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ON THE SPACE OF ANALYTIC FUNCTIONS OF LOGARITHMIC TYPE T

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Abstract. We consider the space of functions analytic in a finite disc. Using the coefficient characterization of the logarithmic type we define a norm and show that the space obtained is a Frechét space. Characterizations for continuous linear functional and proper bases are also obtained.

1. Introduction. The study of spaces of entire functions was initiated by Ganapathy Tyer [**3**]. He introduced the notion of a proper base and established a relationship between proper bases and automorphisms of the space. Arsov [**1**] considered the space of functions analytic in the finite disc $|z| < R$ endowed with the topology of uniform convergence on compact sets and obtained a relationship between proper bases and linear homeomorphisms. Srivastava [**5**] defined a norm on the space of analytic functions with the help of growth parameters and studied the properties of this space.

Let U_R denote the class of all functions f analytic in $|z| < R < \infty$, where $f(z) = \sum_{n=0}^{\infty} a_n z^n$. We set $M(r,f) = M(r) = \max_{|z|=r} |f(z)|$, $0 < r < R$. Then f is said to be of order ρ_0 if

$$\limsup_{r\to R} \frac{\log^+ \log^+ M(r)}{-\log\log(R/r)} = \rho_0, \quad 0 \le \rho_0 \le \infty, \tag{1.1}$$

where $\log^+ x = \max(0, \log x)$ for $x > 0$. If $0 < \rho_0 < \infty$ then the type T_0 of f is defined by

$$\limsup_{r\to R} \left[\log^+ M(r)(\log(R/r))^{\rho_0}\right] = T_0, \quad 0 \le T_0 \le \infty. \tag{1.2}$$

Srivastava [**5**] used the coefficient characterization of the type T_0 to define a norm as follows. It is known [**2**] that

$$\limsup_{n\to\infty} \left\{\left[\log^+ (|a_n|\, R^n)\right]^{\rho_0+1}\right\} n^{-\rho_0} = T_0 A^{\rho_0+1}, \quad \text{where } A = (\rho_0+1)\rho_0^{-\rho_0/(\rho_0+1)}. \tag{1.3}$$

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Let $U_R(\rho_0, T_0)$ denote the class of all functions f, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, analytic in the disc $|z| < R$, having growth parameters not exceeding (ρ_0, T_0). Then for $f \in U_R(\rho_0, T_0)$, we have

$$\lim_{n\to\infty} \sup n^{-\rho_0} \left[\log^+ (|a_n| R^n)\right]^{\rho_0+1} \leq A^{\rho_0+1} T_0. \tag{1.4}$$

For any $\delta > 0$, define

$$\|f; \rho_0, T_0 + \delta\| = |a_0| + \sum_{n=1}^{\infty} |a_n| R^n P(n, \rho_0, T_0 + \delta)$$

where $P(n, \rho_0, T_0 + \delta) = \exp\left[-A n^{\rho_0/(\rho_0+1)} (T_0 + \delta)^{1/(\rho_0+1)}\right]$. Evidently, if $\rho_0 = 0$ or $\rho_0 = \infty$, then the type T_0 can not be defined and consequently the norm above can not be defined either. In this paper, we study the spaces of analytic functions of slow growth, (i.e. when $\rho_0 = 0$). For such functions, the logarithmic order ρ is defined as in [**4**]:

$$\lim_{r\to R} \sup \frac{\log^+ \log^+ M(r)}{\log\log[R/(R-r)]} = \rho, \quad 0 \leq \rho \leq \infty. \tag{1.5}$$

Further, if $0 < \rho < \infty$, the logarithmic type T is defined by

$$\lim_{r\to R} \sup \frac{\log^+ M(r)}{\{\log[R/(R-r)]\}^{\rho}} = T, \quad 0 \leq T \leq \infty.$$

For $1 < \rho < \infty$, the logarithmic type T is given by [**4**; Lemma, p. 448]

$$\lim_{n\to\infty} \sup \left[\log^+ (|a_n| R^n) (\log n)^{-\rho}\right] = T \tag{1.7}$$

We denote by $U_R(\rho, T)$ the class of all functions $f(z) = \sum_{n=o}^{\infty} a_n z^n$ analytic in the disc $|z| < R$, and of logarithmic growth (ρ, T), that is, the logarithmic order of f does not exceed ρ and if f is of logarithmic order ρ, its logarithmic type T does not exceed T, $1 < \rho < \infty$, $0 \leq T < \infty$. From (1.7), it follows that $f \in U_R(\rho, T)$ if and only if

$$\lim_{n\to\infty} \sup \left[\log^+(|a_n| R^n)(\log n)^{-\rho}\right] \leq T. \tag{1.8}$$

From (1.8), we have for any $\varepsilon > 0$ and all $n > n_0(\varepsilon)$

$$|a_n| < R^{-n} \exp\left[(T + \varepsilon)(\log n)^{\rho}\right]. \tag{1.9}$$

For each $f \in U_R(\rho, T)$ we define for $\delta > 0$,

$$\|f; \rho, T + \delta\| = |a_0| + \sum_{n=1}^{\infty} |a_n| R^n \exp\left[-(T + \rho)(\log n)^{\rho}\right]. \tag{1.10}$$

In view of (1.9), (1.10) clearly defines a norm for any $\delta > 0$. We denote by $U_R(\rho, T, \delta)$ the space $U_R(\rho, T)$ equiped with the norm (1.10). Let $U_{R,\lambda}(\rho, T)$ be the space $U_R(\rho, T)$ equiped with the weakest topology which is stronger than the topologies of each $U_R(\rho, T, \delta)$. The equivalent metric on $U_{R,\lambda}(\rho, T)$ can be expressed as

$$\lambda(f, g) = \sum_{q=1}^{\infty} 2^{-q} \frac{\|f - g\|_q}{1 + \|f - g\|_q} \tag{1.11}$$

where $\|f - g\|_q = \|f - g; \rho, T + 1/q\|$, as defined by (1.10).

2. Linear transformations on $U_{R,\lambda}(\rho,T)$. In this section we obtain characterization of continuous linear transformation on $U_{R,\lambda}(\rho,T)$. First we prove

THEOREM 1. *The space $U_{R,\lambda}(\rho,T)$ is a Frechét space.*

Proof. We show that the space $U_{R,\lambda}(\rho,T)$ is complete. Let $\{f_\alpha\}$ be a Cauchy sequence in $U_{R,\lambda}(\rho,T)$. Then it is a Cauchy sequence in each $U_R(\rho,T,\delta)$, $\delta>0$. Hence if we set $f_\alpha(z)=\sum_{n=0}^{\infty} a_n^{(\alpha)} z^n$, then for a given $\eta>0$ and q, there exists a positive integer $m_0=m_0(q,\eta)$ such that $\|f_\alpha - f_\beta\| < \eta$ for $\alpha,\beta \geq m_0$. Thus

$$\left|a_0^{(\alpha)} - a_0^{(\beta)}\right| + \sum_{n=1}^{\infty} \left|a_n^{(\alpha)} - a_n^{(\beta)}\right| R^n \exp\left[-(T+1/q)(\log n)^\rho\right] < \eta \tag{2.1}$$

for $\alpha,\beta \geq m$ and q a fixed positive integer. Hence for $n=0,1,2,\ldots$, we get $\left|a_n^{(\alpha)} - a_n^{(\beta)}\right| < \eta$ for all $\alpha,\beta\geq m_0$. Hence $\left\{a_n^{(\alpha)}\right\}_{\alpha=1}^{\infty}$ is a Cauchy sequence of complex numbers for each $n=0,1,2,\ldots$. Thus there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of complex numbers such that $\lim_{\alpha\to\infty} a_n^{(\alpha)} = a_n$, $n=0,1,2,\ldots$. Let $\beta\to\infty$ in (2.1). Then for $\alpha\geq m_0$ we have

$$\left|a_0^{(\alpha)} - a_0\right| + \sum_{n=1}^{\infty} \left|a_n^{(\alpha)} - a_n\right| R^n \exp\left[-(T+1/q)(\log n)^\rho\right] < \eta. \tag{2.2}$$

Let k be arbitrary integer, $1\leq q<k$. Since $f_{m_0}\in U_{R,\lambda}(\rho,T)$, we have from (1.9),

$$\left|a_n^{(m_0)}\right| < R^{-n} \exp\left[(T+1/k)(\log n)^\rho\right]$$

for $n\geq N_1(k,\eta)$. Also, $|a_n| \leq \left|a_n^{(m_0)}\right| + \left|a_n^{(m_0)} - a_n\right|$ for any n. Hence for $n\geq N_1'$

$$|a_n| < \eta\, R^{-n} \exp\left[(T+1/q)(\log n)^\rho\right] + R^{-n}\exp\left[(T+1/k)(\log n)^\rho\right].$$

Since $\eta>0$ was arbitrary and $k>q$, there exists a positive integer $N_2(q)$ such that for $n>N_2$,

$$\begin{aligned} &|a_n|\, R^n \exp\left[-(T+1/q)(\log n)^\rho\right] \\ &\quad < \eta + R^{-n}\exp\left[\{(T+1/k)-(T+1/q)\}(\log n)^\rho\right] < 1. \end{aligned}$$

Hence $|a_n| < R^{-n}\exp\left[(T+1/q)(\log n)^\rho\right]$, $n>N_2$. Thus the sequence $\{a_n\}$ satisfies (1.9) for every fixed $q=1,2,\ldots$. So, $f\in U_{R,\lambda}(\rho,T)$, where $f(z)=\sum_{n=0}^{\infty} a_n z^n$.

Now from (2.2), for any fixed q, we have $\|f_\alpha - f\|_q < \eta$, $\alpha\geq m_0$. Hence $f_\alpha\to f$ as $\alpha\to\infty$, in each $U_R(\rho,T,1/q)$. Thus $f_\alpha\to f$ in $U_{R,\lambda}(\rho,T)$, and the space $U_{R,\lambda}(\rho,T)$ is complete, and therefore a Frechét space.

In the next theorem, we give a characterization of linear continuous functional on $U_{R,\lambda}(\rho,T)$. We thus have

THEOREM 2. *A continuous linear functional F on $U_{R,\lambda}(\rho,T)$ is of the form $F(f)=\sum_{n=0}^{\infty}a_nC_n$ if and only if*

$$|C_n| \le A\,R^n \exp\left[-(T+1/q)(\log n)^{\rho}\right], \quad n\ge 1, \quad q\ge 1, \tag{2.3}$$

where A is a positive number depending on ρ but not on n and $f(z)=\sum_{n=0}^{\infty}a_nz^n$.

Proof. Let $F\colon U_{R,\lambda}(\rho,T)\to C$ be a continuous linear functional, where C is the field of complex numbers. Then for any sequence $\{f_m\}$, $f_m\in U_{R,\lambda}(\rho,T)$ such that $f_m\to f$ as $m\to\infty$ in $U_{R,\lambda}(\rho,T)$, we have $F(f_m)\to F(f)$. Now let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ where a_n's satisfy (1.9). Then $f\in U_{R,\lambda}(\rho,T)$. Also, for $m=0,1,2,\dots$, let us put $f_m(z)=\sum_{n=0}^{m}a_nz^n$. Then $f_m\in U_{R,\lambda}(\rho,T)$ for each $m=0,1,2,\dots$. Let q be any fixed, positive integer and let $0<\varepsilon<1/q$. Then from (1.9) we can find a positive integer m such that

$$|a_n| < R^{-n}\exp\left[(T+\varepsilon)(\log n)^{\rho}\right], \quad n>m.$$

Now

$$\|f-f_m\|_q = \left\|\sum_{n=m+1}^{\infty}a_nz^n\right\|_q = \sum_{n=m+1}^{\infty}|a_n|\,R^n\exp\left[-(T+1/q)(\log n)^{\rho}\right]$$
$$< \sum_{n=m+1}^{\infty}\exp\left[(\log n)^{\rho}(\varepsilon-1/q)\right] < \varepsilon.$$

for sufficiently large values of m. Hence for these values of m,

$$\lambda(f,f_m)=\sum_{q=1}^{\infty}2^{-q}\frac{\|f-f_m\|_q}{1+\|f-f_m\|_q} < \sum_{q=1}^{\infty}2^{-q}\left(\frac{\varepsilon}{1+\varepsilon}\right) < \varepsilon.$$

Hence $f_m\to f$ in $U_{R,\lambda}(\rho,T)$ as $m\to\infty$. Therefore $\lim_{m\to\infty}F(f_m)=F(f)$. Let us put $C_n=F(z^n)$. Then

$$F(f)=\lim_{m\to\infty}F(f_m)=\lim_{m\to\infty}\sum_{n=0}^{m}a_nC_n=\sum_{n=0}^{\infty}a_nC_n.$$

Further, $|C_n|=|F(z^n)|$. Since F is continuous on $U_{R,\lambda}(\rho,T)$, it is continuous on $U_R(\rho,T,1/q)$ for each $q=1,2,\dots$. Consequently there exists a positive number A independent of q such that

$$|F(z^n)|=|C_n|\le A\,\|\alpha_n\|_q, \quad q\ge 1, \qquad \text{where } \alpha_n(z)=z^n.$$

Now, using the definition of the norm for $\alpha_n(z)$, we get

$$|C_n|\le A\,R^n\exp\left[-(T+1/q)(\log n)^{\rho}\right] \quad \text{for all } n\ge 1,\ q\ge 1.$$

Hence we have

$$F(f)=\sum_{n=0}^{\infty}a_nC_n, \quad \text{where } C_n\text{'s satisfy (2.3).}$$

Conversely suppose that C_n's satisfy (2.3) and for any sequence of complex numbers $\{a_n\}_{n=0}^{\infty}$ let $F(f) = \sum_{n=0}^{\infty} a_n C_n$. Then for $q \geq 1$,

$$|F(f)| \leq A \sum_{n=1}^{\infty} |a_n| R^n \exp[-(T + 1/q)(\log n)^\rho] + |a_0 C_0|.$$

Let us put $A_1 = \max(A, |C_0|)$. Then we have $|F(f)| \leq A_1 \|f\|_q$, $q \geq 1$. Hence F defines a continuous linear functional on $U_R(\rho, T, 1/q)$ for each $q = 1, 2, \ldots$. In view of the metric defined by (1.11), F is continuous linear functional on $U_{R,\lambda}(\rho, T)$. This completes the proof of Theorem 2.

For $f \in U_R(\rho, T)$ and $\delta > 0$, let $\|f; \rho, T + \rho\|$ be defined by (1.10). Then we have the following result.

THEOREM 3. *A necessary and sufficient condition that there exists a continuous linear transformation* $F: U_{R,\lambda}(\rho, T) \to U_{R,\lambda}(\rho, T)$ *with* $F(\alpha_n) = \beta_n$, $n = 0, 1, 2, \ldots$, $\alpha_n = z^n$, $\beta_n \in U_R(\rho, T)$, *is that for every* $\delta > 0$,

$$\lim_{n\to\infty} \sup \frac{(\log n)^\rho}{\log^+ \left\{ \|\beta_n; \rho, T + \delta\|^{-1} R^n \right\}} < \frac{1}{T} \tag{2.4}$$

Proof. Let F be a continuous linear transformation from $U_{R,\lambda}(\rho, T)$ into $U_{R,\lambda}(\rho, T)$ with $F(\alpha_n) = \beta_n$, $n = 0, 1, 2, \ldots$. Then for any given $\delta > 0$ there exists a $\delta_1 > 0$ and a constant $K = K(\delta)$ such that

$$\|F(\alpha_n); \rho, T + \delta\| \leq K \|\alpha_n; \rho, T + \delta_1\|, \quad \text{i.e.}$$
$$\|\beta_n; \rho, T + \delta\| \leq K R^n \exp[-(T + \delta_1)(\log n)^\rho], \quad \text{i.e.}$$
$$\frac{(\log n)^\rho}{\log^+ \left[\|\beta_n; \rho, T + \delta\|^{-1} R^n \right]} < o(1) + (T + \delta_1)^{-1}.$$

Hence

$$\lim_{n\to\infty} \sup \frac{(\log n)^\rho}{\log^+ \left[\|\beta_n; \rho, T + \delta\|^{-1} R^n \right]} \leq \frac{1}{T + \delta_1} < \frac{1}{T}.$$

Conversely suppose that the sequence $\{\beta_n\}$ satisfies (2.4). Then for any given $\eta' > 0$, there exists a positive integer $N_0 = N_0(\eta')$ such that

$$\frac{(\log n)^\rho}{\log^+ \left[\|\beta_n; \rho, T + \delta\|^{-1} R^n \right]} < \frac{1}{T + \eta'} \tag{2.5}$$

for all $n > N_0$ and all $\delta > 0$. Let $f \in U_R(\rho, T)$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$, and let $0 < \eta < \eta'$. Then from (1.8), there exists a positive integer $N_1 = N_1(\eta)$ such that for all $n > N_1$,

$$|a_n| < R^{-n} \exp[(T + \eta)(\log n)^\rho]. \tag{2.6}$$

Let $N = \max(N_0, N_1)$. Then from (2.5) and (2.6), we have for $n > N$,

$$|a_n| \|\beta_n; \rho, T + \delta\| < \exp[(\eta - \eta')(\log n)^\rho].$$

Since $0 < \eta < \eta'$, this inequality implies that the series $\sum_{n=0}^{\infty} a_n\beta_n$ converges absolutely in $U_R(\rho, T, \delta)$ for each $\delta > 0$. Since each $U_R(\rho, T, \delta)$ is complete, we conclude that this series converges to an element of $U_R(\rho, T, \delta)$. Let us define a transformation $F\colon U_{R,\lambda}(\rho, T) \to U_{R,\lambda}(\rho, T)$ by putting $F(f) = \sum_{n=0}^{\infty} a_n\beta_n$ for $f \in U_{R,\lambda}(\rho, T)$. We note that F is linear, $F(\alpha_n) = \beta_n$ and for any $\delta > 0$, there exists a $\delta' > 0$ such that

$$\frac{(\log n)^\rho}{\log^+ \left[\|\beta_n; \rho, T + \delta\|^{-1} R^n\right]} \le \frac{1}{T + \delta'} \quad \text{for } n > N(\delta, \delta'), \quad \text{i.e.}$$

$$\|\beta_n; \rho, T + \delta\| \le K\, R^n \exp\left[-(T + \delta')(\log n)^\rho\right]$$

for all $n \ge 0$, $K = K(\delta)$ being a constant. Hence

$$\begin{aligned}
\|F(\alpha); \rho, T + \delta'\| &\le \sum_{n=0}^{\infty} |a_n|\, \|\beta_n; \rho, T + \delta\| \\
&\le |a_0| + \sum_{n=1}^{\infty} |a_n| K\, R^n \exp\left[-(T + \delta')(\log n)^\rho\right] \\
&\le K'\, \|\alpha_n; \rho, T + \delta'\|, \quad \text{where } K' = \max(1, K^{-1}).
\end{aligned}$$

Hence F is continuous on $U_R(\rho, T, \delta)$ for each $\delta > 0$. Consequently F is continuous on $U_{R,\lambda}(\rho, T)$. This proves Theorem 3.

3. Proper bases. In this section, we will study the properties of bases in $U_{R,\lambda}(\rho, T)$. We give some definitions. Let f_k, $k = 0, 1, \ldots$, be a sequence of functions in $U_R(\rho, T)$. If $\sum_{k=0}^{\infty} C_k f_k = 0 \Rightarrow C_k = 0$ for all $k = 0, 1, 2, \ldots$ and all sequences $\{C_k\}$ of complex numbers for which $\sum_{k=0}^{\infty} C_k f_k$ converges in $U_{R,\lambda}(\rho, T)$, then the sequence $\{f_k\}$ is said to be linearly independent. We say that $\{f_k\}_{k=0}^{\infty}$ spans a subspace $V_{R,\lambda}(\rho, T)$ of $U_{R,\lambda}(\rho, T)$ provided $V_{R,\lambda}(\rho, T)$ consists of all linear combinations $\sum_{k=0}^{\infty} C_k f_k$ where $\{C_k\}_{k=0}^{\infty}$ is any sequence of complex numbers such that $\sum_{k=0}^{\infty} C_k f_k$ converges in $U_{R,\lambda}(\rho, T)$. A sequence $\{f_k\}$ which is linearly independent and spans a closed subspace $V_{R,\lambda}(\rho, T)$ of $U_{R,\lambda}(\rho, T)$ is called a basis of $V_{R,\lambda}(\rho, T)$. Lastly, a basis $\{f_k\}_{k=0}^{\infty}$ of a subspace $V_{R,\lambda}(\rho, T)$ is said to be a proper base if for all sequences of complex numbers $\{C_n\}$, $\sum_{k=0}^{\infty} C_k f_k$ converges if and only if $\sum_{k=0}^{\infty} C_k \alpha_k$ converges in $V_{R,\lambda}(\rho, T)$. From (1.8) we know that $\sum_{k=0}^{\infty} C_k \alpha_k$ converges in $U_{R,\lambda}(\rho, T)$ if and only if

(3.1) $$\lim_{n\to\infty} \sup \left[\log^+ \left(|C_n|\, R^n\right) (\log n)^{-\rho}\right] \le T.$$

Now we prove

THEOREM 4. *The following three conditions are equivalent:*

(3.2) $$\lim_{n\to\infty} \sup \frac{(\log n)^\rho}{\log^+ \left[\|\beta_n; \rho, T + \delta\|^{-1} R^n\right]} < \frac{1}{T}, \quad \delta > 0;$$

(3.3) *for all sequences $\{a_n\}$ of complex numbers the convergence of $\sum_{n=0}^{\infty} a_n\alpha_n$ in $U_{R,\lambda}(\rho, T)$ implies convergence of $\sum_{n=0}^{\infty} a_n\beta_n$ in $U_{R,\lambda}(\rho, T)$;*

(3.4) *for all sequences* $\{a_n\}$ *of complex numbers, convergence of* $\sum_{n=0}^{\infty} a_n \alpha_n$ *in* $U_{R,\lambda}(\rho, T)$ *implies that* $\lim_{n\to\infty} a_n \beta_n = 0$ *in* $U_{R,\lambda}(\rho, T)$.

Proof. In proving the sufficiency part of Theorem 3 we have already shown that (3.2) $\Rightarrow$ (3.3). Further, the implication (3.3) $\Rightarrow$ (3.4) is evident. Thus we have only to show that (3.4) $\Rightarrow$ (3.2). Therefore, let (3.4) be true but for some $\delta > 0$, (3.2) be not satisfied. Then for $\delta = \delta'$ (say), there exists a sequence $\{n_k\}$ of positive integers such that

$$\frac{(\log n_k)^\rho}{\log^+ \left[\|\beta_{n_k}; \rho, T + \delta'\|^{-1} R^{n_k} \right]} > \frac{1}{T + k^{-1}}, \quad \text{for all } k = 1, 2, \ldots. \tag{3.5}$$

We define a sequence $\{a_n\}$ of real numbers as follows:

$$a_n = \begin{cases} \|\beta_n; \rho, T + \delta'\|^{-1}, & n = n_k \\ 0, & n \neq n_k \end{cases}$$

Then for all large values of k, we have

$$\frac{\log^+ [|a_{n_k}| R^{n_k}]}{(\log n_k)^\rho} = \frac{\log^+ \left[\|\beta_{n_k}; \rho, T + \delta'\|^{-1} R^{n_k} \right]}{(\log n_k)^\rho} < T + k^{-1}.$$

Hence

$$\lim_{k\to\infty} \sup \left\{ \log^+ [|a_{n_k}| R^{n_k}] (\log n_k)^{-\rho} \right\} \le T.$$

Thus the sequence $\{a_n\}$ defined as above satisfies (3.1) and hence $\sum a_n \alpha_n$ converges in $U_{R,\lambda}(\rho, T)$. So, by (3.4), we have $\lim_{n\to\infty} a_n \beta_n = 0$. However

$$\|a_{n_k} \beta_{n_k}; \rho, T + \delta'\| = |a_{n_k}| \, \|\beta_{n_k}; \rho, T + \delta'\| = 1.$$

Therefore $\{a_n \beta_n\}$ does not converge to 0 in $U_{R,\lambda}(\rho, T)$. This is a contradiction. Hence (3.2) must hold for all $\delta > 0$. This proves Theorem 4.

Next we prove

THEOREM 5. *The following three conditions are equivalent:*

(a) *for all sequences* $\{a_n\}_{n=0}^{\infty}$ *of complex numbers,* $\lim_{n\to\infty} a_n \beta_n = 0$ *in* $U_{R,\lambda}(\rho, T)$ *implies that* $\sum_{n=0}^{\infty} a_n \alpha_n$ *converges in* $U_{R,\lambda}(\rho, T)$;

(b) *for all sequences* $\{a_n\}$ *of complex numbers, convergence of* $\sum_{n=0}^{\infty}$ *implies that* $\sum_{n=0}^{\infty} a_n \alpha_n$ *converges in* $U_{R,\lambda}(\rho, T)$;

(c) $$\lim_{\delta\to 0} \left[\lim_{n\to\infty} \inf \frac{(\log n)^\rho}{\log^+ \left[\|\beta_n; \rho, T + \delta\|^{-1} R^n \right]} \right] \ge \frac{1}{T}.$$

Proof. Obviously (a) $\Rightarrow$ (b). To prove (b) $\Rightarrow$ (c), we assume that (b) holds but (c) does not. Then we have

$$\lim_{\delta\to 0} \left[\lim_{n\to\infty} \inf \frac{(\log n)^\rho}{\log^+ \left[\|\beta_n; \rho, T + \delta\|^{-1} R^n \right]} \right] < \frac{1}{T}.$$

Hence for any $\delta > 0$,

$$\lim_{n\delta\to\infty} \inf \frac{(\log n)^\rho}{\log^+ \left[\|\beta_n;\rho,T+\delta\|^{-1} R^n\right]} < \frac{1}{T}. \tag{3.6}$$

Let $\eta > 0$ be any fixed number. From (3.6), we can find an increasing sequence $\{n_k\}$ of positive integers such that

$$\frac{(\log n_k)^\rho}{\log^+ \left[\|\beta_{n_k};\rho,T+\delta\|^{-1} R^{n_k}\right]} < \frac{1}{T+\eta}. \tag{3.7}$$

For η_1, $0 < \eta_1 < \eta$, we define a sequence $\{a_n\}$ as follows:

$$a_n = \begin{cases} R^{-n}\exp\left[(T+\eta_1)(\log n)^\rho\right], & n = n_k \\ 0, & n \neq n_k. \end{cases}$$

Then for any $\delta > 0$ we have

$$\sum_{n=0}^{\infty} |a_n| \, \|\beta_n;\rho,T+\delta\| = \sum_{k=1}^{\infty} |a_{n_k}| \, \|\beta_{n_k};\rho,T+\delta\|. \tag{3.8}$$

Now for any $\delta > 0$, we omit those terms of the series on right-hand side for which $\delta < 1/k$. Then the remainder of the series in (3.8) is dominated by $\sum |a_{n_k}| \left\|\beta_{n_k};\rho,T+k^{-1}\right\|$. Consequently by (3.7) we obtain

$$\begin{aligned} &\sum_{k=1}^{\infty} |a_{n_k}| \left\|\beta_{n_k};\rho,T+k^{-1}\right\| \\ \leq &\sum_{k=1}^{\infty} R^{-n_k}\exp\left[(\log n_k)^\rho (T+\eta_1)\right] R^{n_k}\exp\left[-(T+\eta)(\log n_k)^\rho\right] \\ = &\sum_{k=1}^{\infty} \exp\left[(\log n_k)^\rho(\eta_1-\eta)\right]. \end{aligned}$$

Since $0 < \eta_1 < \eta$, the series on the right-hand side is convergent. Since $a_n = 0$ for $n \neq n_k$ the series $\sum_{n=0}^{\infty} a_n\beta_n$ is convergent for the above sequence $\{a_n\}$. Since this is true for every $\delta > 0$, the series $\sum a_n\beta_n$ converges in $U_{R,\lambda}(\rho,T)$. On the other hand, for this sequence $\{a_n\}$, we also have

$$\lim_{n\to\infty} \sup\{\log^+ [|a_n| R^n](\log n)^{-\rho}\} = T + \eta_1 > T, \tag{3.9}$$

which gives a contradiction to (3.1) and consequently to (b). So, we must have (b) $\Rightarrow$ (c). Lastly we prove that (c) $\Rightarrow$ (a). Hence, suppose that (c) holds but (a) does not. Then there exists a sequence $\{a_n\}$ of complex numbers for which $\lim_{n\to\infty} a_n\beta_n = 0$, but $\sum_{n=0}^{\infty} a_n\alpha_n$ does not converge in $U_{R,\lambda}(\rho,T)$. Hence from (3.1) we have

$$\lim_{n\to\infty} \sup\left\{\log^+ [|a_n| R^n] (\log n)^{-\rho}\right\} > T$$

Thus there exists a positive number ε and a sequence $\{n_k\}$ of positive integers such that

$$\log^+\left\{\left[|a_{n_k}|\ R^{n_k}\right](\log n_k)^{-\rho}\right\} > T-\varepsilon.$$

Let $0<\eta<\varepsilon/2$. From (c) we can find a positive number δ such that

$$\lim_{n\to\infty}\inf\frac{(\log n)^\rho}{\log^+\left[\|\beta_n;\rho,T+\delta\|^{-1}R^n\right]}\geq\frac{1}{T+\eta}.$$

Hence there exists an integer $N=N(\eta)$ such that for $n\geq N$,

$$\frac{(\log n)^\rho}{\log^+\left[\|\beta_n;\rho,T+\delta\|^{-1}R^n\right]}\geq\frac{1}{T+2\eta}.$$

Therefore

$$\begin{aligned}\max\left[\|a_n\beta_n;\rho,T+\delta\|\right]&=\max\left[|a_n|\ \|\beta_n;\rho,T+\delta\|\right]\\&\geq\max\left[|a_{n_k}|\ \|\beta_{n_k};\rho,T+\delta\|\right]\\&\geq\exp\left[(\log n_k)^\rho(\varepsilon-2\eta)\right]>1\end{aligned}$$

since $\varepsilon>2\eta$. Hence the sequence $\{a_n\beta_n\}$ does not converge to zero in $U_{R,\lambda}(\rho,T)$. This is a contradiction to (c). So we must have (c) $\Rightarrow$ (a) and proof of Theorem 5 is completed.

Lastly we give a characterization of proper bases. This result follows from the last two theorems.

THEOREM 6. *A base $\{\beta_n\}$ in a closed subspace $V_{R,\lambda}(\rho,T)$ of $U_{R,\lambda}(\rho,T)$ is proper if and only if the conditions* (3.2) *and* (c) *stated above are satisfied.*

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