1}^n y'_i(B_F)_i = D_F + \sum_{i=1}^n y''_i(B_F)_i = D''$$

with all $y'_i, y''_i \geq 0$, so we may apply Theorem 4.8 to take both $(D^{(M)}, B_F)$ and (D_F, B_F) to (D'', B_F) .

Finally, we note that after reorganizing the vertices in each component so that the singular vertices are listed last, the columns in B_E and B_F corresponding to singular vertices are in fact identical, since they are completely determined by the information in Γ because of (III) in the definition of augmented canonical form. These columns will not be affected by the moves we performed, and hence this part of the matrices requires no further attention. The proof is complete. \square

Remark 5.5. It is possible to use the result above in combination with observations in [ER] to reprove the main result of [COR], which is obtained there by

substantially different methods. This work was known to us prior to obtaining the result presented here, and indeed was a key motivation for it.

We are now ready to present our main theorem. We will be deliberately vague about the K -theoretical invariant used – see [ERRS21, Section 2.6] for details.

Corollary 5.6. *Let E and F be graphs with finitely many vertices. Then the following are equivalent*

- (1) E can be obtained from F by moves of the type (O), (I+), (R+), (C+), (P+),
 - (2) $C^*(E) \cong C^*(F)$, and,
 - (3) The filtered, ordered, pointed K -theories of $C^*(E)$ and $C^*(F)$ are isomorphic.
- When (E, F) are in augmented standard form, they are also equivalent to
- (4) E is GL_Γ^+ -equivalent to F .

Proof. We proved (2) \iff (3) in [ERRS21, Corollary 3.6], and (1) \implies (2) was proved in [ER] as noted in Theorem 2.5.

Assuming (2), we note by Lemma 5.2 that we may pass without loss of generality to the case when E and F are in augmented standard form. It is proved in [ERRS21, Theorem 14.6] that (4) then holds. Appealing further to [ERRS21, Section 11-12], we may change the graphs by moves to arrive at two graphs that are SL_Γ^+ -equivalent. Indeed, in these two sections a pair of graphs in standard form are revised by a finite number of changes of the form

- The move (C),
- The move (P),
- Simple expansions by move (R) in reverse.

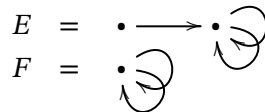
to arrange for the original pair (U, V) giving a GL_Γ -equivalence to be replaced by one with determinants 1 by an inductive procedure. The procedure also involves rearranging for standard form by a number of row operations after each step.

Starting with a pair of GL_Γ^+ -equivalent graphs in augmented standard form, we do the same, but use (C+), (P+), and (R+) instead to obtain a pair that is SL_Γ^+ -equivalent. To do the (R+) move in reverse we have to have an appropriate selection of antennae to delete, but since these changes are only applied to vertices i for which $\gamma(i)$ satisfies (IV) of augmented canonical form, this is easily arranged by Theorem 4.8. Applying the algorithm in Proposition 4.3 from STEP 4 onwards reestablishes augmented standard form without changing SL_Γ^+ -equivalence.

The argument is completed by Theorem 5.4. □

As explained in [ERRS21, Section 14.2], all conditions are decidable because of [BS].

Remark 5.7. With



is it well known that both (filtered, ordered, pointed) K -theories vanish, and hence Corollary 5.6 shows that there is a path through moves from one to the other. Our proof is in principle constructive, but since it requires us to place the pair of graphs in augmented standard form both to apply the results of Section 4 and to appeal to [ERRS21] (because of the use of a similar condition in [Boy02]), this involves passing to rather large graphs as in Example 4.4.

Here, and in many other settings, shortcuts can be obtained by going back into the proofs in Section 3. The objective for this pair is to adjust the D -vector, and because of the essential need for an edge between two non-sources in Proposition 3.3, this is impossible for E and F as given. But outsplitting only once gives the data, respectively

$$(D_{E'}, B_{E'}) = \left(\begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right)$$

$$(D_{F'}, B_{F'}) = \left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right),$$

for which two column additions into the $D_{F'}$ -vector obviously takes one graph into the other. This may be unraveled to the sequence of moves given in the introduction.

References

- [Boy02] BOYLE, MIKE. Flow equivalence of shifts of finite type via positive factorizations. *Pacific J. Math.* **204** (2002), no. 2, 273–317. [MR1907894](#) (2003f:37018), [Zbl 1056.37008](#), doi: [10.2140/pjm.2002.204.273](#). 928, 956
- [BH03] BOYLE, MIKE; HUANG, DANRUN. Poset block equivalence of integral matrices. *Trans. Amer. Math. Soc.* **355** (2003), no. 10, 3861–3886. [MR1990568](#) (2004f:15020), [Zbl 1028.15006](#), doi: [10.1090/S0002-9947-03-02947-7](#). 928
- [BS] BOYLE, MIKE; STEINBERG, BENJAMIN. Decidability of flow equivalence and isomorphism problems for graph C^* -algebras and quiver representations. *Proc. Amer. Math. Soc.*, to appear. [arXiv:1812.04555](#), doi: [10.1090/proc/15148](#). 955
- [COR] CARLSEN, TOKE MEIER; ORTEGA, EDUARD; RESTORFF, GUNNAR. Diagonal preserving isomorphisms between unital graph C^* -algebras. In preparation. 954
- [ERRS18] EILERS, SØREN; RESTORFF, GUNNAR; RUIZ, EFREN; SØRENSEN, ADAM P.W. Geometric classification of graph C^* -algebras over finite graphs. *Canad. J. Math.* **70** (2018), no. 2, 294–353. [MR3759003](#), [Zbl 1396.46047](#), [arXiv:1604.05439](#), doi: [10.4153/CJM-2017-016-7](#). 930, 932, 942, 943, 947, 952
- [ERRS21] EILERS, SØREN; RESTORFF, GUNNAR; RUIZ, EFREN; SØRENSEN, ADAM P.W. The complete classification of unital graph C^* -algebras: geometric and strong. *Duke Math J.* **170** (2021), no. 11, 2421–2517. [MR4302548](#), [Zbl 1476.46064](#), [arXiv:1611.07120](#), doi: [10.1215/00127094-2021-0060](#). 928, 929, 930, 932, 942, 943, 946, 947, 953, 954, 955, 956
- [ER] EILERS, SØREN; RUIZ, EFREN. Refined moves for structure-preserving isomorphism of graph C^* -algebras. Preprint, 2019. [arXiv:1908.03714](#). 929, 930, 932, 954, 955
- [FLR00] FOWLER NEAL J.; LACA, MARCELO; RAEBURN, IAIN. The C^* -algebras of infinite graphs. *Proc. Amer. Math. Soc.* **128** (2000), no. 8, 2319–2327. [MR1670363](#) (2000k:46079), [Zbl 0956.46035](#), doi: [10.1090/S0002-9939-99-05378-2](#). 930

- [Fra84] FRANKS, JOHN. Flow equivalence of subshifts of finite type. *Ergodic Theory Dynam. Systems*, **4** (1984), no. 1, 53–66. [MR0758893](#) (86j:58078) [Zbl 0555.54026](#), doi: [10.1017/S0143385700002261](#). [928](#)
- [Tom03] TOMFORDE, MARK. The ordered K_0 -group of a graph C^* -algebra. *C. R. Math. Acad. Sci. Soc. R. Can.*, **25** (2003), no. 1, 19–25. [MR1962131](#) (2003m:46104), [Zbl 1046.46052](#), [arXiv:math/0204095](#). [933](#)

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