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EXPLICIT SOLUTIONS OF SOME LINEAR MATRIX EQUATIONS

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1. Various methods for obtaining explicit solutions of the linear matrix equation

$$A_1XB_1 + A_2XB_2 + \cdots + A_nXB_n = C \tag{1}$$

were surveyed by Lancaster [**1**]. However, he seemed to be unaware of the method given by Prešić [**2**] for the special case of (1):

$$\sum_{i=0}^{r}\sum_{j=0}^{s} \alpha_{ij}A^iXB^j = 0, \tag{2}$$

though the equation (2) was given considerable space in [**1**].

In this note we suggest a different method for obtaining explicit solutions for a rather wide class of linear equations of the form (1), and in the special case when (1) takes the form (2) we compare the two methods.

2. The method we propose is based upon the following facts. Let V_1 and V_2 be vector spaces over a scalar field S and let $f: V_1 \to V_2$, be a homomorphism. We consider the equation

$$f(x) = c \qquad (c \in V_2 \text{ is given}) \tag{3}$$

together with the corresponding homogeneous equation

$$f(x) = 0. \tag{4}$$

If there exists a function $g: V_2 \to V_1$ such that $g(0) = 0$ and $fgf = f$, i.e. $f(g(f(x))) = f(x)$ for all $x \in V_1$ then:

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(i) The general solution of the equation (4) is

$$x = t - g(f(t)) \qquad (t \in V_1 \text{ arbitrary}); \tag{5}$$

(ii) The equation (3) has a solution if and only if: $f(g(c)) = c$. In that case its general solution is

$$x = g(c) + t - g(f(t)) \qquad (t \in V_1 \text{ arbitrary}). \tag{6}$$

The proof is straightforward. Indeed, suppose that there exists a function $g : V_2 \to V_1$ with the prescribed properties. Then from (5) follows: $f(x) = f(t - g(f(t))) = f(t) - f(g(f(t))) = f(t) - f(t) = 0$, which means that (5) is a solution of (4). On the other hand, if x_0 is a solution of (4), then putting $t = x_0$ in (5) we get: $x = x_0 - g(f(x_0)) = x_0 - g(0) = x_0$, which means that the solution x_0 is contained in the formula (5). Hence, (5) is the general solution of (4).

Regarding the equation (3) we first note that if the condition $f(g(c)) = c$ is fulfilled, then $g(c)$ is clearly a solution of (3). Conversely, if (3) has a solution, then we have: $f(g(f(x))) = f(g(c))$, i.e. $f(x) = f(g(c))$ which together with (3) implies $f(g(c)) = c$. The fact that (6) is the general solution of (3) is easily verified.

Remark. In particular, if the inverse f^{-1} of f exists, then the only solution of $fgf = f$ is $g = f^{-1}$. Hence, in that case the equations (3) and (4) have unique solutions: $f^{-1}(c)$ and 0, respectively.

3. We now apply the above considerations to the equations

$$A_1XB_1 + A_2XB_2 + \cdots + A_nXB_n = 0 \tag{7}$$

and

$$A_1XB_1 + A_2XB_2 + \cdots + A_nXB_n = C, \tag{8}$$

where $A_1, \dots, A_n$ are $l \times l$ and $B_1, \dots, B_n$ are $m \times m$ complex matrices, C is a given $l \times m$ matrix, and X is the $\times m$ unknown matrix.

Suppose that the matrices $A_1, \dots, A_n$ generate a finite semigroup $\{A_1, \dots, A_n, \dots, A_r\}$ and that $B_1, \dots, B_n$ generate a finite semigroup $\{B_1, \dots, B_n, \dots, B_s\}$ both with respect to multiplication. Then, if we define the function $f : M \to M$, where M is the set of all $l \times m$ matrices, by

$$f(X) = A_1XBI + \cdots + A_nXB_n, \tag{9}$$

wee see that for any $k \in N$ the k-th iterate f^k of f has the form

$$f^k(X) = \sum \alpha_{ij}A_iXB_j \quad (i = 1, \dots, r; j = 1 \dots, s; \alpha_{ij} \in C). \tag{10}$$

From the equations (9) and (10), together with

$$i(X) = X \qquad (i \text{ is the identity mapping})$$

we eliminate all the A_iXB_j's ($i = 1, \dots, r; j = 1, \dots, s$); the result of the elimination is a polynomial equation for f:

$$a_p f^p + a_{p-1} f^{p-1} + \cdots + a_1 f + a_0 i = 0, \tag{11}$$

where $p \le rs$, and $a_0, \dots, a_p$ are fixed complex numbers.

We now distinguish between the following cases:

(i) $a_0 \neq 0$. Then there exists the inverse function f^{-1}, since

$$f^{-1} = -a_0^{-1}(a_p f^{p-1} + a_{p-1} f^{p-2} + \cdots + a_i i).$$

Hence, the trivial solution is the only solution of (7), while (8) has the unique solution $X = f^{-1}(C)$.

(ii) $a_0 = 0$, $a_1 \neq 0$. The function g defined by

$$g = -a_1^{-1}(a_p f^{p-2} + a_{p-1} f^{p-3} + \cdots + a_2 i)$$

is such that $fgf = f$ and $g(0) = 0$. Then the general solution of (7) is:

$$X = T + a_1^{-1}(a_p f^{p-1}(T) + \cdots + a_2 f(T)),$$

where $T \in M$ is arbitrary. The equation (8) is possible if and only if: $-a_1^{-1}(a_p f^{p-1}(C) + \cdots + a_2 f(C)) = C$, and if this condition is fulfilled, its general solution is

$$X = -a_1^{-1}(a_p f^{p-2}(C) + \cdots + a_2 C) + T + a_1^{-1}(a_p f^{p-1}(T) + \cdots + a_2 f(T)),$$

where $T \in M$ is arbitrary.

Remark. The case $a_0 = a_1 = 0$ remains undecided. Namely, if $a_0 = a_1 = 0$, then clearly $X = a_p f^{p-1}(T) + \cdots + a_2 f(T)$, with $T \in M$ arbitrary, is a solution of (7), but it need not be the general solution.

Remark. Equations of the form (2) belong to the class considered here.

Example. Consider the equations

$$AX + XB = 0, \tag{12}$$

$$AX + XB + AXB = 0, \tag{13}$$

where $A^2 = B^2 = I$. For the equation (12) we have

$$f(X) = AX + XB; \quad f^2(X) = 2X + 2AXB; \quad f^3(X) = 4AX + 4XB,$$

and so $f^3 - 4f = 0$. Hence, its general solution is:

$$X = T - f^2(T)/4 = (T - = ATB)/2 \quad (T \in M \text{ arbitrary}).$$

Similarly, for the equation (13) we have

$$f(X) = AX + XB + AXB; \quad f^2(X) = 3X + 2AXB + 2XB + 2AXB,$$

which together with $i(X) = X$ implies ${}^2 - 2f - 3i = 0$. Therefore, the trivial solution is the only solution of (13).

4. The method suggested by Prešić [**2**] for the equation (2) consists in forming the matrix $P = \sum \alpha_{ij} \overline{B}^j \otimes \overline{A}^i$, where $\overline{A}$ and $\overline{B}$ denote the companion matrices of the minimal polynomials of A and B. Then, if $t^p + a_{p-1}t^{-1} + \cdots + a_0$ is the minimal polynomial of P, the general solution of (2) can explicitly be written down provided that $a_0 \neq 0$ or $a_0 = 0$ and $a_1 \neq 0$.

If we define f by $f(X) = \sum\limits_{ij} \alpha_{ij} A^i X B^j$, then it seems that the left hand side of the polynomial equation (11) for f obtained by our method *coincides* with the minimal polynominal of Prešić's matrix P, but we have not been able to prove this.

In the example which follows the two polynomials indeed coincide. Moreover, this example shows that it is easier to arrive at the polynomial in question by our method.

Example. The equation

$$AX - XB = 0 \tag{14}$$

where A, B, X are $n \times n$ matrices and $A^2 = A$, $B^2 = B$, was considered as an example in [**2**]: Since

$$\overline{A} = \overline{B} = \left\| \begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix} \right\|,$$

we have

$$P = I \otimes \overline{A} - \overline{B} \otimes I = \left\| \begin{matrix} 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{matrix} \right\|$$

and the minimal polynomial of P is $t^3 - t$.

On the other hand, if we let $f(X) = AX - XB$, then $f^2(X) = AX - 2AXB + XB$, $f^3(X) = AX - XB$. Hence, $f^3 - f = 0$, and the general solution of (14) is:

$$X = T - f^2(T) = T - AT + 2ATB - XT,$$

where T is an arbitrary $n \times n$ matrix.

Following a suggestion made by the referee, we give an example of an equation which is not of the form (2). Suppose that the matrices O, E, F, A, B form the following semigroup:

(15)

	O	E	F	A	B
O	O	O	O	O	O
E	O	E	O	A	O
F	O	O	F	O	B
A	O	O	A	O	E
B	O	B	O	F	O

and consider the equation

$$EXB + AXF + BXA = 0. \tag{16}$$

In this case E, A, B and B, F, A generate the same semigroup (15).

For $f(X) = EXB + AXF + BXA$ we have $f^4 = f$, implying $g = f^2$, and hence the general solution of (16) is

$$X = T - f^3(T) = T - ETE - ETF - FTF,$$

where T is arbitrary.

There exist matrices which form the semigroup (15). An example is provided by:

$$O = \begin{Vmatrix} 0 & 0 \\ 0 & 0 \end{Vmatrix}, \; E = \begin{Vmatrix} 1 & 0 \\ 0 & 0 \end{Vmatrix}, \; F = \begin{Vmatrix} 0 & 0 \\ 0 & 1 \end{Vmatrix}, \; A = \begin{Vmatrix} 0 & 1 \\ 0 & 0 \end{Vmatrix}, \; B = \begin{Vmatrix} 0 & 0 \\ 1 & 0 \end{Vmatrix}.$$

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