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B-CONNECTIONS AND THEIR CONFORMAL INVARIANTS ON CONFORMALLY KAEHLER MANIFOLDS WITH B-METRIC

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Abstract. On a Kaehler manifold there is not a complete analogue of the conformal geometry on a Riemannian manifold. In this paper, we consider a class of complex manifolds with B-metric (including Kaehler manifolds with B-metric). The general conformal group and its special subgroups are determined. The Bochner curvature tensor of the manifold is shown to be a conformal invariant. The zero Bochner curvature tensor is proved to be an integrability condition of a geometrical system of partial differential equations and a characterization condition of a conformally flat manifold. Holomorphically umbilic submanifolds (Holomorphic spheres) are conformal invariants. The manifolds satisfying the axiom of holomorphic spheres are also characterized by zero Bochner curvature tensor. Thus, on the considered manifolds, there is a complete analogue of the conformal geometry on a Riemannian manifold.

1. Introduction

Let (M, ∇) be a differentiable manifold with a symmetric linear connection and corresponding curvature tensor R. The projective curvature tensor $P(R)$ is an invariant of the group of the projective transformations of ∇. The geometric meaning of $P(R)$ is illustrated by the following classical theorem.

THEOREM A. *The manifold* (M, ∇) $(\dim M \geq 3)$ *is projectively flat iff* $P(R) = 0$.

The manifold (M, ∇) $(\dim M = n \geq 3)$ is said to satisfy the axiom of r-planes $(2 \leq r < n)$ if, for each point p and for any r-dimensional subspace E of the tangential space T_pM, there exists an r-dimensional totally geodesic submanifold N containing p such that $T_pN = E$.

Another geometric meaning of $P(R)$ is given by

THEOREM B. *The manifold* (M, ∇) $(\dim M = n \geq 3)$ *satisfies the axiom of* r*-planes* $(2 \leq r < n)$ *iff* $P(R) = 0$.

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There exists a natural analogy between the projective geometry on (M, ∇) and the conformal geometry on a Riemannian manifold (M, g)

Let (M, g) be a Riemannian manifold with Levi-Civita connection ∇ and corresponding curvature tensor R. It is well known that the group of conformal transformations of g induces the group of conformal transformations of ∇. The Weyl conformal tensor $C(R)$ is an invariant of the conformal group and the analogue of Theorem A is

THEOREM C. *The Riemannian manifold* (M, g) $(\dim M \geq 4)$ *is conformally flat iff* $C(R) = 0$.

The manifold (M, g) $(\dim M \geq 3)$ is said to satisfy the axiom of r-spheres $(2 \leq r < n)$ if, for each point p and for any r-dimensional subspace E of the tangential space T_pM, there exists an r-dimensional totally umbilic submanifold N, containing p such that $T_pN = E$. So, the analogue of Theorem B is.

THEOREM D. *The Riemannian manifold* (M, g) $(\dim M \geq 4)$ *satisfies the axiom of* r*-spheres* $(3 \leq r < n)$ *iff* $C(R) = 0$.

The projective geometry on (M, ∇) has its natural analogue on a complex manifold with a symmetric almost complex linear connection.

Let (M, J, ∇) be a complex manifold with a symmetric linear connection ∇ so that $\nabla J = 0$. There arises in a natural way the group of holomorphically projective transformations of ∇ (transformations preserving the holomorphically planar curves of ∇) [5]. The holomorphically projective tensor $H(R)$ associated to the curvature tensor R of ∇ is a holomorphically projective invariant. The next theorem is analogous to theorems A and C.

THEOREM E [6]. *The manifold* (M, J, ∇) $(\dim M \geq 6)$ *is holomorphically projectively flat iff* $H(R) = 0$.

The corresponding analogue of theorems B and D is the following theorem.

THEOREM F [5]. (M, J, ∇) (M, g) $(\dim M = 2n \geq 6)$ *satisfies the axiom of the holomorphic* $2r$*-planes* $(2 \leq r < n)$ *iff* $H(R) = 0$.

In [7] Yano has shown that the Bochner curvature tensor on a Kaehler manifold can be considered in some sense as an analogue of the Weyl curvature tensor. But a natural similarity of the above is not yet known in the Kaehler geometry.

In this paper we find exact analogues of theorems A–F on a class of complex manifolds with B-metric. Theorems 5, 6 and 7 are analogues of theorems A, C and E. Theorem 10 is an analogue of theorems B, D and F.

2. Almost complex manifolds with B-metric

Let (M, J, g) be a $2n$-dimensional almost complex manifold with B-metric, i. e. J is the complex structure and g is the metric on M such that:

$$J^2X = -X; \qquad g(JX, JY) = -g(X, Y)$$

for all vector fields X, Y on M. The associated metric $\tilde{g}$ of the manifold is given by $\tilde{g}(X, Y) = g(JX, Y)$. Both metrics are necessarily of signature (n, n).

Further, X, Y, Z, U will stand for arbitrary differentiable vector fields on M. The Levi-Civita connection of g will be denoted by ∇. The tensor field F of type (0,3) on the manifold is defined by $F(X,Y,Z) = g((\nabla_X J)Y, Z)$. This tensor has the following symmetries:

$$F(X,Y,Z) = F(X,Z,Y); \tag{1}$$

$$F(X,Y,Z) = F(X,JY,JZ). \tag{2}$$

A classification of the almost complex manifolds with respect to the tensor F is given in [2].

Let $\tilde{\nabla}$ be the Levi-Civita connection of $\tilde{g}$. Then, $\tilde{\nabla}_X Y - \nabla_X Y$ is a tensor field of type (1,2) on M. We denote

$$\Phi(X,Y) = \tilde{\nabla}_X Y - \nabla_X Y. \tag{3}$$

This is the fundamental tensor of the manifold.

Since $\tilde{\nabla}$ and ∇ are torsion free, $\Phi(X,Y) = \Phi(Y,X)$. The corresponding tensor of type (0,3) is denoted by the same letter:

$$\Phi(X,Y,Z) = g(\Phi(X,Y),Z).$$

Further, x, y, z, u will stand for arbitrary vectors in the tangential space T_pM to M at an arbitrary point p in M. If $\{e_i\}$ $(i = 1, 2, \ldots, 2n)$ is an arbitrary basis of T_pM, g^{ij} are the components of the inverse matrix of g, then the vector field $\operatorname{tr}\Phi$ is defined by $\operatorname{tr}\Phi = g^{ij}\Phi(e_i, e_j)$. The Lee form θ associated with the tensor F is defined by

$$\theta(x) = -(1/n)g^{ij}F(e_i, e_j, Jx).$$

Using (1), (2) and (3), we obtain the relations between the tensors F and Φ:

$$\Phi(X,Y,Z) = -\frac{1}{2}\{F(JZ,X,Y) - F(X,Y,JZ) - F(Y,JZ,X)\}, \tag{4}$$

$$F(X,Y,Z) = \Phi(X,Y,JZ) - \Phi(X,Z,JY). \tag{5}$$

The Nijenhuis tensor N of the manifold is given by

$$N(X,Y) = [JX,JY] - [X,Y] - J[JX,Y] - J[X,JY].$$

By means of the covariant derivative $(\nabla_X J)\,Y$ of J this tensor is expressed by the equality

$$N(X,Y) = (\nabla_X J)JY - (\nabla_Y J)JX + (\nabla_{JX} J)Y - (\nabla_{JY} J)X \tag{6}$$

The associated tensor $\tilde{N}$ with N is defined by

$$\tilde{N}(X,Y) = (\nabla_X J)JY + (\nabla_Y J)JX + (\nabla_{JX} J)Y + (\nabla_{JY} J)X. \tag{7}$$

Taking into account (4), (5), (6) and (7), we have

$$g(N(X,Y),Z) = 2\Phi(Z,JX\,JY) - 2\Phi(Z,X,Y), \tag{8}$$

$$\tilde{N}(X,Y) = 2\Phi(X,Y) - 2\Phi(JX,JY). \tag{9}$$

Further we have.

LEMMA 1. *On an almost complex manifold with B-metric the following conditions are equivalent:*

1) $\Phi(X,Y) = -\Phi(JX,JY)$, 2) $\Phi(JX,Y) = -J\Phi(X,Y)$, 3) $N(X,Y) = 0$.

Proof. Using the property $\tilde{N}(JX,Y) = -J\tilde{N}(X,Y)$, from (9) we get

$$\Phi(X,Y) - \Phi(JX,JY) - J\Phi(JX,Y) - J\Phi(X,JY) = 0. \tag{10}$$

If $\Phi(X,Y) = -\Phi(JX,JY)$, then (10) implies $\Phi(JX,Y) = -J\Phi(X,Y)$. Hence $\Phi(X,JY,JZ) = \Phi(X,Y,Z)$. Now, taking into account (8), we find $N = 0$. So we proved the implications 1) $\Rightarrow$ 2) $\Rightarrow$ 3). The implication 2) $\Rightarrow$ 1) is trivial; 3) $\Rightarrow$ 2) follows from (8).

From (9) one obtains immediately

LEMMA 2. *On an almost complex manifold with B-metric the following conditions are equivalent:*

1) $\Phi(X,Y) = \Phi(JX,JY)$; 2) $\tilde{N}(X,Y) = 0$.

In [2] the eight classes of almost complex manifolds with B-metric are characterized by conditions for the tensor F. Using lemmas 1 and 2 we obtain characterizations of these classes with respect to the fundamental tensor Φ. Below, we give these two types of characterization conditions.

1. The class of the Kaehler manifolds with B-metric:

I. $F(X,Y,Z) = 0$; II. $\Phi(X,Y) = 0$.

2. The class W_1:

I. $F(X,Y,Z) = \{g(X,Y)\theta(JZ) + g(X,Z)\theta(JY) - g(X,JY)\theta(Z) - g(X,JZ)\theta(Y)\}/2$;

II. $\Phi(X,Y) = \{g(X,Y)\operatorname{tr}\Phi + g(X,JY)J\operatorname{tr}\Phi\}/2n$.

3. The class W_2 of special complex manifolds with B-metric:

I. $F(X,Y,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0$, $\theta = 0$ or $N(X,Y) = 0$, $\theta = 0$.

II. $\Phi(X,Y) = -\Phi(JX,JY)$, $\operatorname{tr}\Phi = 0$; or $\Phi(JX,Y) = -J\Phi(X,Y)$, $\operatorname{tr}\Phi = 0$.

4. The class W_3 of quasi-Kaehler manifolds with B-metric:

I. $F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = 0$, or $\tilde{N}(X,Y) = 0$.

II. $\Phi(X,Y) = \Phi(JX,JY)$.

5. The class $W_1 \oplus W_2$ of the complex manifolds with B-metric:

I. $F(XY,JZ) + F(Y,Z,JX) + F(Z,X,JY) = 0$, or $N(X,Y) = 0$.

II. $\Phi(X,Y) = -\Phi(JX,JY)$, or $\Phi(JX,Y) = -J\Phi(X,Y)$.

6. The class $W_2 \oplus W_3$ of semi-Kachler manifolds with B-metric:

I. $\theta = 0$; II. $\operatorname{tr}\Phi = 0$.

7. The class $W_1 \oplus W_3$:

I . $F(X,Y,Z) + F(Y,Z,X) + F(Z,X,Y) = g(X,Y)\theta(JZ) + g(Y,Z)\theta(JX)$
$+g(X,Z)\theta(JY) - g(X,JY)\theta(Z) - g(Y,JZ)\theta(X) - g(Z,JX)\theta(Y)$.

II. $\Phi(X,Y)-\Phi(JX,JY)=(g(X,Y)\mathrm{tr}\,\Phi+g(X,JY)J\mathrm{tr}\,\Phi)/n$.

8. The class of almost complex manifolds with B-metric:

No conditions.

3. Conformally Kaehler manifolds with B-metric

Let (M,J,g) be an almost complex manifold with B-metric. We consider the following conformal transformations of the metric g:

$$\overline{g}=e^{2u}(\cos 2v g+\sin 2v\tilde{g}),$$

where u, v are differentiable functions on M. By $v=0$, (11) is the usual conformal change of g. The manifold $(M,J,\overline{g})$ is also an almost complex manifold with B-metric.

Next, we take into consideration local conformal transformations.

THEOREM 1. *Let (M,J,g) be a W_1-manifold with Lee form θ and $(M,J,\overline{g})$ be conformally related to (M,J,g) by a transformation* (11). *Then $(M,J,\overline{g})$ is a W_1-manifold with Lee form $\overline{\theta}$, so that*

$$\overline{\theta}=\theta+2du-2dv\circ J.$$

Proof. Let ∇ and $\overline{\nabla}$ be the Levi-Civita connections of g and $\overline{g}$. Then

$$\begin{aligned}g(\overline{\nabla}_XY-\nabla_XY,Z)&=\sin 4v\{F(X,Y,Z)+F(Y,Z,X)-F(Z,X,Y)\}/4\\&-\sin^2 2v\{F(X,Y,JZ)+F(Y,JZ,X)-F(JZ,X,Y)\}/2\\&+du(X)\,g(Y,Z)+du(Y)\,g(X,Z)+dv(X)\,g(Y,JZ)+\\&+dv(Y)\,g(X,JZ)\end{aligned}\tag{13}$$

$$+\{-\cos^2 2v\,du(Z)+\sin 4v\,du(JZ)/2+\sin 4v\,dv(Z)/2-\sin^2 2v\,dv(JZ)\}g(X,Y)$$

$$+\{-\sin 4v\,du(Z)/2+\sin^2 2v\,du(JZ)-\cos^2 2v\,dv(Z)+\sin 4v\,dv(JZ)/2\}g(X,JY).$$

Let $\overline{F}(X,Y,Z)=\overline{g}((\overline{\nabla}_XJ)Y,Z)$. From (13) we obtain

$$\begin{aligned}2\overline{F}(X,Y,Z)&=2e^{2u}\cos 2vF(X,Y,Z)+e^{2u}\sin 2v\{F(JY,Z,X)-F(Y,JZ,X)\\&-F(Z,X,JY)+F(JZ,X,Y)\}+\omega(Y)g(X,Z)+\omega(Z)g(X,Y)\\&+\omega(JY)g(X,JZ)+\omega(JZ)g(X,JY),\end{aligned}\tag{14}$$

where $\omega=d(2e^{2u}\cos 2v)\circ J+d(2e^{2u}\sin 2v)$.

Remark. The formulas (13) and (14) are valid for arbitrary almost complex manifold with B-metric.

Now, let (M,J,g) be a W_1-manifold. Taking into account the form of F, from (14) we get

$$2\overline{F}(X,Y,Z)=\overline{g}(X,Y)\overline{\theta}(JZ)+\overline{g}(X,Z)\overline{\theta}(JY)-\overline{g}(X,JY)\overline{\theta}(Z)-\overline{g}(X,JZ)\overline{\theta}(Y),$$

where $\overline{\theta}=\theta+2\,du-2dv\circ J$, which completes the proof.

This theorem means the class W_1 is closed with respect to the conformal transformations (11).

COROLLARY 1. *Let* (M, J, g) *be a Kaehler manifold with* B*-metric. Every almost complex manifold with* B*-metric* $(M, J, \overline{g})$ *conformally equivalent to* (M, J, g) *by a transformation* (11) *is a* W_1*-manifold with Lee form* $\overline{\theta} = 2\,du - 2\,dv \circ J$.

COROLLARY 2. *Let* (M, J, g) *be a Kaehler manifold with* B*-metric. Every almost complex manifold with* B*-metric* $(M, J, \overline{g})$ *conformally equivalent to* (M, J, g) *by a transformation* $\overline{g} = e^{2u}\,g$ *is a* W_1*-manifold with closed Lee form* $\overline{\theta} = 2\,du$.

Let (M, J, g) be a W_1-manifold. It is necessarily a complex manifold. A pluriharmonic function u on M is characterized by the differential equationd $(du \circ Hj) = 0$. A holomorphic function $u + iv$ on M is characterized bythe condition $dv = -du \circ J$.

Next we consider the following special conformal changes of the metric g:

a) Conformal transformations of type I:

$$\overline{g} = e^{2u}\,g$$

where u is a pluriharmonic function on M.

b) Conformal transformations of type II:

$$\overline{g} = e^{2u}\,(\cos 2v\,g + \sin 2v\,\tilde{g}),$$

where $u + iv$ is a holomorphic function on M.

Definition. A W_1-manifold (M, J, g) with Lee form θ is said to be in the class CK_0 if both forms θ and $\theta \circ J$ are closed.

THEOREM 2. *Let* (M, J, g) *be a* W_1*-manifold. The manifold is in the class* CK_0 *iff it is conformally equivalent to a Kaehler manifold with* B*-metric by a transformation of type I.*

Proof. Let (M, J, g) be a W_1-manifold with Lee form θ. If it is conformally equivalent to a Kaehler manifold with B-metric by a conformal transformation of type I, from Theorem 1 we have $\theta = 2\,du$. Hence, $d\theta = d(\theta \circ J) = 0$.

For the converse, we solve locally the equation $\theta = 2\,du$ (θ and $\theta \circ J$ are closed) and find a pluriharmonic function u. The conformal transformation $e^{-2u}g$ gives rise to a Kaehler manifold with B-metric.

COROLLARY 3. *Let* (M, J, g) *be a* W_1*-manifold with Lee form* θ. *Then every almost complex manifold with* B*-metric* $(M, J, \overline{g})$ *conformally equivalent to* (M, J, g) *by a transformation of type II is a* W_1*-manifold with the same Lee form* θ.

COROLLARY 4. *The class* CK_0 *is closed with respect to conformal transformations of type I or II.*

4. The B-comlection on a W_1-manifold

Let (M, J, g) be a complex manifold with B-metric. The following theorem describes a natural linear connection on M.

THEOREM [4]. *On a complex manifold with B-metric (M, J, g) there exists a unique linear connection D with torsion tensor T such that*

(i) $(D_X g)(Y, Z) = 0$, $(D_X \tilde{g})(Y, Z) = 0$;
(ii) $T(JX, Y) = -T(X, JY)$, *i.e. T is hybrid;*
(iii) $g(T(X, Y), Z) + g(T(Y, Z), X) + g(T(Z, X), Y) = 0$.

This is the B-connection on the manifold. The equality iii) is equivalent to the condition

$$\tilde{g}(T(X, Y), Z) + \tilde{g}(T(Y, Z), X) + \tilde{g}(T(Z, X), Y) = 0,$$

i. e. the connection D is related to both metrics in the same way.

If ∇ is the Levi-Civita connection on (M, J, g), then

$$D_X Y = \nabla_X Y - J(\nabla_X J)\, Y/2.$$

Thus, if (M, J, g) is a W_1-manifold, then its B-connection D is given by

$$D_X Y = \nabla_X Y + \{g(X, Y)Q + g(X, JY)JQ - \theta(Y)X - \theta(JY)JX\}/4 \tag{15}$$

where θ is the Lee form and Q is the Lee vector $(g(Q, X) = \theta(X))$ of the manifold. From this it follows that the torsion tensor T of D has the form:

$$T(X, Y) = \{\theta(X)\, Y - \theta(Y)\, X + \theta(JX)\, JY - \theta(JY)\, JX\}/4.$$

For arbitrary vector fields X and Y, $\theta(T(X, Y)) = 0$.

Further, R will stand for the curvature tensor of ∇, i. e.

$$R(X, Y)\, Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The corresponding tensor of type $(0, 4)$ is denoted by the same letter and is given by $R(X, Y, Z, U) = g(R(X, Y)Z, U)$.

A tensor L of type $(0, 4)$ is said to be a curvature-like tensor if it satisfies the conditions:

i) $L(X, Y, Z, U) = -L(Y, X, Z, U)$;
ii) $L(X, Y, Z, U) + L(Y, Z, X, U) + L(Z, X, Y, U) = 0$;
iii) $L(X, Y, Z, U) = -L(X, Y, U, Z)$.

A curvature-like tensor L is said to be a Kaehler tensor if it satisfies the condition

iv) $L(X, Y, Z, U) = -L(X, Y, JZ, JU)$.

Let S be a tensor of type $(0, 2)$. We consider the following tensors

$$\begin{aligned}\psi_1(S)(X, Y, Z, U) = {} & g(Y, Z)S(X, U) - g(X, Z)S(Y, U) \\ & + g(X, U)S(Y, Z) - g(Y, U)S(X, Z),\end{aligned}$$

$$\psi_2(S)(X,Y,Z,U) = g(JY,Z)\,S(JX,U) - g(JX,Z)\,S(JY,U) \\ + g(JX,U)\,S(JY,Z) - g(JY,U)\,S(JX,Z).$$

We have

LEMMA 3. *Let S be a tensor of type* $(0,2)$. *Then*

a) $\psi_1(S)$ is a curvature-like tensor iff $S(X,Y) = S(YX)$;

b) $\psi_2(S)$ is a curvature-like tensor iff $S(JX,Y) = S(JY,X)$.

The tensors $\pi_1 - \pi_2$ and π_3 are defined as follows:

$$\pi_1(X,Y,Z,U) = \psi_1(g)(X,Y,Z,U)/2 = g(Y,Z)g(X,Y) - g(X,Z)g(Y,U),$$
$$\pi_2(X,Y,Z,U) = \psi_2(g)(X,Y,Z,U)/2 = g(JY,Z)g(JX,U) - g(JX,Z)g(JY,U),$$
$$\begin{aligned}\pi_3(X,Y,Z,U) &= -\,\psi_1(\tilde{g})(X,Y,Z,U) = \psi_2(\tilde{g})(X,Y,Z,U) \\ &= -\,g(Y,Z)g(JX,U) + g(X,Z)g(JY,U) - g(X,U)g(JY,Z) + \\ &\quad + g(Y,U)g(JX,Z).\end{aligned}$$

The tensors $\pi_1 - \pi_2$ and π_3 are Kaehler tensors.

LEMMA 4. *Let (M,J,g) be a W_1-manifold. If R and K are the curvature tensors of ∇ and D respectively, then*

$$K = R + \psi_1(S') + \psi_2(S'') + \theta(Q)\,(\pi_1 - \pi_2)/16 - \theta(JQ)\,\pi_3/16$$

where

$$\begin{aligned}S'(X,Y) &= (\nabla_X\theta)\,Y/4 + \{\theta(X)\,\theta(Y) - \theta(JX)\,\theta(JY)\}/16, \\ S''(X,Y) &= (\nabla_X\theta)\,Y/4 + \{\theta(X)\,\theta(Y) + \theta(JX)\,\theta(JY)\}/8 + \\ &\quad + \{\theta(X)\,\theta(Y) - \theta(JX)\,\theta(JY)\}/16.\end{aligned}$$

The statement follows by a direct computation taking into account (15).

LEMMA 5. *Let (M,J,g) be a W_1-manifold with Lee form θ and Levi-Civita connection ∇. Then*

a) θ is closed iff $(\nabla_\lambda\theta)\,Y = (\nabla_Y\theta)\,X$;

b) $\theta \circ J$ is closed iff $(\nabla_X\theta)\,JY - (\nabla_Y\theta)\,JX = \theta(JX)\,\theta(Y) - \theta(X)\,\theta(JY)$.

THEOREM 3. *Let (M,J,g) be a CK_0-manifold with B-connection D. If K is the curvature tensor of D then K is a Kaehler tensor.*

Proof. Using Lemma 5, we obtain the tensors S' and S'' defined in Lemma 4 satisfy the conditions of Lemma 3. Thus, Lemma 4 implies K is a curvature-like tensor. From the condition $DJ = 0$ it follows immediately that K is a Kaehler tensor.

5. Conformal groups and their invariants

In this section we consider conformal transformations of type I and II of the metric and we find the groups of conformal transformations of the B-connection. The Bochner curvature tensor of the B-connection on a CK_0-manifold is shown to be a conformal invariant of type I or II.

LEMMA 6. *Let (M, J, g) and $(M, J, \overline{g})$ be conformally related W_1-manifolds by a transformation $\overline{g} = e^{2u}(\cos 2v\, g + \sin 2v\, \tilde{g})$ with differentiable functions u, v. The corresponding B-connections D and $\overline{D}$ are related as follows*

$$\begin{aligned} 2\overline{D}_X Y = 2D_X Y &+ 2du(X)Y + 2dv(X)JY + \{du(Y) + dv(JY)\}X \\ &+ \{dv(Y) - du(JY)\}JX - g(X,Y)\{\operatorname{grad} u + J\operatorname{grad} v\} \\ &+ g(JX,Y)\{J\operatorname{grad} u - \operatorname{grad} v\}. \end{aligned} \tag{16}$$

The proof is a straightforward calculation using formulas (13) and (15).

The transformations $\overline{g} = e^{2u}(\cos 2v\, g + \sin 2v\, \tilde{g})$ with differentiable functions u, v on M form the (general) conformal group on M and give rise to the conformal group of transformations of the B-connection on M. The formula (16) is an analytic expression of a conformal transformation of D.

Lemma 6 implies

COROLLARY 5. *Let (M, J, g) and $(M, J, \overline{g})$ be conformally related CK_0-manifolds by a transformation* (11) *and D, $\overline{D}$ be their B-conections.*

If the transformation (11) *is of type* I, *then*

$$2\,\overline{D}_X Y = 2\,D_X Y + 2\,\sigma(X)Y + \sigma(Y)X - \sigma(JY)JX - g(X,Y)S + g(X,JY)JS. \tag{17}$$

If the transformation (11) *is of type* II, *then*

$$\overline{D}_X Y = D_X Y - \sigma(X)Y + \sigma(Y)X - \sigma(JX)JY - \sigma(JY)JX - g(X,Y)S + g(X,JY)JS. \tag{18}$$

Here $\sigma = du$, $g(S,X) = du(X)$.

The transformations (11) of type I (type II) form the conformal group of type I (type II). Formulas (17) and (18) express analitically the conformal groups of conformal transformations of the B-connection of type I and II, respectively.

Using (17) and (18) we obtain

LEMMA 7. *Let (M, J, g) and $(M, J, \overline{g})$ be conformally related CK_0-manifolds by a transformation* (11) *of type I. If K and $\overline{K}$ are the curvature tensors of tlle corresponding B-connections, then*

$$\overline{K} = K - \psi_1(L) + \psi_2(L) \tag{19}$$

where

$$\begin{aligned} 2L(X,Y) =& (\nabla_X \sigma)Y - \sigma(X)\sigma(Y) + \sigma(JX)\sigma(JY) + (g(X,Y)\sigma(S)/2 \\ &- g(X,JY)\sigma(JS)/2 - (g(X,Y)\theta(S) + g(X,JY)\theta(JS))/4 \end{aligned}$$

and θ is the Lee form of (M, J, g).

LEMMA 8. *Let (M, J, g) and $(M, J, \overline{g})$ be conformally related CK_0-manifolds by a transformation* (11) *of type II. If K and $\overline{K}$ are the curvature tensors of tlle corresponding B-connections, then*

$$\overline{K} = K - \psi_1(L) + \psi_2(L) \tag{19}$$

where

$$\begin{aligned} L(X,Y) =&(\nabla_X \sigma)Y - \sigma(X)\sigma(Y) + \sigma(JX)\sigma(JY) + (g(X,Y)\sigma(S) \\ &- g(X,JY)\sigma(JS))/2 - (g(X,Y)\theta(S) + g(X,JY)\theta(JS))/4. \end{aligned}$$

Let (M, J, g) be an arbitrary almost complex manifold with B-metric. If K is a Kaehler tensor over T_pM, $p \in M$ and $\{e_1, \ldots, e_{2n}\}$ is a basis of T_pM, then the Ricci tensor ϱ and the scalar curvatures τ and $\tilde{\tau}$ of K are given by

$$\varrho(xy) = g^{ij} K(e_i, x, y, e_j), \quad \tau = g^{ij} \varrho(e_i, e_j), \quad \tilde{\tau} = g^{ij} \varrho(Je_i, e_j).$$

The associated Bochner curvature tensor $B(K)$ is defined by

$$B(K) = K - \frac{1}{2\,(n-2)}\,(\psi_1 - \psi_2)(\varrho) + \frac{1}{4\,(n-1)\,(n-2)}\{\tau(\pi_1 - \pi_2) + +\tilde{\tau}\pi_3\}, \tag{21}$$

THEOREM 4. *Let (M, J, g) be a CK_0-manifold with B-connection D and corresponding curvature tensor K. Then the Bochner curvature tensor $B(K)$ is a conformal invariant of type I or II.*

Proof. Let $(M, J, \overline{g})$ be conformally related to (M, J, g) by a transformation (11) of type I or II. If $\overline{D}$ and $\overline{K}$ are the B-connection and its curvature tensor of $(M, J, \overline{g})$, then (19) and (20) imply

$$L = \frac{1}{2\,(n-2)}\,(\varrho - \overline{\varrho}) - \frac{1}{8\,(n-1)\,(n-2)}\{\tau g + \tilde{\tau}\tilde{g} - \overline{\tau}\,\overline{g} + \tilde{\overline{\tau}}\,\tilde{\overline{g}}\},$$

where $\overline{\varrho}$, $\overline{\tau}$, $\tilde{\overline{\tau}}$ are the associated Ricci tensor and the scalar curvatures of $\overline{K}$. Substituting L into (19), respectively (20), and taking into account (21), we obtain

$$B(\overline{K}) = e^{2u}\, B(K),$$

respectively

$$B(\overline{K}) = e^{2u}\{\cos 2v\, B(K) + \sin 2v\, B(\tilde{K})\}, \tag{23}$$

where $\tilde{K}$ is the Kaehler tensor, given by $\tilde{K}(X, Y, Z, U) = K(X, Y, Z, JU)$. Thus, if $B(K)$ and $B(\overline{K})$ are the corresponding tensors of type $(1, 3)$, then (22) and (23) imply $B(K) = B(\overline{K})$.

6. The Bochner curvature tensor and integrability conditions

In this section we show the zero Bochner curvature tensor of a B-connection is an integrability condition for a system of PDE describing conformally flat (of type I or Il) CK_0-manifolds.

THEOREM 5. *Let (M, J, g) ($\dim M \geq 8$) be a Kaehler manifold with B-metric and vanishing Bochner curvature tensor of Levi-Civita connection ∇. Then, (M, J, g) is conformaly related to a CK_0-manifold $(M, J, \overline{g})$ by a transformation of type I, so that the B-connection of $(M, J, \overline{g})$ is flat.*

Proof. Let $\overline{g} = e^{2u}g$, with an unknown pluriharmonic function u on M. Using (19), we obtain the B-connection $\overline{D}$ on $(M, J, \overline{g})$ is flat iff

$$(24)\quad \begin{aligned}&(\nabla_X\sigma)Y - \sigma(X)\sigma(Y) + \sigma(JX)\sigma(JY) + g((X,Y)\sigma(S) - g(X,JY)\sigma(JS))/2\\ &= \frac{1}{(n-2)}\varrho(X,Y) - \frac{1}{4\,(n-1)(n-2)}\{\tau g(X,Y) - \tilde{\tilde{\tau}} g(x,JY)\},\end{aligned}$$

where $\sigma = du$, $g(S,X) = du(X)$, ϱ, τ, $\tilde{\tau}$ are the Ricci tensor and the scalar curvatures of the curvature tensor R of ∇.

Now we shall show $B(R) = 0$ is an integrability condition for the system (24). Denoting the right hand side of (24) by $L(X, Y)$, we have $L(X, Y) = L(Y, X)$ and $L(X, Y) = -L(JX, JY)$. Applying the Ricci identity $(\nabla_X\nabla_Y\sigma)Z - (\nabla_Y\nabla_X\sigma)Z = -\sigma(R(X,Y)Z)$ to the left hand side of (24) and using $B(R) = 0$, we find the system is integrable iff

$$(25)\qquad (\nabla_X L)(Y,Z) = (\nabla_Y L)(X,Z)$$

To prove (25) we use the equality $R = (\psi_1 - \psi_2)(L)$ and apply the second Bianchi identity for R. After a contraction we obtain

$$(2n-5)\{\nabla_{JX}L)(JY,Z) - (\nabla_{JY}L)(JX,Z)\} + (\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z) = 0$$

and by the substitution $X \to JX$, $Y \to JY$

$$(2n-5)\{\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z)\} + (\nabla_{JX}L)(JY,Z) - (\nabla_{JY}L)(JX,Z) = 0$$

From the last equalities it follows that

$$(n-2)(n-3)\{(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z)\} = 0.$$

Hence (25) is a consequence of $B(R) = 0$ and the system (24) is integrable. It follows immediately that every solution u ($du = \sigma$) of (24) is a pluriharmonic function. The change $\overline{g} = e^{2u}g$ (u — a solution of (24) gives rise to a CK_0-manifold $(M, J, \overline{g})$ with flat B-connection and this completes the proof.

If $2n = 6$ and $B(R) = 0$, the integrability condition (25) is not a consequence of $B(R) = 0$. In this case $B(R) = 0$ implies only

$$(\nabla_X L)(Y,Z) - (\nabla_Y L)(X,Z) = (\nabla_{JX}L)(JY,Z) - (\nabla_{JY}L)(JX,Z).$$

THEOREM 6. *Let (M, J, g') $(dim M \geq 8)$ be a CK_0-manifold with B-connection D' and vanishing Bochner curvature tensor of D'. Then, (M, J, g') is conformaly related to a CK_0-manifold $(M, J, \overline{g})$ by a conformal transformation of type II, so that the B-connection $\overline{D}$ of $(M, J, \overline{g})$ is flat.*

Proof. Let θ' be the Lee form of (M, J, g'). Solving (locally) the equation $\theta' = 2du'$, we obtain a Kaehler manifold with B-metric (M, J, g) with $g = e^{-2u'}g'$. From Theorem 4 it follows that the Bochner curvature tensor $B(R) = 0$, R beingthe curvature tensor of the L vi-Civita connection ∇ of (M, J, g). Applying Theorem 5 we find a pluriharmonic function u and the manifold $(M, J, \overline{g})$, $\overline{g} + e^{2u}g$ is of flat B-connection. Thus, the conformal change $\overline{g} = e^{2(u-u')}g'$ is of type I and transforms (M, J, g') into $(M J, \overline{g})$.

Now we shall consider the conformal transformations of type II.

THEOREM 7. *Let (M, J, g') $(dim M \geq 8)$ be a CK_0-manifold with B-connection D' and vanishing Bochner curvature tensor of D'. Then, (M, J, g') is conformaly related to a CK_0-manifold $(M, J, \overline{g})$ by a conformal transformation of type II, so that the B-connection $\overline{D}$ of $(M, J, \overline{g})$ is flat.*

Proof. Let $\overline{g} = e^{2u}(\cos 2v\, g' + \sin 2v\, \tilde{g'})$ with unknown functions u and v such that $dv = -du \circ J$. Apllying Theorem 2 we obrain a pluriharmonic function u on M, so that (M, J, g) $(g = e^{-2u'}g')$ is a Kaehler manifold with B-metric. From Theorem 4 it follows that $B(R) = 0$, where R is the curvature tensor of the Levi-Civita connection ∇ of (M, J, g). By a simple calculation using (17) and (18), we find

$$\begin{aligned}\overline{D}_X Y =& \nabla_X Y + du(X)Y - du(JX)JY + du'(X)Y \\ &+ \alpha(Y)X - \alpha(JY)JX - g(X,Y)A + g(X, JY)JA,\end{aligned} \tag{26}$$

where $\alpha = du + du'/2$, $g(A, X) = \alpha(X)$.

We need the following well known propositions.

LEMMA A. *Let ∇' and ∇'' be linear connections on a differentiable manifold M such that $\nabla''_X Y = \nabla'_X Y + \beta(X)Y$, where β is a 1-form. Then ∇' and ∇'' have the same curvature tensor iff $d\beta = 0$.*

LEMMA B. *Let ∇' and ∇'' be linear connections on an almost complex manifold (M, J) such that $\nabla' J = 0$, $\nabla''_X Y = \nabla'_X Y + \gamma(JX)JY$, where γ is a 1-form. Both connections have the same curvature tensor iff $d(\gamma \circ J) = 0$.*

From lemmas A, B and (26) it follows that the curvature tensors $\overline{K}$ and R of $\overline{D}$ and ∇ respectively, are related as follows:

$$\overline{K} = R - \psi_1(L) + \psi_2(L),$$

where

$$L(X,Y) = (\nabla_X \alpha)Y - \alpha(X)\alpha(Y) + \alpha(JX)\alpha(JY) + (g(X,Y)\alpha(A) - g(X,JY)\alpha(JA))/2.$$

As in the proof of Theorem 5, we have $\overline{K} = 0$ iff
(27)
$$(\nabla_X \alpha)Y - \alpha(X)\alpha(Y) + \alpha(JX)\alpha(JY) + (g(X,Y)\alpha(A) - g(X,JY)\alpha(JA))/2$$
$$= \frac{1}{2(n-2)}\varrho(X,Y) - \frac{1}{8\,(n-1)(n-2)}\{\tau g(X,Y) - \tilde{\tau} g(X,JY)\},$$

where ϱ, τ and $\tilde{\tau}$ are the Ricci tensor and the scalar curvatures of R. The condition $B(R) = 0$ implies the system (27) is completely integrable. Let u'' be a (local)solution of (27) ($du'' = \alpha$). Then $u = u'' - u'/2$ is a pluriharmonic function. Solving locally $dv = -du \circ J$, we obtain a holomorphic function $u + iv$ so that the conformal transformation $\overline{g} = e^{2u}(\cos 2v\, g' + \sin 2v\, \tilde{g}')$ is of type II and the manifold $(M, J, \overline{g})$ is of flat B-connection $\overline{D}$.

This theorem has the following immediate

COROLLARY. *Let (M, J, g) ($\dim M \geq 8$) be a Kaehler manifold with B-metric and vanishing Bochner curvature tensor. Then, (M, J, g) is conformally rela- ted to a Jlat Kaehler manifold with B-metric by a transformation of type II.*

Remark. The statement of this corollary (without the condition for the $\dim M$) has been announced in [8] without proof.

Applying theorems 2, 4 and the last corollary, we check

THEOREM 8. *Let (M, J, g) ($\dim M \geq 8$) be a CK_0-manifold with B-connection D and vanishing Bochner curvature tensor of D. Then there exist pluriharmonic functions u and v on M, such that the conformal transformation $\overline{g} = e^{2u}(\cos 2v\, g + \sin 2v\, \tilde{g})$ gives rise to a flat Kaehler manifold with B-metric (M, J, g).*

7. The holomorphic sphere axiom

In this section, we prove an analogue of theorems B, D, F on manifolds of the class CK_0.

Let (M, J, g) ($\dim M = 2n$) be a W_1-manifold. A submanifold N ($\dim N = 2r$) of M is said to be a holomorphic submanifold if the restriction of g on N has a maximal rank and $JT_pN = T_pN$, $p \in N$. The restrictions of J and g on N are denoted by the same letters. Then (N, J, g) is an almost complex manifold with B-metric. If ∇ is the Levi-Civita connection on M and ∇' is the induced connection on N, the Gauss equation is $\nabla_X Y = \nabla'_X Y + \sigma(X,Y)$ for all X, Y tangent to N. It follows immediately that every holomorphic submanifold (N, J, g) of (M, J, g) is also a W_1-manifold and has the same Lee form θ as M. The Lee vector Q' of (N, J, g) is the restriction on N of the Lee vector Q of (M, J, g). The mean curvature vector H on N is defined by $H = (1/2r)\, tr\, \sigma$.

A holomorphic submanifold N of M is said to be holomorphically umbilic (h-umbilic) if in every point of N
$$\sigma = Hg - JH\tilde{g}.$$

H-umbilic submanifolds of a Kaehler manifold with B-metric are considered in [3].

The following statement is easy to prove.

LEMMA 9. *Let (M, J, g) be a W_1-manifold with Levi-Cirita connction ∇. A holomorphic submanifold N of M is h-umbilic iff for every vector fields X, Y tangent to N and such that $g(X,Y) = g(X, JY) = 0$, the vector field $\nabla_X Y$ is tangent to N.*

THEOREM 9. *H-umbilic submanifolds of W_1-manifolds are eonformal invariants.*

Proof. Let (M, J, g) and $(M, J, \overline{g})$ be W_1-manifolds conformally related by a transformation (11) and ∇, $\overline{\nabla}$ be the corresponding Levi-Civita connections. Let us consider a holomorphic shbmanifold N of M, which is h-umbilic with respect to g, and choose vector fields X, Y tangent to N, such that $\overline{g}(X,Y) = \overline{g}(X, JY) = 0$. The last condition is equivalent to $g(X,Y) = g(X, JY) = 0$. From this and (13) we find $\overline{\nabla}_X Y$ is tangent to N. According to Lemma 9, N is h-umbilic with respect to $\overline{g}$.

A W_1-manifold (M, J, g) $(\dim M = 2n \geq 6)$ is said to satisfy the axiom of holomorphic $2r$-spheres $(2 \leq r < n)$ if, for each point p and for any $2r$-dimensional J-invariant subspace E of T_pM such that rank $g/_E = 2r$, there exists a $2r$-dimensional h-umbilic submanifold N containing p and $T_pN = E$.

THEOREM 10. *Let (M, J, g) $(\dim M = 2n \geq 8)$ be a CK_0-manifold. The manifold satisfies the axiom of holomorphic $2r$-spheres $(3 \leq r < n)$ iff the Bochner curvature tensor of its B-connection is zero.*

Proof. Let (M, J, g) satisfy the axiom of holomorphic $2r$-spheres and K, $B(K)$ be the curvature tensor and the Bochner curvature tensor associated to its B-connection. To prove $B(K) = 0$, we take an arbitrary orthonormal totally real quadruple $\{x, y, z, u\}$ in T_pM, $p \in M$, i. e. $\{x, y, z, u\}$ spans a four-dimensional space E, such that $E \perp JE$. If N is an h-umbilic submanifold, such that T_pN contains x, y, z and $T_pN \perp u$, Ju, from the Codazzi formula we obtain

$$(R(x,y)z)^{\perp} = g(y,z)D_x^{\perp}H - g(x,z)D_y^{\perp}H + g(y,Jz)D_x^{\perp}JH$$
$$-g(x,Jz)D_y^{\perp}JH = 0,$$

where R is the curvature tensor of the Levi-Civita connection on M and $D^{\perp}$ is the natural normal connection. This implies that $R(x,y)z$ is tangent to N and hence $R(x,y,z,u) = 0$. Lemma 4 implies $K(x,y,z,u) = 0$. In [1, theorem 5] it is proved that the last condition is equivalent to $B(K) = 0$.

For the converse, let $B(K) = 0$. From Theorem 7 it follows that (M, J, g) is conformally equivalent to a flat Kaehler manifold with B-metric. Every flat Kaehler manifold with B-metric is holomorphicaly isometric to (R^{2n}, J, g) with the canonical structures, and hence it satisfies the axiom of holomorphic $2r$-planes. Now, the proposition follows from Theorem 9.

COROLLARY. *Let (M, J, g) $(\dim M = 2n \geq 8)$ be a Kaehler manifold with B-metric. The manifold satisfies the axiom of holomorphic $2r$-spheres $(3 \leq r < n)$ iff its Bochner curvature tensor vanishes.*

REFERENCES

[1] A. Borisov, G. Ganchev, *Curvature properties of Kaehlerian manifolds with B-metric*, Proc. XIV Spring Conf. UBM, Sunny Beach, (1985), 220–226.

[2] G. Granchev, A. Borisov, *Note on almost complex manifolds with Norden metric* [2] C.R. Acad. Bulg. Sci. **39** (1986), 31–34.

[3] G. Ganchev, K. Gribachev, V. Mihova, *Holomorphic hypersurfaces of Kaehler manifolds with Norden metric* (to appear).

[4] G. Ganchev, V. Mihova, *Characterization of some linear connections on almost complex manifolds with metric* (to appear).

[5] S. Ishihara, *Holomorphically projective changes and their groups in an almost complex manifold*, Tohoku Math. J. **9** (1959), 273–297.

[6] Y. Tashiro, *On holomorphically projective correspondence in an almost complex space*, Math. J. Okayama Univ. **6** (1957),167–172.

[7] K. Yano, *On complex conformal connections*, Kodai Math. Sem. Rep. **26** (1975), 137–151.

[8] Е. Павлов, М. Василева, *Конформно голоморфное соотвецтвие между В-многообразиями*, Изв. ВУЗ, Математика, **10** (1981), 187–196.

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