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ON THE FOURIER COEFFICIENTS OF A FUNCTION OF Λ – BOUNDED VARIATION

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1. One of generalization of a concept of bounded variation is studied by S. J. Perlman [6], R. Pleissner [5], R. Pleissner [5], S. J. Perlman and D. Waterman [7] and D. Waterman [8] [9] [10] [11] [12].

DEFINITION. *f is of Λ-bounded variation on the interval $I = [a, b]$, $(\Lambda - BV)$, if*

$$\sum_{i=1}^{\infty} |f(I_i)|/\lambda_i < \infty$$

for any decomposition $\{I_i\}$ of I, where $\Lambda = \{\lambda_i\}$ is an increasing sequence of positive numbers such that $\sum \lambda_i^{-1} = \infty$ and

$$f(I_i) = f(b_i) - f(a_i) \text{ for } I_i = [a, b_i].$$

The fundamental prorerties of function of this class are given in the following.

[I] $\Lambda - BV \subset L^{\infty}$.

[II] *The function of $\Lambda - BV$ has only discontinuous points of the first kind, so, at most denumerable.* $\Lambda - BV \subset W$, (c.f. B. I. Golubov [2]).

[III] *The Helly's selection theorem holds for these functions.*

[IV] *The followings are equivalent.*

(i) $f \in \Lambda - BV$.

(ii) *There exists a $M > 0$ such that $\sum |f(I_i)|/\lambda_i < M$ for every decomposition $\{I_i\}$ of I,*

(iii) *There exists a $M > 0$ such that for every finite collection $\{I_i\}$ $(i = 1, 2, \ldots, N) \subset I$,*

$$\sum_{I}^{N} |f(I_i)|/\lambda_i < M.$$

[V] $\Lambda - BV$ *is a Banach space with the norm*

$$\|f\|_{\Lambda - BV} + |f(a)| \leq V_\Lambda(b),$$

where $V_\Lambda(b) = \sup\{\sum |f(I_i)|/\lambda_i;\ \{I_i\}$ *such that* $I = \cup I_i\}$.

[VI] *If* $\{\lambda_i\}$ *is a strctly sequence,* $BV \underset{\neq}{\subset} \Lambda - BV$.

[VII] $BV = \cup\{\Lambda - BV; \Lambda\}$.

[VIII] $\Lambda - BV \cap C$ *is a closed subspace of* $\Lambda - BV$.

2. Let f be an 2π-periodic integrable function on $[0, 2\pi)$ and $\{a_n\}$ and $\{b_n\}$ are Fourier coefficients of f. At first we show the order of the magnitude $\{a_n\}$ and $\{b_n\}$ of $f \in \Lambda - BV$.

LEMMA. *If* $A \in \Lambda - BV$, *then*

$$a_n, b_n = O(\lambda_n/n). \tag{1}$$

COROLLARY. If $f \in \{n^\alpha\} - BV$, $0 \leqq \alpha \leqq 1$, then

$$a_n, b_n = O(1/n^{1-\alpha}). \tag{2}$$

PROOF OF LEMMA. From (iii) of [IV], we have

$$\sum_1^{2N} f(I_i^x)/\lambda_i < M$$

for some $M > 0$, where $I_i^x = x + (i-1)\pi/N,\ x + i\pi/N]$ $(i = 1, 2, \ldots 2N)$, that is

$$\sum_1^{2N} |f(I_i^x)| = O(\lambda_{2N}).$$

From the properties of λ_n, we assume that $\lambda_{2n} = O(\lambda_n)$, so,

$$\sum_1^{2N} |f(I_i^x)| = O(\lambda_N). \tag{3}$$

It is well known (c.f. N. K. Bari [1] and M. and S. Izumi [3])

$$\begin{aligned} |a_N| &\leq (1/2\pi) \int_0^{2\pi} |f(x + \pi/N) - f(x)| dx \\ &\leq (1/2\pi) \int_0^{2\pi} |f(I_i^x)| dx, \quad (i = 1, 2, \ldots, 2N). \end{aligned}$$

Adding such inequalities for $i = l, 2, \dots, 2N$, we have (1) by (3). Similarly, we have $b_n = O(\lambda_n/n)$.
Now, we give the necessary condition for continuity of $\Lambda - BV$.

$$I_N = (N/\lambda_N) \sum_1^\infty \rho_n^2 \sin^2(n\pi/2N) = o(1). \tag{5}$$

$$J_N = (N/\lambda_N)^{-1} \sum_1^N n^2 \rho_n^2 = o(1). \tag{6}$$

$$T_N = N^{-1}\lambda_N^{-1/2} \sum_1^N n\rho_n = o(1). \tag{7}$$

$$S_N = (\log N)^{-1}\lambda_N^{-1/2} \sum \rho_n = o(1). \tag{8}$$

$$H_N = N\lambda_N^{-1} \sum_1^N \rho_N^2 = o(1). \tag{9}$$

THEOREM 1. *If $f \in \Lambda - BV$, then we have*

(i) $f \in C \Rightarrow (5) \Rightarrow (6) \Rightarrow (7) \Rightarrow (8)$.

(ii) $(9) \Rightarrow (6)$,

where $p_n = \{a_n^2 + b_n^2\}^{1/2}$.

$$I_N^{(\alpha)} = N^{1-\alpha} \sum_1^\infty \rho_n^2 \sin^2(n\pi/2N) = o(1). \tag{5$'$}$$

$$J_N^{(\alpha)} = N^{-(1+\alpha)} \sum_1^N n^2 \rho_n^2 = o(1). \tag{6$'$}$$

$$T_N^{(\alpha)} = N^{-(1+\alpha/2)} \sum_1^N n\rho_n = o(1). \tag{7$'$}$$

$$S_N^{(\alpha)} = \left\{ \begin{array}{ll} N^{-\alpha/2} \sum_1^N \rho_n; & 0 < \alpha < 1 \\ (\log N)^{-1} \sum_1^N \rho_n; & \alpha = 1 \end{array} \right\} = o(1) \tag{8$'$}$$

$$H_N^{(\alpha)} = N^{1-\alpha} \sum_N^\infty \rho_n^2 = o(1). \tag{9$'$}$$

COROLLARY 2. *If $f \in \{n^\alpha\} - BV$ $(0 \le \alpha \le 1)$, then we have*

(i) $f \in C \Rightarrow (5') \Rightarrow (6) \Rightarrow (7') \Rightarrow (8')$.
(ii) $(9') \Rightarrow (6')$.

THEOREM 2. *If* $f \in \{n^\alpha\} - BV$ $(0 \leq \alpha < 1/2)$ *and*

$$\hat{J}_N^{(\alpha)} = N^{-(1+\alpha)} \sum_1^{[N^\beta]} n^2 \rho_2^n = o(1) \tag{10}$$

for same $\beta > (1-\alpha/1-2\alpha)$, *then we have* $(6')$.

REMARK 1; For $f \in BV$, these results have been got by N. Wiener [13] and S. M. Lozinskii [4].

REMARK 2; If f is of rth bounded variation, the similar results are given by B. I. Golubov [2].

PROOF OF THEOREM 1

(i) $f \in C \Rightarrow (5)$; From (iii) of [IV], we have

$$\sum_1^{2N} |f(I_i^x)|^2 = \sum_1^{2N} |f(I_i^x)|/\lambda_i \cdot \lambda_i |f(I_i^x)| < M\lambda_2 N\omega_f(\pi/N)$$

where $\omega_f(.)$ is a modulous of continuity of f. Then, from $\lambda_{2N} = O(\lambda_N)$, we get

$$2N \int_0^{2\pi} |f(I_i^x)|^2 dx = O(\lambda_N \omega_j(\pi/N)),$$

where $I^x = [x-\pi/N, x+\pi/N]$. By Parseval's equality,

$$I_N = (N/\lambda_N) \sum_1^\infty \rho_n^2 \sin^2(n\pi/2N) = O(\omega_f(\pi/N)) = o(1).$$

$(6) \Rightarrow (7)$; From Schwartz's inequality and (6), we get

$$T_N^2 = (N^2\lambda_N)^{-1} \left(\sum_1^N n\rho_n \right)^2 < (N\lambda_N)^{-1} \sum_1^N n^2 \rho_n^2 = J_N = o(1).$$

$(7) \Rightarrow (8)$; Putting $u_N = \sum_1^N n\rho_n$, then $u_N = o(N\lambda_N^{1/2})$ and

$$\begin{aligned} \sum_1^N \rho_n &= \sum_1^N \frac{1}{n}(u_n - u_{n-1}) \\ &= (u_N/N) + \sum_1^{N-1} (n+1)^{-1}(u_n/n) \\ &= o(\lambda_N^{1/2}) + o(\lambda_N^{1/2}) \sum_1^{N-1} (1/n+1) \\ &= o(\lambda_N^{1/2} \cdot \log N). \end{aligned}$$

(ii) (9) $\Rightarrow$ (6); Putting $A_N = \sum_N^\infty \rho_n^2$, we hawe $A_N = o(\lambda_N/N)$ from (9). So,

$$\begin{aligned} J_N &= (N\lambda_N)^{-1}\sum_1^N n^2\rho_n^2 \\ &= (N\lambda_N)^{-1}\left\{N^2A_N - \sum_1^{N-1}(2n+1)A_n\right\} \\ &= o(1) + o\left(1/N\lambda_N\cdot\sum_1^{N-1}(2n+1)(\lambda_n/n)\right) = o(1). \end{aligned}$$

PROOF OF THEOREM 2.

$$\begin{aligned} I_N^{(\alpha)} &= N^{1-\alpha}\sum_1^\infty n^2\sin^2(n\pi/2N) \\ &= N^{1-\alpha}\left\{\sum_1^{[Nx]}\rho_n^2\sin(n\pi/2N) + \sum_{[Nx]+1}^\infty \rho_n^2\sin^2(n\pi/2N\right\} \\ &= I_{N,1}^{(\alpha)} + I_{N,2}^{(\alpha)} \text{ for some } x>0. \end{aligned}$$

From Corollary 1, we have $\rho_n = O(n^{\alpha-1})$. So, accounting of $0 \leqq \alpha < 1/2$,

$$\begin{aligned} I_{N,2}^{(\alpha)} &= O\left(N^{1-\alpha}\sum_{[Nx]+1}^\infty n^{2\alpha-2}\right) \\ O &= \left(N^{1-\alpha}\int_{Nx}^\infty t^{2\alpha-2}dt\right) \\ O &= (N^{1-\alpha}(Nx)^{2\alpha-1}) = O(N^\alpha x^{2\alpha-1}). \end{aligned}$$

Putting $x = N^{\beta-1}$, then

$$I_{N,2}^{(\alpha)} = O(N^{(1-\alpha)-\beta(1-2\alpha)}) = o(1).$$

Further,

$$\begin{aligned} I_{N,1}^{(\alpha)} &= N^{1-\alpha}\sum_{n=1}^{[N\beta]}\rho_n^2(n\pi/2n)^2 \\ &= O\left(N^{-(1+\alpha)}\sum_1^{[N\beta]}\rho_n^2n^2\right) = O(\hat{J}_N^{(\alpha)}) = o(1). \end{aligned}$$

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