PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série tome 45 (59), 1989, pp. 17-26

ORDER PRESERVING OR INCREASING MAPPINGS FREEDOM OR INCOMPARABILITY PRESERVING MAPPINGS

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Summary. One recalls the definitions of increasing, SI, (s. 2:0) ASI(s. 3:0) mappings of ordered sets and introduces FP mappings (s. 4:0). Main theorems 2:2, 2:2:7, 3:1, 3:5:1, 4:8 are established.

0. Introduction

0:0. In 1937:4 was introduced a very important notion of increasing (decreasing) mappings between ordered sets accompanied by statements–solution of some problems which were put earlier. At the same time were submitted the papers 1937:2, 1940:1, 1940:2, 1941:1, 1945:1, concerning ASI mapings (sf. no 3:0). It was proved that every uncountable tree in which there exists a real strictly increasing transformation is equinumerous to a free subset.

0:1. In the present paper analogous statements are proved for SI transformations of trees into linearly ordered sets L. Almost SI transformations from T into L are examined as well and in this area a very interesting theorem 3:5:1 is found showing a great difference in the behavior of SI and ASI transformations of ordered sets. In particular, the transfer of the main corolary 2:2:7 concerning SI transformations to the statement 3:5. concerning ASI transformations $T \to L$ has a postulational character.

0:2. Terminology and notations are as in other author's papers. In particular, T and L denote any tree and any chain (=linearly ordered set) respectively; unless otherwise stated, T is assumed to be infinite.

0:3. In particular the rank or the height γT is defined as the first ordinal which is not embeddable into T; one has the fundamental partition $T = \cup R_i T, (i < \gamma T)$ into rows or levels $R_i T$ of T; one puts

0:4. $mT := \sup pR_i T$, $(i < \gamma T)$; pX denotes the power (=cardinality) of X.

AMS Subject Classification (1980): Primary 06 A 10, Secondary 05 C 05

0:5. $(i)(E, \leq)$ is said to be degenerate or a d-set if for every $x \in E$ the coresponding cone $Ea := E(\cdot, a] \cup E[a, \cdot)$ is a chain; $E[a, \cdot) := \{x; x \in E,\ a \leq x\}, E(\cdot, a] := (E, \geq)[a, \cdot)$. The vacuous set is denoted by v or $\varnothing$. If E is finite or if (i) contains a d-subset of power pE, we say that (i) is d-reflexive. A free subset of (i) is any subantichain of (i). $b(E, \leq) := \sup\{pD : D \text{ is degenerate in } (E, \leq)\}$.

1. Generalities

1:0. Lemma. *Every tree T satisfies $pT \leq mT \cdot p\gamma T$; if T is infinite, then $pT = mT \cdot p\gamma T$ and $pT \in \{mT,\ p\gamma T\}$.*

The proof is obvious.

1.1. Lemma. *Let T be infinite; If c is any cardinal number $< pT$, then T contains a D-subset X such that $pX = c$.*

Proof. By L. 1:0 one has $pT = mT$ or $pT = p\gamma T$. If $pT = mT$, then the relations $c < pT = mT$ and $mT = \sup pR_iT(i < \gamma T)$ imply that some $i < \gamma T$ satisfies $pR_iT \geq c$. If $c < pT = p\gamma T$, then for the first ordinal $i < \gamma T$ such that $pi = c$ and for every $x \in R_iT$ the left cone $T(\cdot, x)$ is a chain of power c.

1:1:1. Corollary. *If T is infinite, then $bT = pT$ or $pT = (bT)^+$; the former holding for every limit pT.*

1.2. Lemma. *If (0) $pT[x, \cdot) < pT(x \in T)$, then is d-reflexive.*

Proof. The disjoint partition (1) $T = \cup T[x, \cdot)(x \in RT_0)$ and (0) imply that (2) $pR_0T = pT$ or at least (3) $pR_0 \geq cf\ pT := n$. If (2), everything is done; in particular, if pT is regular, then necessarily (2) holds. Therefore, there remains the case that pT is singular and that (2) does not hold; then $n \leq pR_0T < pT$ and $\sup pT[x, \cdot) = pT(x \in R_0T)$; therefore, there exists a set $A \subset R_0T$ such that $pA = n$ and $\sup pT(a, \cdot) = pT(a \in A)$. Let $(a_i;\ i < n)$ be a well ordering of A and $(k_i,\ i < n)$ an n-sequence of isolated stictly increasing cardinals such that $\sup k_i = pT$ thus also $\sup k_i^- = pT(i < n)$. Let b_0 be the first a_i such that $pT[a_i, \cdot) > k_0$; if for every $0 < j < n$ and every $i < j$ a member b_i of A is determined such that $pT[b_i, \cdot) > k_i$, let us define also b_j as the first member in the well-ordering of A such that $b_j \neq b_i(i < j)$ and (3) $pT[b_j, \cdot) > k_j$. Of course, b_j exists; so by (transfinite) induction we have an n-subsequence $b_j(j < n)$ of the n-sequence $a_i(i < n)$ such that (3) holds. Now, in virtue of Lemma 1:1, the relation (3) implies that $(3)_1$ contains a d-subset D_j for every $j < n$; then the union $D := \cup D_j,\ (j < n)$, is a required d-subset of T such that $pD = pT$.

1:3. Lemma. *If T is infinite and $mT > p\gamma T$, then T is equinumerous to a free subset A.*

Proof. Let $U := \{x : x \in T,\ pT[x, \cdot) < pT\}$. If $pU = pT$, then (v. L. 1:2) T is equinumerous to a free subset D. The equality $pT = pD$, the disjoint partition of D into chains $D[x, \cdot)(x \in R_0D)$ and the relation $pT = mT < p\gamma T$ imply $pR_0D = pD = pT$. If $pU < pT$, then $V := T \backslash U$ satisfies (0) $pV(x, \cdot) = mV = pV = pT$ for every $x \in V$. The case when $mT (= mV)$ is regular is settled like in the proof in no 1:2. If mV is singular, then some $i < \gamma V$ satisfies $pR_iV \geq n$; let then $A = (a_i, i < n)$ be a subset of R_iV of power n. Since $mV(x,\cdot) = pV = pT = mT = mV = m(x \in V)$, for any fixed cardinal $c < m$ there is a free subset $A(x)$ in $V(x,\cdot)$ such that $pA(x) \geq c$. By arguments like those in no. 1:2 one constructs the free sets D_i in $V(a_i,\cdot)$ of power $\geq c_i$, and the free subset $D = \cup A_i \subset V$ such that $pD = mV = pT$.

1:3:1. COROLLARY. *If T is infinite and $pT > p\gamma T$, then T is equinumerous to a free subset.*

1:4. LEMMA. *If $cf\ \gamma T \in \{1, \aleph_0\}$. then T is d-reflexive* (cf. 1935:2,3 no. 11:2a)).

Proof. In virtue of 1:3 Lemma, it is sufficient to settle the case when $pT = p\gamma T$ and $pR_iT < pT(i < \gamma T)$. In addition we can suppose, like in the proof of 1:3 Lemma, that the corresponding set U satisfies $pU < pT$. Thus $V := T\backslash U$ satisfies 1:3:(0). Let $\alpha_i (i < \omega)$ be a strictly increasing sequence of ordinals $\to \gamma T$. Let $x_i \in R_iT$, $(i < \omega)$ be a strictly increasing sequence in T; the existence of such a sequence is obvious (by induction argument); then $L := \cup T(\cdot, x_i](i < \omega)$ is a chain in T of power $p\gamma T (= pT)$.

1:5. *Remark*. Unless stated otherwise, we shall assume in the sequel that $pT = p\gamma T \geq \aleph_0$ and that every subchain of T is $< pT$.

2. Increasing and strictly increasing mappings.

2:0. *Definition*. Let $((E, \leq), (F, \leq_F))$ be a 2-un of ordered sets; every mapping $f : E \to F$ such that $x \leq y[x < y]$ in $(E, \leq)$ implies $fx \leq_F fy[fx <_F fy]$ in $(F, \leq_F)$ is called increasing or orderpreserving [strictly increasing, SI, or strictly orderpreserving mapping] from $(E, \leq)$ into $(F, \leq_F)$ (cf. Kurepa 1937:4, 1940:1,2, 1941:1, 1945:1). E. g. each constant automapping of $(E, \leq)$ is increasing. For every T the mapping $x \in T \to \gamma(x,T)$ where $x \in R_{\gamma(x,T)}T$ is SI, from T onto the section $O[0, \gamma T)$ of all ordinals $< \gamma T$) It is interesting to notice the following.

2:1. THEOREM. *If there is a SI selfmapping f of an infinite T into a subchain $L \subset T$, then T is not only d-reflexive, but in addion T is equinumerous: to a free subset A (case $mT > p\gamma T$) or to L (case $mT \leq p\gamma T$). Let $F_i := fF_iT$, $c_i = \inf F_i (i < \gamma T)$; then $c_i < c_j$ for $i < j < \gamma T$; the set $L_0 = \cup T(\cdot, o_i], (i < \gamma T)$ is a branch of T such that $L_0 \cap R_iT \neq v\, (i < \gamma T)$. Although L is a universal chain in T—for every chain K in T, f/K is an isomorphism of K onto the part fK of L—L need not be a branch in T. The sets L, L_0 and $C := \{o_i : i < \gamma T\}$ are cofinal.*

Proof. First of all, if $i < \gamma T$, F_i is a nonempty part of the given wellordered subset L of T; therefore, c_i is the minimal point of F_i. Let us prove that $c_i < c_j$ for $i < j < \gamma T$. As a matter of fact, let $y \in R_jT$ such that $fy = c_j$; since $i < j$ there is a unique $x \in R_iT$ such that $x <_T y$ and $x \in R_iT$; thus $c_i \leq_T fx <_T fy = c_j$, and $c_i <_T c_j$; $C := \{c_i : i < \gamma T\}$ is a chain in T and its order type is γT; therefore, in particular, (0) $pT \geq pC = p\gamma T$ and the well-ordered sets $C, L, L_0, O[0, \gamma T)$ are pairwise order-isomorphic; therefore C, L, L_0 are cofinal, i. e. if $X, Y \in \{C, L, L_0\}$ then $X = \cup X(\cdot, y]$, $(y \in Y)$.

What about pT? Since T is infinite, $pT = mT$ or $pT = p\gamma T$. If $mT > p\gamma T$, then $pT = mT$ and, in virtue of L. 1:3, T contains a free set A of power pT. If $mT \leq p\gamma T$, then (1) $pT = p\gamma T$; therefore (0) yields $pT = pC = pL$. This completes the proof of 2:1 Theorem.

2:1:1. COROLLARY. *An SI mapping $f : T \to L \subset T$ exists if and only if T is attained in the sense that T contains a chain intersecting every level of T.*

2:2. MAIN THEOREM. *Let $\aleph_\sigma$ be any aleph and $(L, \leq_L)$ any ordered chain such that the density (=separability) number dL equals $\aleph_\sigma$. Every tree T of power $pT > \aleph_\sigma$ such that there exists an SI mapping f of E into L contains a free subset mA of power pT (for the case $\sigma = 0$ see Kurepa* 1937:4 *Th. I,* 1941:1 *Th.* 6).

The proof of 2:2, is implied by the following facts 2:2:0—2:2:6.

2:2:0. LEMMA. *If D is a d-subset of T of power pT, then $A : R_0D$ is a required free subset A of T og power pT.*

As a matter of fact, every summand $a' := D[a, \cdot)$ in $D = \cup D[x, \cdot), (x \in R_0D)$, is order-similar to the well-ordered subset fa' of L; therefore $pa' \leq dL$ and consequently (0) $pt = pD \leq pR_0D \cdot dL$.

Now, $pR_0D = pT$. In the opposite case one would have $pR_0T < pT$ and therefore $pD < pT$ because both factors in the last term of the relation (0) are $< pT$.

2:2:1. In virtue of Lemma 1:3 we may suppose that $mT \leq p\gamma T$ and sonsequently (T being infinite) $pT = p\gamma T$. Now, T contains no chain C of cardinality $p\gamma T$, because otherwise fC would be a well-ordered subset of L of power $p\gamma T = pT$; this is impossible because every well-ordered subset of L is $\leq dL < pT$.

2:2:2. Let $U := \{x : x \in T,\ pT[x, \cdot) < pT\}$. If $pU = pT$, then, by L. 1:2, U (and a fortiori T) is d-reflexive. If $pU < pT$, the tree $V := T \backslash U$ is of power pT and satisfies $pV(a, \cdot) = pV = pT$, $(a \in V)$. Therefore, there is no restriction to assume that U =empty (it is sufficient to change the notation to write T instead of $T \backslash U$). In order words, we have just proved the following.

2:2:3. LEMMA. In order to prove the Main Theorem 2:2 it is sufficient to prove the statement 2:2 under the following conditions (0)—(4):

(0) $pT = \aleph_\tau,\ \gamma T = \omega_\tau$

(1) $pR_iT < pT\ \ (i < \gamma T)$

(2) Every chain in T is $< \aleph_\tau$;

(3) $pT[x, \cdot) = pT(x \in T)$;

(4) There is an SI mapping f of T into a chain L such that $dL = \aleph_\sigma < pT = \aleph_\tau$.

2:2:4. LEMMA. *A consequnce of* (0)—(3) *is the following.*

(5) $mT :=\sup pR_iT,\ (i<\gamma T)$, *is* $\geq n^- := (cf\, p\gamma T)^-$.

As a matter of fact, if $mT < n^-$, then, by Theor. 5 bis in 1935:2,3 p. 80, T would contain a chain of power $p\gamma T$, contrary to (2).

2:2:5. LEMMA. *T which satisfies* (0)—(5) *contains a free subset A_0 of power* $n := cf\, p\gamma T$.

Proof. Let $r_j (j < \omega_\sigma)$ be a normal one-to-one well-order of a density base S of L. Thus S is a subset of L of minimal power dS such that every non-empty open interval of L contains a point of S. Let g be a mapping of T such that $gt \in R_1T(t,\cdot)(t \in T)$; then obviously $ft <_L fgt\, (t \in T)$. For every $j < \omega_\sigma$ let

(6) $T^j = \{t : t \in T,\ ft \leq_L r_j <_L fg^2t\}$.

Then $T^j \neq v \neq L(ft, fg^2t)(t \in T)$ and

(7) $T = \cup T^j (j < \omega_\sigma)$.

I. *First case*: γT is regular: $n = \aleph_\tau$. Since, by assumption (4), $\tau > 0$, the partition (7) implies the existence of a $j < \omega_\sigma$ such that

(8) $pT^j = pT$.

Therefore it sufficies to prove that T^j contains a free set A_0 of power n. If some row R of T^j has n points, it is sufficient to put $A_0 := R$. Therefore, let us suppose that $pR_iT^j < n(i < \gamma T^j)$ and consequently

(9) $\gamma T^j = \gamma T = \omega_\tau$.

By induction procedure, we are going to define a 1–1 sequence

(10) $(a_i, i < \omega_\tau)$ of incomparable points of T^j such that $\gamma a_i (i < \omega_\tau)$, where $a_i \in R_{\gamma a_i}T$, is SI and $\to \omega_\tau$ and

(11) $\gamma a_i < \gamma g a_i < \gamma a_{i+1} (i < \omega_\tau)$.

To start with, let a_0 be a point in R_0T^j. Let ν be any ordinal such that $0 < \nu < \omega_\tau$ and that the ν-initial segment of (10) is defined in such a way that the conditions (11) for $i < \nu$ are satisfied. Then we consider the ordinal $\beta := \sup \gamma a_i (i < \nu)$; since $\nu < n$ and since n is regular, one has $\beta < n$; therefore, the level $R_{\beta+2}T^j$ is $\neq \nu$ (cf. (9)). We denote by a_ν any point of this level. Consequently, the induction procedure of the construction of (10) is going on for every $i < \omega_\tau$ and the conditions (11) are satisfied. Let us prove that the points $g^2a_i (i < \omega_\tau)$ are incomparable. First, the ω_τ-sequence $\gamma g^2 a_i (i < \omega_\tau)$ is SI: if $x < y < \omega_\tau$, then $\gamma g^2 a_x < \gamma g^2 a_y$. Therefore, one does not have $g^2 a_y \leq g^2 a_x$. One has

(12) $g^2 a_x \leq g^2 a_y$ neither. In the opposite case, the relation (12) would be possible and the point $g^2 a_y$ would be preceded by a_y as well as by $g^2 a_x$. Therefore, the points $g^2 a_x, a_y$ would be comparable; now, for their ranks $\gamma g^2 a_x, \gamma a_y$, in virtue of (12), one has (because $x < y$) $\gamma g^2 a_x < \gamma a_y$; therefore, the relation $a_y \leq g^2 a_x$ is excluded; one would have $g^2 a_x <_T a_y$ and $fg^2 a_x <_L f a_y$; the last inequality with $f a_y \leq_L r_y <_L fg^2 y$ (cf(6)) would imply $fg^2 a_x \leq_L r_y$, contrary to the defining relation (6) for every element $a_x \in T^j$.

II *Second case*: $\gamma T = \omega_\tau$ is singular: $n < \aleph_\tau$. Since by condition (4), $dL = \aleph_\sigma < \aleph_\tau$ there is a regular $\aleph_\rho < \aleph_\tau$ which is $> n$, dL; in particular, the tree

$X := T^j(\cdot, \omega_\rho) := \cup R_i T^j (i < \omega_\rho)$ is a tree satisfying (0)—(4) with ρ instead of $\tau; \gamma$ is regualar; and the above first case of L. 2:2:5. applied to this set X yields an antichain A_0 in $X \subset T^j \subset T$ of power n. This proves L. 2:2:5 completely.

2:2:6. Final step in the proof of the Main Theorem 2:2. From the free subset $A_0 \subset T$ of cardinality $n := \mathrm{cf} pT$ it is easy to deduce a free subset $A \subset T$ of cardinality pT. If pT is regular, it sufficies to put $A := A_0$. If pT is singular, let $A_0 = (a_i, i < \omega_{(n)})$ be a 1–1 well-ordering of the free subset $A_0 \subset T$ of cardinality n (s. L. 2:2:5). Let $(c_i, i < \omega_{(n)})$ be an SI $\omega_{(n)}$-sequence of cardinals $< pT$ such that $\sup c_i = pT$; let $b_i \in R_{\omega(ci)} T(a_i, \cdot)$; then $D := \cup T(a_i, b_i)(i < \omega_{(n)})$ is degenerate of power pT; by *L.* 2:2:0 the first level $R_0 D$ is a free subset of T of power pT as was required in the Main Theorem 2:2. Q. E. D.

2:2:7. Main Corollary= Wording obtained from 2:2 on replacing "free subset A" by"degenerate subset D".

3. Almost Strictly Increasing (ASI) Mappings.

3:0. *Definition.* An increasing mapping $f : (E, \leq_E) \to (F, \leq_F)$ such that $x \in E$, $pE[x, \cdot) > 1$ implies $pfE[x, \cdot) > 1$ is said to be ASI (Almost Strictly Increasing); in other words, unless x is a terminal point of E there is some $x <_E y \in E$ such that $fx <_F fy$. The notion was introduced at the same time when was introduced the notion of increasing and strictly increasing [SI] mappings (s. Definition 2:0).

Here is a theorem concerning a connection between ASI and SI mappings of trees T on chains L.

3:1. THEOREM. *Let* $f : (T, \leq) \to (L, \leq_L)$ *be* ASI *and*

(0) $Tf := R_0(T, \leq) \cup R_0(T, \geq) \cup \underset{c}{\cup} R_0\{y : C <_T y \in T \& fC <_L fy\}$, C *running through the class* IT *of all subchains of* T.

(1) *The set* Tf *is the most extensive subset* X *of* T *such that* $f \mid X$ *is* SI;

(2) Tf *is cofinal with* T, *i. e.* $T = \cup T(\cdot, x]$ $(x \in Tf)$.

Proof of (1). First, f is SI in Tf: if $x < y$ in Tf, then $fx <_L fy$ in L. As a matter of fact, $fx \leq_L fy$. Now, since x, $y \in Tf$ and $x < y$, the set $\cup T(\cdot, t](t < y$ such that $ft <_L fy)$ is a chain C; one has $x \in C < y$ and $fC <_L fy$, thus $fx <_L fy$.

Secondly, assume that there exists a subset $X \subset T$ such that $Bf \subset\neq Tf$ and that $f \mid X$ is SI; thus there would exist a point (3) $x \in X \backslash Tf$. The point x is neither initial nor final in T; thus the chain $T(f)(\cdot, x)$ is $\neq \varnothing$; the more is $T(\cdot, x) \neq v$; let $C := C(x)$ denote the most exstensive initial section of $T(\cdot, x)$ such that $C <_L fx$. The set Y of all points $t \in T$ such that $C < t$ is well determined: so is $R_0 T$ as well. By definition of Tf this set is a part of Tf; therefore, the unique point x' in $R_0 Y$ which is $< x$ is a well determined point in Tf, thus also in X. Consequently, x', x would be two points in X such that $x' < x$. Since f is SI in $X, fx' <_L fx$; therefore, by definition of $C(x), x' \in C$, contrary to the fact $C < Y$ and in particular to the fact that $C < x' \in R_0 Y$. This contradiction eliminates the assumption (3) as false.

Proof of (2): if $t \in T$ then some $x \in Tf$ satisfies $t \leq x$. First, if t is a terminating point in T i. e. if $t \in R_0(T, \geq)$. then by definition of Tf one has $t \in Tf$. If $t \notin R_0(T, \geq)$, then by definition of the ASI f there exists a $y \in T$ such that $t < y \in T$, $fT <_L fy$; the first point x of the well ordered set $T(t, y]$ for which $ft <_L fx$ is a required member of Tf such that $t \leq x$. This finishes the proof.

3:2. THEOREM. *Let $f : T \to L$ be ASI; whenever $pTf > dL$, the set Tf is not only d-reflexive but also equinumerous to a free subset of T* (cf. 3:1(0)).

Proof. The d-reflexivity of Tf is implied by the Main Corollary 2:2:7 and the Theorem 3:1. Thus there is a d-set D in Tf such that $pTf = pD$. We claim that $pD = pR_0D$. This is implied by the decomposition $D = \cup D[a, \cdot)(a \in R_0D)$ of D into disjoint chains and the fact that each summand is $\leq dL$, whence one has $pD \leq pR_0D \cdot dL$; therefore if $pR_0D < pD$, the number $pTf (= pD)$ would be $\leq$ the product of numbers pR_0D, dL each $\leq pTf$, contrary to the hypothesis that $pTf > dL$.

3:3. THEOREM. *Let f: $T \to L$ be ASI and $pT > dL$; if pT is regular, then T is equinumerous to a free subset.*

Proof. Due to the decomposition 3:1:(2) one has cf $pT \leq pTf$ (reall that by remark 1:5 we assume that every chain in T is $< pT$) i. e. $pT = pTf$ and $pTf > dL$; therefore, one can apply the Main Theorem 2:2 for the tree Tf and get a free subset F of $Tf \subset T$ such that $pF = pTf = pT$.

3:4. THEOREM. *Let T be a sequence-tree (i. e. $\gamma T = \gamma T(t)$, where $T(t) = T(\cdot, t] \cup T[t, \cdot)$ for every $t \in T$); if f: $T \to L$ is ASI and $pTf > dL$, then T is d-reflexive.*

Proof. Since f is SI in Tf and since $pT > dL$, the Main Theorem 2:2 yields a free subset D of Tf such that $pD = pTf$. As above in 2:2:0 one proves that $pR_0D = pD$. On the other hand, the decomposition 3:1 (2) implies that $n : cf\ pT \leq pTf$; thus $n < pR_0D$. Let A be any subset of R_0D such that $pA = n$; let $\beta_i := \omega_{(c_j)}(i < n)$ be an n-sequence of ordinals $\to \gamma T$; for every $a \in A$ let $b(a) \in R_{\beta_i}T(a, \cdot)$; then $\cup b(a)$ $(a \in A)$ is a requered d-subset of T of power pT.

3:5. Proposition P_{49} is the statement obtained from the statement of the Main Theorem 2:2 writing ASI instead of SI and a degenerate subset D instead of a free subset A; thus

3:5:0. *Definition* of P_{49}. Let $\aleph_\sigma$ be any aleph and $(L, \leq_L)$ any linearly ordered set such that the density number dL equals $\aleph_\sigma$. Every tree T of power $pT > \aleph_\sigma$ such that there exists an ASI mapping $f : E \to L$ contains a degenerate subset D of power pT.

3:5:1. THEOREM. *P_{49} and the RH (Ramification Hypothesis) are equivalent.*

The implication $RH \Rightarrow P_{49}$ being obvious, let us prove the converse implication $P_{49} \Rightarrow RH$.

1. If this implication were false, there would exist an infinite tree S in which evert d-subset is $< pS$; in particular every subchain and every free set of S would

be $< pS$ and necessarilly $cf\,\gamma S > \aleph_0$ (cf. no 11:3 pp. 108–109 Kurepa 1935:2,3; s. also the above 1:4. Lemma).

2. Let $S' := \cup R_{i+1}S(i < \gamma T)$. Let $La(a \in S')$ be an S'— un of disjoint well ordered sets of order type β each, where $\beta := \omega(2^{pS}) :=$ the first ordinal of cardinality 2^{pS}. Let $Z := S \cup La(a \in S')$; we order Z in such a way that $Z(a^-, a) := La(a \in S')$ and that for incomparable points a, b in S one has $\gamma(a, S) = \gamma(b, S) \Rightarrow La \| Lb$ in Z. Then one checks readily that Z is a tree such that $\gamma Z = \beta,\ mZ = mS = (p\gamma S)^-$; in addition, S is cofinal to Z.

3. Z is not d-reflexive.

In the opposite case there would exist a d-subset D of Z such that $pD = pZ$ and $pR_0D \geq cf\,p\gamma Z = cf\,p\gamma S := n$. If then for every $x \in R_0D$ one denotes by gx a point of S such that $x \leq gx$, then the set $A := \{gx, x \in R_0D\}$ would be an antichain in S such thau (0) $pA \geq n$.

The last relation does not hold if γS is regular because by definition of S every antichain in S is of a power $< p\gamma S$. The relation (0) holds neither if γS is singular because in this case one would establish (by usual procedure) a d-subset A' of $\cup S[a, \cdot)(a \in A)$ such that $pA' = pS$, i. e. S would be d-reflexive, contary to the initial assumption.

4. On the other hand, let us define a mapping $f : Z \to L := O[0, \gamma S)$ by $fx = \gamma(x, S)(x \in S)$, $fx = \gamma(a, S)(x \in La,\ a \in S')$. One checks readily that f is ASI in Z. In addition $pZ = 2^{pS} > pS = pL$. Thus we should be allowed to apply the statement P_{49} and conclude that Z would be d-reflexive, contrary to the fact 3. This contradiction proves the requered implication $P_{49} \Rightarrow RH$.

4. Freedom (Incomparability or Antijoin) Preserving [FP] mappings between ordered sets.

4:0. *Definition.* A mapping $f : (E, \leq_E) \to (F, \leq_F)$ is said to be FP provided $x \| y$ in $(E, \leq_E)$ implies $fx \| fy$ in $(E, \leq_F)$.

Consequently, in every free subset $A \subset E$ the FP mapping f is bijective; on any chain $L \subset E$, f could be even constant.

4:1. LEMMA. *Let $a(E, \leq)$ denote the system of all antichains of $(E, \leq)$; $a(E, \leq)$ is monotone additive in the sense that for any linearly ordered subsystem $(M, \subset)$ of $a(E, \leq)$ the union $\cup M$ is an antichain.*

The proof is straighforward because it a, b are 2 distinct points of $\cup M$ let $A, B, \in M$ be such that $a \in A, b \in B$; then $A \subset B$ thus $\{a, b\} \subset B$ or $B \subset A$ thus $\{a, b\} \subset A$; consequently in either case, a, b belong to a member of M, and therefore $a \| b$.

4:2. LEMMA. *The system $a(E, \leq)$ contains various disjoint subsystems D such that $\cup D = \cup a(E, \leq) = E$.*

Proof. Such a system is the system of all singletons $\{x\}(x \in E)$. One can proceed also in the following typical way. Let D_0 be a *maximal* antichain in $(E, \leq)$;

let D_1 be a maximal antichain in $(E\backslash D_0, \leq)$; if disjoint antichains (1) $D_i(i < j)$ are formed; let us consider the set (2) $E\backslash \cup D_i(i < j)$; if (2) is v, then (1) is a required disjoint system of antichains exhausting E; if (2) $\neq v$, let D_j be a maximal antichains of (2). By induction procedure one gets in this way a maximal sequence of disjoint nonempty antichains.

Similary one proves the following.

4:3.LEMMA. *The system $l(E, \leq)$ of all chains of $(E, \leq)$ contains various subsystems of pairwise disjoint chians exhausting E; in particular, there is a disjoint system F of chains exhausting F and such that $pT = st(E, \leq) :=$ the least cardinal c such that there exists a system F of subchains such that $pF = n$ and $\cup F = E$.*

Proof of the last phrase of the Lemma. Let G be a system of chains exhausting E and such that $pG = st(E, \leq)$. Let (0) $g_i(i < \beta)$ be a normal well-order of G. Let h_0 be a maximal chain $\supset g_0$; assume $0 < \alpha < \beta$ and that disjoint chains $h_i(i < \alpha)$ are formed such that $h_i \supset g_{ni}$; let us define h_α: let g_{n_α} be the first member of (0) such that g_{n_α} is not contained in (1) $\cup g_{n_i}(i < \alpha)$; we denote by h_α any maximal chain L such that $g_{n_\alpha} \subset L \subset E\backslash(1)$. The procedure is going on for every $\nu < \beta$ because otherwise if it stopped for some $\gamma < \beta$, the system of sets $g_{n_i}(i < \gamma)$ would exhaust E and would be of a power $< st\, E$ and this is a contradiction.

4:4. THEOREM. *Given $((E, \leq), (F, \leq_F))$, if $(F, \leq_F)$ contains an antichain M of power $st(E, \leq)$, then there exists a freedom preserving mapping f of $(E, \leq)$ into $(F, \leq_F)$ such that $fE \subset M$.*

Proof. Let H be any disjoint system of chains exhausting E and such that $pH = stE$; let h be a one-to-one mapping of H into M; if fot every $e \in E$ we define $fe := h(eH)$ where $e \in eH \in H$, the mapping $f \mid E$ is FP. As a matter of fact, if $a \|_e b$ then a, b belong to distinct members aH, bH of H, thus $h(aH) := fa$, $h(bH) = fb$ are distinct members of M.

4:5. *Remark*. All preceding considerations are transferable to binary graphs, where "sub chain" should be replaced by "complete subgraphs".

4:6. *Problem*. Is it legitimate to replace in the wording of the theorem 4:4 the phonem $st(E, \leq)$ by $p_s(E, \leq)$?

Let us examine this for trees.

If $p_s(T, \leq)$ is finite, then $p_s = st(T)$, and everything is O. K. If $p_s(T)$ is infinite and attained then RH implies $p_sT = st\,T$ and everything is O.K.

4:7. Statement TFPSFS (Tree FP Selfmapping into Free Subset): *For any tree T there is an FP selfmapping g into a free subset A of $(T, \leq)$.*

4:8. THEOREM. *TFPSFS is a consequence of the RH and is independent of the usual axioms of the Set Theory.*

Proof. According to the theorem 4:4, statement 4:7 holds for every tree T containing a free subset M of power $st(T, \leq)$. Now, the last condition is verified if γT is finite or countable. If $\gamma T = \omega_1$, then $st\,T = p_sT$ if and only if "The answer to the Suslin problem is affirmative" (s. 1963:3 Theor. 3:3); and one knows that this answer SH (Suslin Hypothesis) is a postulate. On the other hand, TFPSPS implies that the free number $p_s T$ is attained for every T; (obviously, gT shoud be an antichain of power $p_s T$). Now, the last fact is provable for every T for which $p_s T$ is not a regular infinite limit cardinal (cf. Kurepa 1987:1 Theor. 2:4). The attainability of $p_s T$ for the case when $p_s T$ is regular limit non countable is implied by the RH and in this case T is a union of $p_s T$ chains and one can apply the theorem 4:4.

4:9. The dual of TFPSFS obtained by substitutions FP|SI, Free subset | chain does not hold: it is violated each time when γT is not attained (s. 2:1 Theorem, 2:1:1 Corollary). Such is the case e. g. for the tree $w(Q, \leq)$: = set of all well-ordered subset of $(Q, \leq)$ ordered by the relation "to be an initial segment of".

4:10. *Remark*. ASI [FP] mappings are a particular case of Chain [Antichain] Preserving mapping carrying every chain [antichain]$\subset$ $(E, \leq)$ into a chain [antichain]: one agrees that $\varnothing$ and every singleton are chains and antichains. In a next paper we shall examine such transformations.

REFERENCES

1935:2 *Ensembles ordonnés et ramifiés*, Thèse de doctorat soutenue à la Faculté des Sciences à Paris le 1935:12:19; pp. VI+138+2. (Deuxième thèse:*Itération des fractions rationneles*).

1935:3 =1935:2. PMU 4(Beograd 1935) 1–138. JMF 61:2 (1935) 980 (Aumann) Zbl. 14 (1936) 394 (B. Knaster).

1937:2 *Ensembles linéaires et une classe de tableaux ramifiés. Tableaux ramifiés de M. Aronszajn*, P. M. 6 (Beogra 1937) 129–160. JMF 64 (1940) 37 (Aumann) Zbl. 20 (1940) 108 (Blumberg).

1937:4 *Transformations monotones des ensembles partiellement ordonnés*, C. R. 205 (Paris 1935) 1033–1035. JMF 64 (1940) 173 (Seebach) Zbl. 17 (1938) 300 (B. Knaster).

1941:1 (continuation of 1940:2) No 437, año 43 (Lima 1941) 483–500. M. R. 3 (1942) p. 225 (J. W. Tuckey).

1945:1 *Le poblème de Suslin et les espaces abstraits.* Revista de Ciencias, 47 (Lima 1945) No 453, pp. 457–488. MR. 7 (1946), p. 4. (J. Todd). Zbl. 60 (1957) p. 396.

1953:12 *Sur un principe de la théorie des espaces abstraits.* CR. 256 (Paris 1953:02:16) 655–657. MR. 15 (1954) 409 (Viola, T.). RŽM (1953) 141. Zbl. 51 (1954) 290 (Neumer, W.).

1963:3 *Star number and antistar number of ordered sets and graphs. Zvjezdani broj i antizvjezdani broj uređenih skupova i grafova.* Glasnik Mat. Fiz. Astr. (2), 18 (Zagreb 1963) 27–36–37. MR 30 (1965)# 1945 (L. Gillman), RŽM 1964:12 A 273 (Fedorčuk), Zbl. 127 (1967) 250 (A. H. Kruse).

1987:1 *On a tree statement* (=1986:2) J. Symb. Log. **52** (1986) 1071-2.

1987:2 *Free power or Width of some kinds of mathematical structures.* Publ. Math. **42** (**56**) (1987), 3–12.

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(Received 15 03 1989)