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COMMUTATIVE WEAK GENERALIZED INVERSES OF A SQUARE MATRIX AND SOME RELATED MATRIX EQUATIONS

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Abstract. The chief concern of this paper is the existence and the construction of all weak generalized inverses which commute with the original matrix; in other words we are concerned with the system $AXA = A$, $AX = XA$. Some other matrix systems and equations are also considered.

1. The (unique) generalized inverse of an arbitrary $m \times n$ complex matrix A is defined (see [**1**]) as the $n \times m$ matrix A^+ which satisfies the conditions:

$$AA^+A = A,\ \ A^+AA^+ = A^+,\ \ AA^+ \text{ and } A^+A \text{ are Hermitian.} \tag{1}$$

However, in various applications (particularly in solving linear matrix equations) it is not necessary to use the generalized inverse A^+. Instead, it is enough to take a matrix which satisfies only the first of the conditions (1), i.e. a matrix $\overline{A}$ such that

$$A\overline{A}A = A. \tag{2}$$

We note in passing that Bjerhammar [**2**] defined by the first equality of (1) the generalized inverses, and by the first two equalities of (1) the reciprocal inverses of the given matrix A.

A matrix $\overline{A}$ satisfying (2) will be called here a weak generalized inverse of A (w.g.i. of A). Unless A is a regular matrix (in which case the only w.g.i. is the inverse A^{-1}), any matrix A has an infinity of w.g.i.'s. We shall first investigate whether among them there exists a w.g.i. $\overline{A}$ which commutes with A, i.e. whether there exists a matrix $\overline{A}$ which satisfies (2) and also

$$A\overline{A} = \overline{A}A. \tag{3}$$

Notice that possible commutative reciprocal inverses of A, i.e. solutions of the system

$$AXA = A, \quad XAX = X, \quad AX = XA$$

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were considered e.g. in [**3**] or [**4**].

We shall characterize commutative w.g.i.'s, and more generally solutions of the system in X:

$$AXA = A, \quad A^k X = X A^k \qquad (k \in N)$$

by means of the coefficients of the minimal polynomial of A. Such an approach was not, as far as we know, employed before.

The existence of a commutative w.g.i. facilitates certain problems. For instance, in that case a w.g.i. of A^n is $\overline{A}^n$ $(n \in N)$; the equations $A^n X = 0$ and $AX = 0$ are equivalent for all $n \in N$; equalities of the form $A^m Y = A^n Z$ can be canceled by A $(m, n$ integers $> 1)$, and so on.

Naturally, the search for commutative w.g.i.'s restricts us to square matrices. We also exclude from our considerations regular matrices.

2. Suppose that A is a singular square matrix. We may take that the minimal polynomial of A has the form

$$m(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1 \lambda \tag{4}$$

since the existence of the constant term $\alpha_0 \neq 0$ would imply that A is regular.

THEOREM 1. *The matrix A with the minimal polynomial* (4) *has a commutative w.g.i. if and only if $\alpha_1 \neq 0$.*

Proof. If $\alpha_1 \neq 0$, it is easily verified that the matrix $\overline{A}$ defined by

$$\overline{A} = -(1/\alpha_1)(A^{n-2} + \alpha_{n-1}A^{n-3} + \cdots + \alpha_2 I)$$

is a commutative w.g.i. of A.

Conversely, suppose that $\overline{A}$ is a commutative w.g.i. of A and that $\alpha_1 = 0$. Then from the equality

$$A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_2 A^2 = 0.$$

after multiplying by $\overline{A}$, and noting that $\overline{A}A^k = A^{k-1}$ $(k = 2, 3, \dots)$ follows

$$A^{n-1} + \alpha_{n-1}A^{n-2} + \cdots + \alpha_2 A = 0,$$

implying that (4) is not the minimal polynomial of A. This completes the proof.

As a direct consequence of the above theorem we obtain

THEOREM 2. *If $\overline{A}$ is a commutative w.g.i. of A, then there exist a commutative w.g.i. $\overline{A}_0$ and a polynomial P such that $\overline{A}_0 = P(A)$.*

Proof. If $\overline{A}$ is a commutative w.g.i. of A, then the minimal polynomial of A has the form (4), with $\alpha_1 \neq 0$. But then the polynomial $P(\lambda) = -\alpha_1^{-1}(\lambda^{n-2} + \alpha_{n-1}\lambda^{n-3} + \cdots + \alpha_2)$ is such that $\overline{A}_0 = P(A)$ is a commutative w.g.i. of A.

The above theorem suggests the question: If $\overline{A}$ is a commutative w.g.i. of A, does there exist a polynomial P such that $\overline{A} = P(A)$? The answer is negative. Indeed, the matrix

$$\overline{A} = \frac{1}{9}\begin{Vmatrix} 4 & 4 & -5 \\ -2 & -2 & 7 \\ 1 & 1 & 1 \end{Vmatrix}$$

is a commutative w.g.i. of

$$A = \begin{Vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{Vmatrix}$$

but it cannot be expressed in the form of a polynomial in A. In fact, since $A^2 = 3A$, any polynomial in A can be reduced to a polynomial of the form $\alpha A + \beta I$. But then it easily verified that there are no α and β such that $\overline{A} = \alpha A + \beta I$.

At the end of this section we deduce a formula which enables us to write down all commutative w.g.i.'s of a given matrix, provided that one of them is known.

THEOREM 3. *Suppose that $\overline{A}$ is a commutative w.g.i. of A. Then all the solutions of the system in X:*

$$AXA = A, \quad AX = XA \tag{5}$$

are given by

$$X = \overline{A}A\overline{A} + T - \overline{A}AT - TA\overline{A} + \overline{A}ATA\overline{A}, \tag{6}$$

where T is an arbitrary matrix.

Proof. The proof is based on the fact that the general solutions of the equations in X:

$$AXA = A, \quad AXB = 0$$

are given by

$$X = \overline{A}A\overline{A} + U - \overline{A}AUA\overline{A}, \quad X = U - \overline{A}AUB\overline{B},$$

respectively, where $\overline{A}, \overline{B}$ are w.g.i's of A and B, and U is an arbitrary matrix.

In order to solve (5), we substitute the general solution

$$X = \overline{A}A\overline{A} + U - \overline{A}AUA\overline{A} \tag{7}$$

where U is arbitrary, of the first equation into the second equation of the system, to obtain the following equation in U:

$$AU - AUA\overline{A} = UA - \overline{A}AUA. \tag{8}$$

The equation (8), when multiplied by A from the right becomes

$$UA^2 - \overline{A}AUA^2 = 0,$$

and again multiplying by $\overline{A}$ we obtain

$$UA - \overline{A}AUA = 0, \quad \text{i.e.,} \quad (I - \overline{A}A)UA = 0,$$

and since I is a w.g.i. of $I - \overline{A}A$, its general solution is

$$U = V - (I - \overline{A}A)VA\overline{A}, \tag{9}$$

where V is an arbitrary matrix. We now substitute (9) into (8), to obtain the following equation in V:

$$AV - AVA\overline{A} = 0,$$

i.e.

$$AV(I - A\overline{A}) = 0.$$

The general solution of the last equation is

$$V = T - \overline{A}AT(I - A\overline{A}), \tag{10}$$

where T is an arbitrary matrix. From (7), (9) and (10) we conclude that (5) implies (6). Conversely, it is easily verified that (6) is a solution of (5), and the proof is complete.

3. As we have seen, a matrix A need not have a commutative w.g.i. We therefore investigate whether for a given matrix A it is possible to find a matrix X which satisfies the weaker conditions:

$$AXA = A, \quad A^k X = XA^k, \tag{11}$$

for some $k > 1$. Such matrices are called k-commutative w.g.i.'s of A (of course, we suppose that k is the smallest positive integer such that (11) holds). Note that systems of the form

$$AXA = A, \quad XAX = X, \quad A^k X = XA^k, \quad AX^k = X^k A$$

were considered by Erdelyi [**5**].

As before, we suppose that

$$\lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_1\lambda \tag{12}$$

is the minimal polynomial of A.

THEOREM 4. *If the matrix A has a k-commutative w.g.i. then at least one of the coefficients $\alpha_1, \ldots, \alpha_k$ differs from zero.*

Proof. The proof is similar to the proof of Theorem 1. Namely, suppose that $\alpha_1 = \cdots = \alpha_k = 0$, so that

$$A^n + \alpha_{n-1}A^{n-1} + \cdots + \alpha_{k+1}A^{k+1} = 0. \tag{13}$$

If there exists a matrix X which satisfies (11), then multiplying (13) by X, and noting that from (11) follows $XA^m = A^{m-1}$ for $m \geq k+1$, we see that (13) reduces to $A^{n-1}+\alpha_{n-1}A^{n-2}+\cdots+\alpha_{k+1}A^k=0$, implying that (11) is not the minimal polynomial of A.

THEOREM 5. *Suppose that k is the smallest positive integer* $(1\le k\le n-1)$ *such that $\alpha_k\neq 0$. Then there exists a k-commutative w.g.i. of A.*

Proof. Since the minimal polynomial of A is

$$\lambda^n+\alpha_{n-1}\lambda^{n-1}+\cdots+\alpha_k\lambda^k \quad (\alpha_k\neq 0),$$

the matrix A is similar to a Jordan matrix J, which can be written in the form $J=T\oplus R$, where T is a Jordan matrix of order $\ge k$, with zeros on the diagonal, and R is a regular matrix. Moreover, $T^k=0$, and so $J^k=0\oplus R^k$

Therefore, if T is a w.g.i. of T, then $\overline{J}=\overline{T}U\oplus R^{-1}$ is a w.g.i. of J, and $\overline{J}$ commutes with J^k since

$$J^k\overline{J}=(0\oplus R^k)(\overline{T}\oplus R^{-1})=0\oplus R^{k-1}=\overline{J}J^k.$$

Now since A is similar to J, there exists a regular matrix S such that $A=SJS^{-1}$. But then the matrix X defined by $X=S\overline{J}S^{-1}$ satisfies both equations (11), which is easily verified.

From Theorems 4 and 5 we obtain

THEOREM 6. *Let $1\le k\le n-1$ and suppose that $\alpha_1=\cdots+\alpha_{k-1}=0$. Then the matrix A has a k-commutative w.g.i. if and only if $\alpha_k\neq 0$.*

Again, we can deduce a formula which gives all k-commutative w.g.i.'s of A, provided that one of them is known.

THEOREM 7. *If $\overline{A}$ is a k-commutative w.g.i. of A, then all the solutions of the system (11) are given by*

$$X=\overline{A}A\overline{A}+T-TA^k\overline{A}^k+\overline{A}ATA^k\overline{A}^k-\overline{A}^kA^kT+\overline{A}^kA^kTA\overline{A}-\overline{A}ATA\overline{A}. \tag{14}$$

where T is an arbitrary matrix.

Proof. The proof is similar to the proof of Theorem 3, and we therefore omit it. It should only be noted that from $A\overline{A}A=A$, $A^k\overline{A}=\overline{A}A^k$ follows that $\overline{A}^k$ is a w.g.i. of A^k.

Remark. For $k=1$ formula (14) reduces to formula (6).

4. Notice that Theorems 3 and 7 can be carried over to arbitrary rings. Indeed, if $(R,+,\cdot)$ is a ring, and if $\overline{a}$ is a solution of the system in x:

$$axa=a,\quad a^kx=xa^k \qquad (k\in N, a\in R \text{ fixed}),$$

then all the solutions of that system are given by

$$x=\overline{a}a\overline{a}+t-ta^k\overline{a}^k+\overline{a}ata^k\overline{a}^k-\overline{a}^ka^kt+\overline{a}^ka^kta\overline{a}-\overline{a}ata\overline{a},$$

where $t \in R$ is arbitrary.

5. Commutative and k-commutative w.g.i.'s can be used to solve various matrix equations. As an example we consider the equation in X:

$$A^m X A^n = cA^p, \tag{15}$$

where m, n are nonnegative integers, p is a positive integer, and c is a complex number. If $\overline{A}$ is a commutative w.g.i. of A, then the general solution of the equation (15) is given by

$$X = c\overline{A}^m A^p \overline{A}^n + T - \overline{A}^m T A^n \overline{A}^n, \tag{16}$$

where T is an arbitrary matrix.

However, if A does not have a commutative w.g.i., but only a k-commutative w.g.i. ($k > 1$), then the equation (15) can be solved by this method provided that one of the following conditions is fulfilled:

(i) $k \leq \min(m, n, p)$;
(ii) $k \leq \min(m, p) \quad n \in \{0, 1\}$;
(iii) $k \leq \min(n, p), \quad m \in \{0, 1\}$;
(iv) $k \leq p \quad m, n \in \{0, 1\}$,

and the general solution of (15) in all those cases is again (16).

The equation (15) can be treated analogously in an arbitrary ring, provided that c is an integer.

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