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ON GENERAL SOLUTIONS OF SOME FUNCTIONAL EQUATIONS

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There exist certain functional equations which are so "general" that their general solutions (though they can be explicitly written) have little or no practical value. One such example is provided by the equation which we considered in [**1**]:

$$f(g(x)) = f(x),$$

where g is a given bijection of a nonempty set E and $f :\to S$ is the unknown function.

In fact, general solutions can be written down for very wide classes of equations. For instance, let E and S be nonempty sets and let f, g map E into S. For the equation in x:

$$f(x) = g(x) \tag{1}$$

define the function $F : E^2 \times S^2 \to E$ by

$$F(x, y, u, v) = \begin{cases} x, & u = v \\ y, & u \neq v. \end{cases} \tag{2}$$

Suppose now that the equation (1) is possible, i.e. that it has a solution x_0. Then the general solution of (1) is

$$x = F(t, x_0, f(t), g(t)), \tag{3}$$

where $t \in E$ is arbitrary.

Clearly, the solution (2) – (3) of the equation (1) is not particularly useful, but the equation (1) is so general that nothing more can be said about its solutions.

Remark. The solution (2) – (3) of (1) is a slight modification of the well known formula due to Löwenheim.

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In this note we shall show how the solution (2) – (3) of the equation (1) can be applied, with some simple modifications, to the functional equation for generalized inverses

$$f(g(f(x))) = f(x)$$

considered in [**2**].

Let $f : X \to Y$ and $g : Y \to X$ where X and Y are nonempty sets. Consider first the equation

$$f(g(y)) = y \qquad (\text{for all } y \in Y) \tag{4}$$

where f is given, and g is the unknown function.

Clearly, the equation (4) is possible if and only if f is a surjection. In that case, the existence of a solution g of (4) is equivalent to the Axiom of choice (see [**3**] or [**4**]).

If we suppose that g_0 is a solution of the equation (4), then we obtain the following general solution of (4):

$$g(y) = F(H(y), g_0(y), f(H(y)), y),$$

where $H : Y \to X$ is arbitrary, and $F : X^2 \times Y^2 \to X$ is defined by (2).

Consider now the equation

$$f(g(f(x))) = f(x) \qquad (\text{for all } x \in X), \tag{5}$$

where f and g are as before.

If f is a surjection, then the equation (5) reduces to (4). Hence, we suppose that f is not a surjection, i.e. that $Y \setminus f(X) \neq \emptyset$. Again, it is easily shown that the existence of a solution g of (5) follows from the Axiom of choice.

Obviously, for $y \in Y \setminus f(X)$, the value $g(y)$ is arbitrary, since it has no effect on the equation (5). Therefore, if g_0 is a solution of (5), then its general solution is

$$g(y) = \begin{cases} H(y), & y \in Y \setminus f(X) \\ F(H(y), g_0(y), f(H(y)), y), & y \in f(x) \end{cases} \tag{6}$$

where $H : Y \to X$ is arbitrary, and $F : X^2 \times Y^2 \to X$ is defined by (2).

The equation (5) was considered in [**2**] and the following general solution was established

$$g(y) = F(k(y), g_0(y), h(g_0(y)), f(g_0(y)), f(h(g_0(y))), y), \tag{7}$$

where $k : Y \to X$, $h : X \to X$ are arbitrary functions, and $F : X^3 \times Y^3 \to X$ is defined by:

$$F(x, y, z, u, v, w) = \begin{cases} x, & u \neq w \\ y, & u = w, \ u \neq v \\ z, & u = v = w. \end{cases} \tag{8}$$

We now demonstrate that the solutions (6) and (7) – (8) of the equation (5) are equivalent.

Since g_0 is, by supposition, a solution of (5), then clearly the conditions $y \in Y \setminus f(X)$ and $f(g_0(y)) \neq y$ are equivalent. Hence (6) can be written in the form

$$(9) \qquad g(y) = \begin{cases} H(y), & f(g_0(y)) \neq y \\ g_0(y), & f(g_0(y)) = y, \ f(H(y)) \neq y \\ H(y), & f(g_0(y)) = y, \ f(H(y)) = y. \end{cases}$$

Now, the function $H : Y \to X$ is arbitrary and so we may put

$$H(y) = \begin{cases} k(y), & y \in \setminus f(X) \\ h(g_0(y)), & y \in f(X) \end{cases}$$

where $k : Y \to X$ and $h : X \to X$ are arbitrary. Then (9) reduces to

$$g(y) = \begin{cases} k(y), & f(g_0(y)) \neq y \\ g_0(y), & f(g_0(y)) = y, \ f(h(g_)(y))) \neq y \\ h(g_0(y)), & f(g_0(y)) = f(h(g_0(y))) = y \end{cases}$$

and this is precisely the solution (7) – (8) obtained in [**2**].

We have therefore shown that the general solution of (5) is obtained as a simple consequence of the obvious fact that (2) – (3) is the general solution of (1). Moreover, the general solution of (5) contains only one arbitrary function; namely, the function h appearing in (7) is superfluous.

Remark. The formulas for the general solutions of many other functional equations are also direct consequences of the fact that (2) – (3) is the general solution of (1). Examples are provided by the solutions obtained in [**5**] or in [**1**].

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