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SUBSPACES OF RECURRENT FINSLER SPACES

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1. Introduction. In the Finsler space F_n the metric function is $L(x, \dot{x})$. Let us define the m vector fields $B_a^\alpha(x, \dot{x})$ and $n-m$ vector fields $N_k^\alpha(x, \dot{x})$

$$\begin{aligned} &\alpha, \beta, \gamma, \delta, \varepsilon, \varkappa, \dots = 1, 2, \dots, n \\ &a, b, c, d, e, f, \ \dots = 1, 2, \dots, m \\ &k, l, m, n, p, q, \dots = m+1, \dots, n \end{aligned}$$

in such a way that these vector fields are linearly independent at each $(x, \dot{x})$ and satisfy the relations

$$g_{\alpha\beta} = B_a^\alpha N_k^\beta = 0 \tag{1.1}$$

for each $a = 1, 2, \dots, m \quad k = m+1, \dots n$. Let us define

$$g_{ab} = g_{\alpha\beta} B_{ab}^{\alpha\beta} \tag{1.2}$$

$$g_{kl} = g_{\alpha\beta} N_k^\alpha N_l^\beta \tag{1.3}$$

$$B_\beta^b = g^{ab} g_{\alpha\beta} B_a^\alpha \tag{1.4}$$

$$N_\alpha^k = g^{km} g_{\alpha\beta} N_m^\beta \tag{1.5}$$

$g_{\alpha\beta}$, B_a^α and N_k^α have zero degree of homogenity in $\dot{x}$, g^{ab} and g^{km} are inverse matrices of g_{ab} and g_{km}, respectively. From (1.3) and (1.5) we have

$$N_\alpha^k N_p^\alpha = g^{kl} g_{\alpha\beta} N_l^\beta N_p^\alpha = g^{kl} g_{lp} = \delta_p^k. \tag{1.6}$$

As usually

$$\delta_\beta^\alpha = B_a^\alpha B_\beta^a + N_k^\alpha N_\beta^k \tag{1.7}$$

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Let us denote the absolute differential which corresponds to the motion from $(x, \dot{x})$ to $(x + dx, \dot{x} + d\dot{x})$ by D. Then we may write

$$dx^\alpha = B^\alpha_a du^a + N^\alpha_k dv^k, \quad (1.8)$$
$$\dot{x}^\alpha = B^\alpha_a \dot{u}^a + N^\alpha_k N^\alpha_k \dot{v}^k.$$

If $\xi^\alpha(x, \dot{x})$ is a vector field in F_n, then $\xi^\alpha = B^\alpha_a + N^\alpha_k \xi^k$. The induced differentials $\overline{D}\xi^a$, $\overline{D}\xi^k$ are defined by $B^a_\alpha D\xi^\alpha = \overline{D}\xi^a$, $N^k_\alpha D\xi^\alpha = \overline{D}\xi^k$ and

$$D\xi^\alpha = B^\alpha_a \overline{D}\xi^a + N^\alpha_k \overline{D}\xi^k \quad (1.9)$$

We shall use the notation $l^\alpha = L^{-1}(x, \dot{x})\dot{x}^\alpha = L^{-1}(B^\alpha_a \dot{u}^a + N^\alpha_k \dot{v}^k) = B^\alpha_a + N^\alpha_k l^k$ where $l^a = L^{-1}\dot{u}^a$, $l^k = L^{-1}\dot{v}^k$.

From (1.8) we have

$$Dl^\alpha = B^\alpha_a \overline{D}l^a + N^\alpha_k \overline{D}l^k \quad (1.10)$$

2. The connection coefficients for the recurrent Finsler space. We shall suppose that the metric tensor is determined by

$$g_{\alpha\beta}(x, \dot{x}) = \dot{\partial}_\alpha \dot{\partial}_\beta L^2(x, \dot{x})/2 \quad (2.1)$$

and that space F_n is recurrent, i.e.

$$Dg_{\alpha\beta} = K(x, \dot{x})g_{\alpha\beta} \quad (2.2)$$

As

$$Dg_{\alpha\beta} = g_{\alpha\beta|\gamma}dx^\gamma + g_{\alpha\beta}|_\gamma Dl^\gamma$$

(2.2) is valid if

$$g_{\alpha\beta|\gamma} = \lambda_\gamma(x, \dot{x})g_{\alpha\beta} \quad (2.3)$$
$$g_{\alpha\beta}|_\gamma = \mu_\gamma(x, \dot{x})g_{\alpha\beta} \quad (2.4)$$
$$K(x, \dot{x}) = \lambda_\gamma(x, \dot{x})dx^\gamma + \mu_\gamma(x, \dot{x})Dl^\gamma. \quad (2.5)$$

The absolute differential of $g_{\alpha\beta}$ is

$$Dg_{\alpha\beta} - dg_{\alpha\beta} - (\Gamma^{*\delta}_{\alpha\gamma} + \Gamma^{*\delta}_{\beta\gamma}dx^\gamma) - (A^\delta_{\alpha\gamma}g_{\delta\beta} + A^\delta_{\beta\gamma}g_{\alpha\delta})Dl^\gamma. \quad (2.6)$$

We shall determine connection coefficients under condition (2.2) and

$$\Gamma^{*\delta}_{\alpha\gamma} = \Gamma^{*\delta}_{\gamma\alpha} \quad (2.7)$$
$$A^\delta_{\alpha\gamma} = A^\delta_{\gamma\alpha} \quad (2.8)$$

which have a well known geometric interpretation. Substituting

$$(2.9)\quad dg_{\alpha\beta} = \partial_\gamma g_{\alpha\beta}dx^\gamma + \dot{\partial}_\gamma g_{\alpha\beta}d\dot{x}^\gamma =$$
$$= \partial_\gamma g_{\alpha\beta}dx^\gamma + \dot{\partial}_\gamma g_{\alpha\beta}(L_D l^\gamma + l^\gamma DL - \Gamma^{*\gamma}_\delta dx^\delta - LA^\gamma_{0\delta}Dl^\delta)$$

into (2.6) and using (2.3), (2.4) we get

$$g_{\alpha\beta|\delta} = \partial_\gamma g_{\alpha\beta} - \dot{\partial}_\delta g_{\alpha\beta}\Gamma^{*\delta}_{\gamma} - \Gamma^{*\delta}_{\alpha\gamma} g_{\delta\beta} - \Gamma^{*\delta}_{\beta\gamma} g_{\alpha\delta} = \lambda_\gamma g_{\alpha\beta} \tag{2.10}$$

$$g_{\alpha\beta|\gamma} = L\dot{\partial}_\delta g_{\alpha\beta}(\delta^\delta_\gamma - A^\delta_{0\gamma}) - A^\delta_{\alpha\gamma} g_{\delta\beta} - A^\delta_{\beta\gamma} g_{\alpha\delta} = \mu_\gamma g_{\alpha\beta}. \tag{2.11}$$

Using (2.7) and (2.10) from the expression for $g_{\alpha\beta|\gamma} + g_{\beta\gamma|\alpha} - g_{\gamma\alpha|\beta}$ we obtain

$$2\Gamma^*_{\alpha\beta\gamma} = (\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma}) - -(\dot{\partial}_\delta g_{\alpha\beta}\Gamma^{*\delta}_{\gamma} + \\ + \dot{\partial}_\delta g_{\beta\gamma}\Gamma^{*\delta}_{\alpha} - \dot{\partial}_\delta g_{\alpha\gamma}\Gamma^{*\delta}_{\beta}) - (\lambda_\gamma g_{\alpha\beta} + \lambda_\alpha g_{\beta\gamma} - \lambda_\beta g_{\alpha\gamma}) \tag{2.12}$$

In the same way using (2.11) and (2.8) we have

$$2A_{\alpha\beta\gamma} = L(\dot{\partial}_\alpha g_{\beta\gamma} + \dot{\partial}_\gamma g_{\alpha\beta} - \dot{\partial}_\beta g_{\alpha\gamma}) - L(\dot{\partial}_\delta g_{\alpha\beta} A^\delta_{0\alpha} + \\ + \dot{\partial}_\delta g_{\alpha\beta} A^\delta_{0\gamma} - \dot{\partial}_\delta g_{\alpha\gamma} A^\delta_{0\beta}) - (\mu_\alpha g_{\beta\gamma} + \mu_\gamma g_{\alpha\beta} - \mu_\beta g_{\alpha\gamma}) \tag{2.13}$$

On the right-hand side of (2.12) are the undetermined quantities

$$\Gamma^{*\delta}_{\alpha} = L\Gamma^{*\delta}_{0\alpha}.$$

Multiplying (2.12) by l^α we get

$$2\Gamma^*_{0\beta\gamma} = (\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma}) l^\alpha - L\dot{\partial}_\delta g_{\beta\gamma}\Gamma^{*\delta}_{00} - (\lambda_\gamma l_\beta + \lambda_0 g_{\beta\gamma} - \lambda_\beta l_\gamma) \tag{2.14}$$

Multiplying (2.14) by l^γ we obtain

$$2\Gamma^*_{0\beta 0} = (\partial_\gamma g_{\alpha\beta} + \partial_\alpha g_{\beta\gamma} - \partial_\beta g_{\alpha\gamma}) l^\alpha l^\gamma - (2\lambda_0 l_\beta + \lambda_\beta) \tag{2.15}$$

and now $\Gamma^*_{\alpha\beta\gamma}$ is determined by (2.12), (2.14) and (2.15).

In the same way from (2.13) we obtain

$$2A_{\overline{0}\beta\gamma} - L\dot{\partial}_\delta g_{\beta\gamma} A^\delta_{00} - (\mu_\alpha l^\alpha g_{\beta\gamma} - \mu_\gamma l^\beta - \mu_\beta l_\gamma) \tag{2.16}$$

$$2A_{0\beta 0} = -(2\mu_0 l_\beta - \mu_\beta). \tag{2.17}$$

3. The induced connection for the subspace of the recurrent space. Using the method of O. Varga we define DB^α_a and DN^α_k as vectors in F_n which may be decomposed at each lineelement $(x, \dot{x})$ in the direction of the vectors B^α_a and N^α_k

$$DB^\alpha_a = \overline{w}^d_a(d) B^\alpha_d + \overline{w}^m_a(d) N^\alpha_m \tag{3.1}$$

where

$$\overline{w}^d_a(a) = \overline{\Gamma}^{*d}_{ab} du^b + \overline{\Gamma}^{*d}_{ak} dv^k + \overline{A}^d_{ab}\overline{D}l^b + \overline{A}^d_{ak}\overline{D}l^k, \tag{3.2}$$

$$\overline{w}^m_a(a) = \overline{\Gamma}^{*m}_{ab} du^b + \overline{\Gamma}^{*m}_{ak} dv^k + \overline{A}^m_{ab}\overline{D}l^b + \overline{A}^m_{ak}\overline{D}l^k, \tag{3.3}$$

and

(3.4) $$DN_k^\alpha = \overline{w}_k^d(d)B_d^\alpha + \overline{w}_k^m(d)N_m^\alpha,$$

where

(3.5) $$\overline{w}_k^d(d) = \overline{\Gamma}_{kb}^{*d}du^b + \overline{\Gamma}_{kn}^{*d}dv^n + \overline{A}_{kb}^d\overline{D}l^b + \overline{A}_{kn}^d\overline{D}l^n$$

(3.6) $$\overline{w}_k^m(d) = \overline{\Gamma}_{kb}^{*m}du^b + \overline{\Gamma}_{kn}^{*m}dv^n + \overline{A}_{kb}^m\overline{D}l^b + \overline{A}_{kn}^m\overline{D}l^n.$$

From (1.1) and (2.2) we have

$$D(g_{\alpha\beta}B_a^\alpha N_k^\beta) = g_{\alpha\beta}(DB_a^\alpha)N_k^b eta + g_{\alpha\beta}B_a^\alpha DN_k^\beta = 0$$

from which

$$g_{mk}(\overline{\Gamma}_{ab}^{*m}du^b + \overline{\Gamma}_{an}^{*m}dv^n + \overline{A}_{ab}^m\overline{D}l^b + \overline{A}_{an}^m\overline{D}l^n) = \\ -g_{ad}(\overline{\Gamma}_{kb}^{*d}du^b + \overline{\Gamma}_{kn}^{*d}dv^n + \overline{A}_{kb}^d\overline{D}l^b + \overline{A}_{kn}^d\overline{D}l^n)$$

follows.

From the relation above we obtain

(3.7) $$\begin{aligned} a)\quad & \overline{\Gamma}_{akb}^* = -\overline{\Gamma}_{kab}^* \\ b)\quad & \overline{\Gamma}_{akn}^* = -\overline{\Gamma}_{kan}^* \\ c)\quad & \overline{A}_{akb} = -\overline{A}_{kab} \\ d)\quad & \overline{A}_{akn} = -\overline{A}_{kan} \end{aligned}$$

i.e.

(3.8) $$\overline{w}_{ak} = -\overline{w}_{ka} \quad (g_{km}\overline{w}_b^m = -g_{ab}\overline{w}_k^b).$$

After some calculation using (1.8) and (1.16) we obtain

(3.9) $$DB_a^\alpha = (B_{a|\beta}^\alpha du^b + B_a^\alpha|_\beta \overline{D}l^b)B_b^\beta + \\ + (B_{a|\beta}^\alpha dv^k + B_a^\alpha|_\beta \overline{D}l^k)N_k^\beta$$

where

(3.10) $$B_{a|\beta}^\alpha = \partial_\beta B_a^\alpha - \dot{\partial}_\gamma B_a^k \Gamma_\beta^{*\gamma} + \Gamma_{\gamma\beta}^{*\alpha}B_a^\gamma$$

(3.11) $$B_a^\alpha|_\beta = L\dot{\partial}_\delta B_a^\alpha(\delta_\beta^\delta - A_{0\beta}^\delta) + A_{0\beta}^\alpha B_a^\delta$$

Comparing (3.9) and (3.1) using (3.2) and (3.3) we obtain

(3.12) $$\begin{aligned} \text{a)}\quad & \overline{\Gamma}_{acb}^* = g_{\alpha\delta}B_c^\delta B_b^\beta B_{a|\beta}^\alpha \\ \text{b)}\quad & \overline{\Gamma}_{ack}^* = g_{\alpha\delta}B_c^\delta N_k^\beta B_{a|\beta}^\alpha \\ \text{c)}\quad & \overline{A}_{acb} = g_{\alpha\delta}B_c^\delta B_b^\beta B_a^\alpha|_\beta \\ \text{d)}\quad & \overline{A}_{ack} = g_{\alpha\delta}B_c^\delta N_k^\beta B_a^\alpha|_\beta \\ \text{e)}\quad & \overline{\Gamma}_{anb}^* = g_{\alpha\delta}N_n^\delta B_b^\beta B_{a|\beta}^\alpha \\ \text{f)}\quad & \overline{\Gamma}_{ank}^* = g_{\alpha\delta}N_n^\delta N_k^\beta B_{a|\beta}^\alpha \\ \text{g)}\quad & \overline{A}_{anb} = g_{\alpha\delta}N_n^\delta B_b^\beta B_a^\alpha|_\beta \\ \text{h)}\quad & \overline{A}_{ank} = g_{\alpha\delta}N_n^\delta N_k^\beta B_a^\alpha|_\beta \end{aligned}$$

On the other hand we have for the vector field N_k^α

(3.13) $$DN_k^a lpha = (N_{k|\beta}^\alpha du^b + N_k^\alpha|_\beta \overline{D}l^b)B_b^\beta + (N_k^\alpha|_\beta dv^m + N_k^\alpha|_\beta \overline{D}l^m)N_m^\beta,$$

where

(3.14) $$N_{k|\beta}^\alpha = \partial_\beta N_k^\alpha - \dot{\partial}_\gamma N_k^\alpha \Gamma_\beta^{*\gamma} + \Gamma_{\gamma\beta}^{*\alpha} N_k^\gamma$$

(3.15) $$N_k^\alpha|_\beta = L\dot{\partial}_\delta N_k^\alpha(\delta_\beta^\delta - A_{0\beta}^\delta) + A_{\delta\beta}^\alpha N_k^\delta$$

multiplying (3.4) and (3.13) first with $g_{\alpha\delta}B_c^\delta$ and then with $g_{\alpha\delta}N_n^\delta$, and comparing the coefficients of du^b, dv^l and $\overline{D}l^l$ in these relations we get

(3.16)
$$\begin{aligned}
&\text{a)}\quad \overline{\Gamma}^*_{kcb} = g_{\alpha\delta}B_c^\delta B_b^\beta N_{k|\beta}^\alpha && \text{e)}\quad \overline{\Gamma}^*_{knb} = g_{\alpha\delta}N_n^\delta B_b^\beta N_{k|\beta}^\alpha\\
&\text{b)}\quad \overline{\Gamma}^*_{kcl} = g_{\alpha\delta}B_c^\delta N_l^\beta N_{k|\beta}^\alpha && \text{f)}\quad \overline{\Gamma}^*_{knl} = g_{\alpha\delta}N_n^\delta N_l^\beta N_{k|\beta}^\alpha\\
&\text{c)}\quad \overline{A}_{kcb} = g_{\alpha\delta}B_c^\delta B_b^\beta N_k^\alpha|_\beta && \text{g)}\quad \overline{A}_{knb} = g_{\alpha\delta}N_n^\delta B_b^\beta N_k^\alpha|_\beta\\
&\text{d)}\quad \overline{A}_{kcl} = g_{\alpha\delta}B_c^\delta N_l^\beta N_k^\alpha|_\beta && \text{h)}\quad \overline{A}_{knl} = g_{\alpha\delta}N_n^\delta N_l^\beta N_k^\alpha|_\beta
\end{aligned}$$

4. Recurrent subspaces of recurrent Finsler spaces. From (1.1), (1.2) and (1.3) it follows that

(4.1) $$g_{\alpha\beta} = g_{ab}B_{\alpha\beta}^{ab} + g_{nk}N_\alpha^n N_\beta^k$$

From (4.1) we obtain

(4.2)
$$\begin{aligned}
Dg_{\alpha\beta} &= dg_{ab}B_{\alpha\beta}^{ab} + g_{ab}DB_\alpha^a B_\beta^b + g_{ab}B_\alpha^a DB_\beta^b +\\
&+ dg_{nk}N_\alpha^n N_\beta^k + g_{nk}DN_\alpha^n N_\beta^k + g_{nk}N_\alpha^n DN_\beta^k
\end{aligned}$$

Using (1.7) we have

(4.3) $$0 = DB_\beta^b B_b^\gamma + B_\beta^b DB_b^\gamma + DN_\beta^n N_n^\gamma + N_\beta^n DN_n^\gamma.$$

On the other hand

(4.5)
$$\begin{aligned}
g_{ab}B_\beta^b &= g_{\alpha\varkappa}B_{ab}^{\alpha\chi}B_\beta^b = g_{\alpha\varkappa}B_a^\alpha(\delta_\beta^\chi - N_n^x N_\beta^n) = g_{\alpha\beta}B_a^\alpha.\\
g_{kn}N_\beta^k &= g_{\alpha\varkappa}N_k^\alpha N_n^\varkappa N_\beta^k = g_{\alpha\varkappa}N_n^\varkappa(\delta_\beta^\alpha - B_a^\alpha B_\beta^a) = g_{\beta\varkappa}N_n^\varkappa.
\end{aligned}$$

Using (4.3), (4.4) and (4.5) we have

(4.6)
$$\begin{aligned}
g_{ab}DB_\alpha^a B_\beta^b + g_{nk}DN_\alpha^n N_\beta^k &= g_{\varkappa\beta}B_a^\varkappa DB_\alpha^a + g_{\varkappa\beta}N_n^\varkappa DN_\alpha^n =\\
&= g_{\varkappa\beta}(B_\alpha^a DB_a^\varkappa + N_\alpha^n DN_n^\varkappa)
\end{aligned}$$

(4.7)
$$\begin{aligned}
g_{ab}B_\alpha^a DB_\beta^b + g_{nk}N_\alpha^n DN_\beta^k &= g_{\alpha\varkappa}B_b^\varkappa DB_\beta^b + g_{\alpha\varkappa}N_k^\varkappa DN_\beta^k =\\
&= g_{\alpha\varkappa}(B_\beta^b DB_b^\varkappa + N_\beta^k DN_k^\varkappa)
\end{aligned}$$

Substituting (4.6) and (4.7) into (4.2) and using (3.1) and (3.4) we get

(4.8)
$$\begin{aligned}
Dg_{\alpha\beta} &= B_{\alpha\beta}^{ab}dg_{ab} - g_{\beta\varkappa}B_\alpha^a(B_a^\varkappa \overline{w}_a^d + N_k^\varkappa \overline{w}_a^k) - g_{\beta\varkappa}N_\alpha^n(B_d^\varkappa \overline{w}_n^d + N_k^\varkappa \overline{w}_n^k) +\\
&+ N_\alpha^n N_\beta^k dg_{nk} - g_{\alpha\varkappa}B_\beta^b(B_d^\varkappa \overline{w}_b^d + N_k^\varkappa \overline{w}_\beta^k) - g_{\alpha\varkappa}N_\beta^k(B_d^\varkappa \overline{w}_k^d + N_n^\varkappa \overline{w}_n^k).
\end{aligned}$$

Using the relations

$$(4.9)\qquad \begin{aligned} &\text{a)}\quad \dot{d}u^a = L\overline{D}l^a - \overline{\Gamma}^{*a}_{c} du^c - \overline{\Gamma}^{*a}_{k} dv^k + l^a dL \\ &\text{b)}\quad \dot{d}v^k = L\overline{D}l^k - \overline{\Gamma}^{*k}_{c} du^c - \overline{\Gamma}^{*k}_{l} dv^l + l^k dL \end{aligned}$$

we obtain

$$(4.10)\qquad \begin{aligned} dg_{ab} = \partial_c g_{ab} du^c + \partial_k g_{ab} dv^k + \dot{\partial}_c g_{ab}(L\overline{D}l^c - \overline{\Gamma}^{*c}_{d} du^d - \overline{\Gamma}^{*c}_{k} dv^k + l^c dL) + \\ + \dot{\partial}_k g_{ab}(L\overline{D}l^c - \overline{\Gamma}^{*k}_{d} du^d - \overline{\Gamma}^{*k}_{t} dv^t + l^k dL) \end{aligned}$$

(4.10) is valid if g_{ab} is replaced by g_{nk}. Using again (4.4) and (4.5) some terms in (4.8) become zero, i.e.

$$(4.11)\qquad \begin{aligned} &- g_{\beta\varkappa} B^a_\alpha N^\varkappa_k \overline{w}^k_a - g_{\alpha\varkappa} N^k_\beta B^\varkappa_d \overline{w}^d_k = \\ &= -g_{tk} B^a_\alpha N^t_\beta \overline{w}^k_a - g_{da} N^k_\beta B^a_\alpha \overline{w}^d_k = B^a_\alpha N^t_\beta(-\overline{w}_{at} - \overline{w}_{ta}) = 0 \end{aligned}$$

because of (3.8). Similarly we get

$$(4.12)\qquad -g_{\beta\varkappa} N^n_\alpha B^\varkappa_d \overline{w}^d_n - g_{\alpha\varkappa} B^b_\beta N^\varkappa_k \overline{w}^k_b = -N^n_\alpha B^b_\beta(-\overline{w}_{nb} - \overline{w}_{bn}) = 0.$$

AS $g_{ab} = g_{\alpha\beta} B^\alpha_a B^\beta_b$ is homogeneous of degree zero in $\dot{x}$ so $\dot{\partial}_\varepsilon g_{ab} \dot{x}^\varepsilon = 0$ or

$$(4.13)\qquad \dot{\partial}_\varepsilon g_{ab}(B^\varepsilon_c \dot{u}^c + N^\varepsilon_k \dot{v}^k) = \dot{\partial}_c g_{ab} \dot{u}^c + \dot{\partial}_k g_{ab} \dot{v}^k = 0$$

Substituting (4.10), (4.11) (4.12) and (4.13) into (4.8) using (4.4), (4.5) we obtain

$$(4.14)\qquad \begin{aligned} &Dg_{\alpha\beta} = B^{ab}_{\alpha\beta}(g_{ab\top c} du^c + g_{ab\top k} dv^k + g_{ab}\top_c \overline{D}l^c + g_{ab}\top_k \overline{D}l^k) + \\ &N^{nt}_{\alpha\beta}(g_{nt\top c} du^c + g_{nt\top k} dv^k + g_{nt}\top_c \overline{D}l^c + g_{nt}\top_k \overline{D}l^k), \end{aligned}$$

where

$$(4.15)\qquad \begin{aligned} &\text{a)}\ g_{ab\top x} = \partial_x g_{ab} - \dot{\partial}_d g_{ab} \overline{\Gamma}^{*d}_{x} - \dot{\partial}_m g_{ab} \overline{\Gamma}^{*m}_{x} - g_{db} \overline{\Gamma}^{*d}_{ax} - g_{ad} \overline{\Gamma}^{*d}_{ax}, \\ &\quad x = c \text{ or } x = k, \\ &\text{b)}\ g_{mt\top x} = \partial_x g_{nt} - \dot{\partial}_d g_{ab} \overline{\Gamma}^{*d}_{x} - \dot{\partial}_m g_{nt} \overline{\Gamma}^{*m}_{x} - g_{mt} \overline{\Gamma}^{*m}_{nx} - g_{nm} \overline{\Gamma}^{*m}_{tx}, \\ &\quad x = c \text{ or } x = k, \\ &\text{c)}\ g_a \top_{bx} = L\dot{\partial}_x g_{ab} - g_{db} A^d_{ax} - g_{ad} A^d_{bx},\ \ x = c \text{ or } x = k, \\ &\text{d)}\ g_{nt} \top_x = L\dot{\partial}_x g_{nt} - g_{mt} A^m_{nx} - g_{nm} A^m_{tx},\ \ x = c \text{ or } x. = k, \end{aligned}$$

Using the relation

$$(4.16)\qquad Dg_{\alpha\beta} = B^\delta_c g_{\alpha\beta|\delta} du^c + N^\delta_k g_{\alpha\beta|\delta} dv^k + B^\delta_c g_{\alpha\beta}|_\delta Dl^c + N^\delta_k g_{\alpha\beta}|_\delta \overline{D}l^k$$

from (4.14) we obtain

$$(4.17)\qquad B^\delta_c g_{\alpha\beta\top\delta} = B^{ab}_{\alpha\beta} g_{ab\top c} + N^{nt}_{\alpha\beta} g_{nt\top c}$$

$$(4.18)\qquad N^\delta_k g_{\alpha\beta|\delta} = B^{ab}_{\alpha\beta} g_{ab\top k} + N^{nt}_{\alpha\beta} g_{nt\top k}.$$

Multiplying (4.17) by B^c_γ and (4.18) by N^k_γ and adding these relations we get

(4.19) $$g_{\alpha\beta|\gamma} = B^{abc}_{\alpha\beta\gamma} g_{ab\top c} + B^{ab}_{\alpha\beta} N^k_\gamma g_{ab\top k} + N^{nt}_{\alpha\beta} B^c_\gamma g_{nt\top c} + N^{ntk}_{\alpha\beta\gamma} g_{nt|k}.$$

In the same way we obtain

(4.20) $$B^\delta_c g_{\alpha\beta}|_\delta = B^{ab}_{\alpha\beta} g_{ab} \top_c + N^{nt}_{\alpha\beta} g_{nt} \top_c$$

(4.21) $$N^\delta_k g_{\alpha\beta}|_\delta = B^{ab}_{\alpha\beta} g_{ab} \top_k + N^{nt}_{\alpha\beta} g_{nt} \top_k$$

(4.22) $$g_{\alpha\beta}|_\gamma = B^{abc}_{\alpha\beta\gamma} g_{ab} \top_c + B^{ab}_{\alpha\beta} N^k_\gamma g_{ab} \top_k + N^{nt}_{\alpha\beta} B^c_\gamma g_{nt} \top_c + N^{ntk}_{\alpha\beta\gamma} g_{nt} \top_k.$$

From (4.19) we have

Lemma 4.1. *If*

(4.23) $$g_{ab|c} = \lambda_c g_{ab}, \quad g_{ab|k} = \lambda_k g_{ab}, \quad g_{nt|c} = \lambda_c g_{nt}, \quad g_{ab|k} = \lambda_k g_{nt}$$

then $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ *where,*

(4.24) $$\lambda_\gamma = B^c_\gamma \lambda_c + N^k_\gamma \lambda_k.$$

and λ_c, λ_k, λ_γ *are covariant vectors.*

Proof. Substituting (4.23) and (4.24) into (4.19) we have

$$g_{\alpha\beta|\gamma} = B^{ab}_{\alpha\beta}(B^c_\gamma \lambda_c g_{ab} + N^k_\gamma \lambda_k g_{ab}) + N^{nt}_{\alpha\beta}(B^c_\gamma \lambda_c g_{nt} + N^k_\gamma \lambda_k g_{nt} =$$
$$\lambda_\gamma B^{ab}_{\alpha\beta} g_{ab} + \lambda_\gamma N^{nt}_{\alpha\beta} g_{nt} = \lambda_\gamma g_{\alpha\beta}.$$

In the same way from (4.22) we have

Lemma 4.2. *If*

(4.25) $$g_{ab|c} = \mu_c g_{ab}, \quad g_{ab|k} = \mu_k g_{ab}, \quad g_{nt|c} = \mu_c g_{nt}, \quad g_{ab|k} = \mu_k g_{nt}$$

then $g_{\alpha\beta|\gamma} = \mu_\gamma g_{\alpha\beta}$ *where,*

(4.26) $$\mu_\gamma = B^c_\gamma \mu_c + N^k_\gamma \mu_k.$$

From (4.19) we have

Lemma 4.3. *If* $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ *then* $g_{ab|c} = \lambda_c g_{ab}$, $g_{nt|c} = \lambda_c g_{nt}$, $g_{nt|k} = \lambda_k g_{nt}$ *where:*

(4.27) $$\lambda_c = B^\gamma_c \lambda_\gamma, \quad \lambda_k = N^\gamma_k \lambda_\gamma$$

Proof. Multiplying (4.19) by $B^\gamma_d\ B^\beta_e\ B^\alpha_f$ and using the relations:

(4.28) $$N^k_\alpha B^\alpha_c = 0 \quad B^\alpha_a B^c_\alpha = \delta^c_a \quad N^\alpha_k N^n_\alpha = \delta^n_k$$

we get

$$g_{\alpha\beta|\gamma} B^{\gamma\beta\alpha}_{def} = \delta^a_f \delta^b_e \delta^c_d g_{ab|c} \Rightarrow \lambda_\gamma B^\gamma_d g_{\alpha\beta} B^{\alpha\beta}_{fe} = \lambda_\gamma B^\gamma_d g_{je} = g_{fe|d}.$$

Multiplying (4.19) by $B_f^\alpha\ B_e^\beta\ N_n^\gamma$ and using (4.28) we have

$$g_{\alpha\beta|\gamma}B_f^\alpha B_e^\beta N_n\gamma = \delta_{fen}^{abk} g_{ab|k} \Rightarrow \lambda_\gamma g_{\alpha\beta} B_f^\alpha B_e^\beta N_n^\gamma = \lambda_\gamma N_n^\gamma g_{fe} = g_{fe|n}.$$

Multiplying (4.19) by $N_{kl}^{\alpha\beta}\ B_d^\gamma$ and using (4.28) we have

$$g_{\alpha\beta|\gamma}N_{kl}^{\alpha\beta}B_d^\gamma = \delta_k^n\delta_l^t\delta_d^c g_{nt|c} \Rightarrow \lambda_\gamma g_{\alpha\beta} N B_{kl}^{\alpha\beta} B_d^\gamma = \lambda_\gamma B_d^\gamma g_{kl} = g_{kl|d}.$$

Multiplying (4.19) by $N_{plm}^{\alpha\beta\gamma}$ and using (4.28) we obtain

$$g_{\alpha\beta|\gamma}N_{plm}^{\alpha\beta\gamma} = \delta_{plm}^{ntk} g_{nt|k} \Rightarrow \lambda_\gamma g_{\alpha\beta} N plm^{\alpha\beta\gamma} = \lambda_\gamma N_m^\gamma g_{pl} = g_{pl|d}.$$

LEMMA 4.4 *If* ${}_{\alpha\beta}|_\gamma = \mu_\gamma g_{\alpha\beta}$, *then* $g_{ab}|_c = \mu_c g_{ab}$, $g_{ab}|_k = \mu_k g_{ab}$, $g_{nt}|_c = \mu_c g_{nt}$, $g_{nt}|_k = \mu_k g_{nt}$, *where* $\mu_c = B_c^\gamma \mu_\lambda$, $\mu_k = N_k^\lambda \mu_\lambda$.

The proof is similar to that of Lemma 4.3 if we start from (4.22).

From lemma 4.1 and 4.3 we have

THEOREM 4.1. *The necessary and sufficient conditions for* $g_{\alpha\beta|\gamma} = \lambda_\gamma g_{\alpha\beta}$ *are* (4.23) *and* (4.24)

From lemmas 4.2 and 4.4 we have

THEOREM 4.2. *The necessary and sufficient conditions for* $g_{\alpha\beta|\gamma} = \mu_\gamma g_{\alpha\beta}$ *are* (4.25) *and* (4.26)

REFERENCES

[1] H. Rund, *The differential geometry of Finsler spaces*, Springer-Verlag, 1959.

[2] A. Moór, it Über tensoren von rekurrenter kovarianter Ableitung, Publ. Math. Debrecen, **9** (1962), 81–93.

[3] A. Moór, *Über eine Übertragungstheorie der metrischen Linienelementräume mit rekurentem Grundtensor*, Tensor **29** (1975), 47–63.

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