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A NOTE ON LENGTH DISTORTION FOR CERTAIN CLASSES OF ANALYTIC FUNCTIONS

Shigeyoshi Owa and Shotaro Ogawa

Abstract. Some interesting results on length distortion for certain classes of analytic functions have been proved recently by Y. Komatu. We derive some results involving analytic functions satisfying $Re\{f(z)/z\} > \alpha$, starlike functions of order α, and convex functions of order α.

Introduction. Let A denote the class of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots$$

which are analytic in the unit disk $U = \{z : \mid z \mid < 1\}$. Komatu [**1**] introduced a linear operator L of the form

$$Lf(z) = \int_0^1 t^{-1} f(zt) d\sigma(t)$$

defined on A where $\sigma(t)$ denotes a probability measure supported by the interval $I = [0, 1]$. He has shown in [**1**] that there exiats a family of operators $\{L^\lambda\}$ depending on a continuous parametar λ such that

$$L^\lambda L^\mu = L^{\lambda+\mu} \qquad \text{where } L^0 = \mathrm{id}$$

and, under certain conditions on σ,

$$f_\lambda(z) \equiv L^\lambda f(z) = \int_I t^{-1} f(zt) d\sigma_\lambda(t) \tag{1.1}$$

where σ_λ is a probability measure supported by I.

Let $R(\alpha) \subset A$ be the class of all functions satisfying the condition

$$Re\{f(z)/z\} > \alpha \qquad (f(z) \in A)$$

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for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in U$.

The main tool in deriving our results for length distortion is based on the following lemma [**2**].

LEMMA 1. *Let $L_\lambda(r)(\lambda \geq 0)$ be the length of the image-curve of $\{z : \mid z \mid = r < 1\}$ by a mapping $w = f_\lambda(z)/z$ caused by (1.1) with any $f(z) \in R(0)$. Then we have*

$$L_\lambda(r) \leq \int_I L(rt) d\sigma_\lambda(t) \tag{1.2}$$

where $L = L_0$ satisfies $L(r) \leq 4\pi r/(1-r^2)$.

2. The class $\mathbf{R}(\alpha)$. In view of Lemma 1, we state and prove

THEOREM 1. *Let $L_\lambda(r)(\lambda \geq 0)$ be the length of the image-curve of $\{z : \mid z \mid = r < 1\}$ by a mapping $w = f_\lambda(z)/z$ caused by (1.1) with any $f(z) \in R(\alpha)$. Then we have the estimation (1.2) where $L = L_0$ satisfies*

$$L(r) \leq 4\pi(1-\alpha)r/(1-r^2). \tag{2.1}$$

Proof. The estimation (1.2) is obtained by the method of proof in [**2**]. For a function $f(z)$ belonging to the class $R(\alpha)$, the Herglotz representation implies

$$\frac{f(z)}{z} - \alpha = (1-\alpha)\int_{-\pi}^{\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\tau(\varphi) \tag{2.2}$$

with a probability measure τ supported by $(-\pi, \pi]$. It follows from (2.2) that

$$g(z) \equiv \frac{f(z)}{z} = \int_{-\pi}^{\pi} \frac{e^{i\varphi}+(1-2\alpha)z}{e^{i\varphi}-z} d\tau(\varphi).$$

Therefore, we obtain

$$\begin{aligned}
L(r) =& r\int_{-\pi}^{\pi} \mid g'(re^{i\theta}) \mid d\theta \\
\leq& 2(1-\alpha)r \int_{-\pi}^{\pi} d\theta \int_{-\pi}^{\pi} \mid e^{i\varphi} - re^{i\theta} \mid^{-2} d\tau(\varphi \\
=& 2(1-\alpha)r \int_{-\pi}^{\pi} d\tau\varphi \int_{-\pi}^{\pi} \mid e^{i\varphi} - re^{i\theta} \mid^{-2} d\theta \\
=& 2(1-\alpha)r \int_{-\pi}^{\pi} 2\pi(1-r^2)^{-1} d\tau(\varphi) \\
=& 4\pi(1-\alpha)r/(1-r^2)
\end{aligned}$$

whish gives (2.1). Furthermore, the bound in (2.1) is attained by the extremal function given by $f(z) = (z + (1-2\alpha)\varepsilon z)/(1-\varepsilon z) \qquad (\mid \varepsilon \mid = 1)$.

Remark 1. By letting $\alpha = 1/2$ in Theorem 1, we have the corresponding result due to Komtu [**2**, Theorem 2].

3. The class $S^*(\alpha)$. A function $f(z)$ belonging to the class A is said to be starlike of order $\alpha(0 \leq \alpha < 1)$ if it satisfies $Re(zf'(z)/f(z)\} > \alpha$ for some $\alpha(0 \leq \alpha < 1)$ and for all $z \in U$. We denote by $S^*(\alpha)$ the class of all functions in A which are starlike of order α. We note that $S^*(\alpha) \subseteq S^*(0) \equiv S^*$ for $0 \leq \alpha < 1$.

For the function $f(z)$ in the class $S^*(\alpha)$, we prove

THEOREM 2. *Let* $L_\lambda(r)(\lambda \geq 0)$ *be the length of the image-curve of* $\{z : \mid z \mid = r < 1\}$ *by a mapping* $w = f_\lambda(z)/z$ *caused by* (1.1) *with any* $f(z) \in S^*(\alpha)$. *Then we have the estimation* (1.2) *where* $L = L_0$ *satisfies*

$$L(r) \leq \frac{8(1-\alpha)r}{(1-r)^{2(1-\alpha)}(1+r)} K\left(\frac{2\sqrt{r}}{1+r}\right),$$

where K *denotes the complete elliptic integral of the first kind with modulus* $2\sqrt{r}/(1+r)$.

Proof. Again we use the estimation (1.2). Applying Herglotz representation, we have, for $g(z) = f(z)/z$ and $f(z) \in S^*(\alpha)$,

$$\begin{aligned} g'(z) =& f(z)z^{-2}\{(zf'(z)/f(z) - \alpha) - (1-\alpha)\} \\ =& \frac{g(z)}{z}\left\{(1-\alpha)\int_{-\pi}^{\pi} \frac{e^{i\varphi}+z}{e^{i\varphi}-z} d\tau(\varphi) - (1-\alpha)\right\} \\ =& 2(1-\alpha)g(z)\int_{-\pi}^{\pi} (e^{i\varphi}-z)^{-1} d\tau(\varphi) \end{aligned} \tag{3.1}$$

with a probability measure τ supported by $(-\pi, \pi]$, Thus from (3.1) we derive

$$\begin{aligned} L(r) =& r\int_{-\pi}^{\pi} \mid g'(re^{i\theta}) \mid d\theta \\ =& 2(1-\alpha)r\int_{-\pi}^{\pi} \mid g(re^{i\theta}) \mid d\theta \mid \int_{-\pi}^{\pi} (e^{i\varphi} - re^{i\theta})^{-1} d\tau(\varphi) \mid \\ \leq& 2(1-\alpha)r\int_{-\pi}^{\pi} d\tau(\varphi) \int_{-\pi}^{\pi} \left| \frac{g(re^{i\theta})}{e^{i\varphi} - re^{i\theta}} \right| d\theta \end{aligned}$$

We note that a function $f(z)$ belonging to the class $S^*(\alpha)$ satisfies [**4**]:

$$r(1+r)^{-2(1-\alpha)} \leq \mid (re^{i\theta}) \mid \leq r(1-r)^{-2(1-\alpha)}$$

which implies

$$\mid g(re^{i\theta}) \mid \leq (1-r)^{-2(1-\alpha)}, \tag{3.2}$$

and we have the following explicit expression [**2**]:

$$J(r) = 2\int_0^{\pi} (1 + r^2 - 2r\cos\theta)^{-1/2} d\theta = \frac{4}{1+r} K\left(\frac{2\sqrt{r}}{1+r}\right). \tag{3.3}$$

Consequently, by using (3.2) and (3.3), we obtain

$$L(r) \leq \frac{2(1-\alpha)r}{(1-r)^{2(1-\alpha)}} \int_{-\pi}^{\pi} \frac{1}{\mid 1-re^{i\theta} \mid} d\theta = \frac{2(1-\alpha)r}{(1-r)^{2(1-\alpha)}} K\left(\frac{2\sqrt{r}}{1+r}\right).$$

This completes the proof of Theorem 2.

Remark 2. For $\alpha = 0$, Theorem 2 implies the result for $f(z) \in S^*$ proved in [**2**].

4. The class $K(\alpha)$. A function $f(z) \in A$ is said to be convex of order $\alpha (0 \leq \alpha < 1)$ if it satisfies $Re\{1 + zf''(z)/f'(z)\} > \alpha$ for some $\alpha (0 \leq \alpha < 1)$ and for all $z \in U$. We denote by $K(\alpha)$ the class of all functions in A which are convex of order α. Note that $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, and that $K(\alpha) \subseteq K(0) \equiv K$ for $0 \leq \alpha < 1$.

In order to derive our theorem for $f(z) \in K(\alpha)$, we recall here the following lemma by MacGregor [**3**].

LEMMA 2. *If $f(z) \in K(\alpha)$, then $f(z) \in S^*(\beta(\alpha))$, where $\beta(\alpha)$ is given by*

$$\beta(\alpha) = \begin{cases} (2\alpha - 1)/2(1 - 2^{1-2\alpha}) & (\alpha \neq 1/2) \\ 1/2 \log 2 & (\alpha = 1/2). \end{cases}$$

Commbining Theorem 2 and Lemma 2, we have

THEOREM 3. *Let $L_\lambda(r)$ $(\lambda \geq 0)$ be the length of the image-curve of $\{z : \mid z \mid = r < 1\}$ by mapping $w = f_\lambda(z)/z$ caused by* (1.1) *with any $f(z) \in K(\alpha)$. Then we have the estimation* (1.2) *where $L = L_0$ satisfies*

$$L(r) \leq \frac{8(1-\beta(\alpha))r}{(1-r)^{2(1-\beta(\alpha))}(1+r)} K\left(\frac{2\sqrt{r}}{1+r}\right).$$

where K denotes the complete elliptic integral of the first kind with modulas $2\sqrt{r}/(1+r)$.

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Department of Mathematics
Kinki University
Higashi-Osaka, Osaka 577
Japan

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