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SOME REMARKS ON THE WEAK TOPOLOGY OF LOCALLY CONVEX SPACES

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Abstract. In this note we show that if $(E, \|\cdot\|)$ is a Banach space of infinite dimension then the spaces $(E, \sigma(E, E'))$ and $(E', \sigma(E', E))$ are not spaces of type DF.

In [3] Komura has proved the following theorem: any separated locally convex space is a closed linear subspace of some barrelled space. From this it follows that a closed linear subspace of a countably barralled (countably quasibarrelled) space need not be of the same sort, if there exists a separated locally convex space which is not countably quasibarrelled. In [6, (iii)] Iyahen has proved the following result: let: (E, u) be c_0 with the supremum norm u and let ν be the associated weak topology on c_0; then (E, ν) is not a countably quasibarrelled space. In this note we show that this result of Iyahen is true for every normed space of infinite dimension.

Throughout this paper (E, t) will denote a separated locally convex space over K, where K is the field of real or complex numbers. In general, we follow [5] for definitions concerning locally convex spaces. We shall need the following definitionns of [1], [2] and [4]. A separated locally convex space (E, t) with dual E' is called countably barrelled (countably quasibarrelled) if every weakly (strongly) bounded subset of E' which is a countable union of t-equicontinuous subsets of E' is t-equicontinuous. A locally convex space (E, t) is a space of the type DF if it is countably quasibarrelledwith a fundamental sequence of bounded sets. A barrel U in the space (E, t) is a b-barrel if $U \cap B$ is a neighborhood of origin in B, for every t-bounded absolutely convex subset of E. The space (E, t) is b-barrelled if every b-barrel is a t-neighborhood of origin.

We start with the following result:

THEOREM 1. *If $(E, \|\cdot\|)$ is a normed space of infinite dimension, then $(E, \sigma(E, E'))$ is not a countably quasibarrelled space.*

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Proof. Assume that $(E, \sigma(E, E'))$ is a countably quasibarrelled space. First we show that $\beta(E', E)$-bounded sets are $\beta(E', E)$-precompact. Suppose the contrary and let A be a $\beta(E', E)$-bounded set wich is not $\beta(E', E)$-precompact. There exists a $\beta(E', E)$-neighborhood of the origin U and a sequence $\{x_n\} \subset A$ such that, $n \neq m$ implies $x_n - x_m \notin U$. The sequence $\{x_n\}$ is not $\beta(E', E)$-precompact, but it is $\beta(E', E)$-bounded, and hence it is $\sigma(E, E')$-equicontinuous. This is a contradiction, since on $\sigma(E, E')$-equicontinuous sets the weak and strong topologies agree. Since $\beta(E', E)$-bounded sets are $\beta(E', E)$-precompact, it follows that E', i.e. E, is a space of finite dimension, but this contradicts our assumption. This completes the proof.

Remark 1. A separated locally convex space (E, t) is σ-quasibarrelled if every strongly bounded sequence of E' is t-equicontinuous. Form the proof of Theorem 1 it follows that $(E, \sigma(E, E'))$ is not even σ-quasibarrelled. But it is easy to verify that for each locally convex space (E, t) the associated space $(E, \sigma(E, E'))$ is countably quasibarrelled if and only if it is σ-quasibarrelled.

The following theorem and corollary are of standard interest:

THEOREM 2. *If the locally convex space (E, t) is a space of type DF and t-bounded sets are t-precompact, then it is a quasibarrelled space.*

Proof. Since $\beta(E', E) = E'_p$, where E'_p is a topology of uniforme convergence on the family of all precompact subset of E, then as in Theorem 1 we show that $\beta(E', E)$-bounded sets are $\beta(E', E)$-precompact By [4, Theorem 3.1.2] the space (E, t) is b-barrelled; hence, according to [4, Theorem 1.1.7] $\beta(E', E)$-precompact sets are t-equicontinuous. This shows that (E, t) is a quasibarrelled space.

COROLLARY 1. *if $(E, \|\cdot\|)$ is a Banach space of infinite dimension, then $(E', \sigma(E', E))$ is not a countably quasibarrelled space.*

Proof. If $(E', \sigma(E', E))$ is a countably quasibarrelled space, then by Theorem 2, it is quasibarrelled, i.e. a barrelled space. From this it follows that $\sigma(E', E) = \beta(E', E)$, but this is imposible for the Banach space $(E', \beta(E', E))$ of infinite dimension.

Remark 2. If $(E, \|\cdot\|)$ is a normed space which is not barrelled, then Corollary 1 is not always true. Indeed, if E is the space of all sequences containing only a finite number of non-zero entries, with the supremum norm, then $\sigma(E', E) = \tau(E', E) = \beta^*(E', E)$, i.e. $(E', \sigma(E', E))$, is a quasibarrelled space, but it is not countably barrelled, i.e. a barrelled space.

Instead of Corollary 1 we have:

COROLLARY 1$'$. *If $(E, \|\cdot\|)$ is a normed space which is not barrelled, then $(E', \sigma(E', E))$ is not a countably barrelled space.*

From the following theorem it follows that Iyahen's result is true for some spaces of type DF.

THEOREM 3. *If the locally convex space (E, t) is a space of type DF and*

t-bounded sets are not precompact, then the space $(E, \sigma(E, E'))$ is not countably quasibarrelled, i.e. a space of type DF.

Proof. Since $\sigma(E, E')$-bounded sets are $\sigma(E, E')$-precompact, it is clear that $\sigma(E, E') < t$. Now, let $(E, \tau(E, E'))$ be a space of type DF. Then by Theorem 2 it is a quasibarrelled space, i.e. $t \leq (E, E') = \sigma(E, E') = \beta^*(E, E')$; but this is a contradiction.

COROLLARY 2. *Let (E, t) be a sequentially complete locally convex space of type DF such that t-bounded sets are not t-precompact; then the space $(E', \sigma(E', E))$ is not of type DF.*

Proof. Indeed, if $(E', \sigma(E', E))$ is a space of type DF, then by Theorem 2, it is barrelled i.e. $\sigma(E', E) = \tau(E', E) = \beta(E', E)$. This implies that t-bounded sets are of finite dimension, hence t-precompact. This is a contradiction and the proof is complete.

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