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A NOTE ON APPROXIMATION BY BLASCHKE–POTAPOV PRODUCTS

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Abstract. A theorem on approximation of bounded operator functions by finite Blaschke–Potapov products is proved, where a quantity defined in terms of the Potapov–Ginzburg factorization is simultaneously approximated.

Let H be a fixed separable (non-trivial) Hilbert space and C, S_1 the spaces of all bounded, respectively nuclear operators on H. We will denote by $\|\cdot\|$ the norm in C (the uniform norm) and by $\|\cdot\|_1$ the norm in S_1 (the trace norm). The identity operator on H will be denoted by I. By D we will denote the unit disc $|z| < 1$ in the complex plane.

According to [**1**], let G be the class of operator functions $\theta\colon D \to C$ analytic on D (in the sense of the uniform norm), such that:

(1) $\theta(z)^*\theta(z) \le I$, $z \in D$; (2) there exists $\theta(0)^{-1} \in C$; (3) $\theta(0) - I \in S_1$.

The Blaschke–Potapov products and the Potapov multiplicative integrals are important examples of G functions [**2**], [**1**]. An operator function $B\colon D \to C$ is called a *Blaschke–Potapov product* if

$$B(z) = \overset{\curvearrowleft}{\prod}{}_{j=1}^{q} b(P_j; a_j, z) := \overset{\curvearrowleft}{\prod}{}_{j=1}^{q} \left[\frac{|a_j|}{a_j} \frac{a_j - z}{1 - \bar{a}_j z} P_j + (I - P_j) \right], z \in D, \qquad (1)$$

where: $q \le \infty$, $0 < |a_j| < 1$, P_j are orthogonal projections, $\operatorname{Tr} P_j = \dim P_j H =: p_j < \infty$, $\sum (1 - |a_j|)\, p_j < \infty$. Thereby, it is understood that the partial products converge to the Blaschke–Potapov product $B(z)$ in the sense of the trace norm, uniformly on compact subsets of D. If $q < \infty$, then B is a finite Blaschke–Potapov product.

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A function $F: D \to C$ is called a *Potapov multiplicative integral* if

$$F(z) = \overset{\curvearrowleft}{\int_0^c} \exp\{-v(y(x),z)dE(x)\}, \quad z \in D, \tag{2}$$

where: $v(t,z) = \left(1 + e^{-it}z\right)\left(1 - e^{-it}z\right)^{-1}$, y is a nondecreasing scalar function ($0 \le y(x) \le 2\pi$), $E: [0,c] \to S_1$ is an hermitian-increasing operator function satisfying $\operatorname{Tr} E(x) = x$, $x \in [0,c]$. It is understood here that the integral products converge to the Potapov multiplicative integral $F(z)$ in the sense of the trace norm, uniformly on compact subsets of D. The function y in (2) can be chosen to be left continuous and to take the value 2π only at the point $x = c$ or nowhere on $[0,c]$. Such function y will be called *canonical*.

Each product of a Potapov multiplicative integral and a Blaschke–Potapov product is a G function. The converse is also true, in a sense. Namely, if $\theta \in G$, then there exist a Potapov multiplicative integral F, a Blaschke–Potapov product B and a unitary operator U on H, with $U - I \in S_1$, such that

$$\theta(z) = F(z)UB(z), \quad z \in D \tag{3}$$

[**1**]. If the function y in (2) is canonical, then c, y, a_j and $p_j (j = 1, 2, \ldots, q)$ are uniquely determined by θ.

Remark 1. The factorization (3) implies that $\theta(z) - I \in S_1$, $z \in D$, whenever $\theta \in G$. It follows that $\det\theta(z)$ exists for every $z \in D$ [**3**, p. 199–206]. One can easily see that this determinant can be expressed in terms of the factorization (3):

$$\det\theta(z) = \det F(z)\det(UB(z)), \tag{4}$$

$$\det F(z) = \exp\left\{ -\int_0^c v(y(x),z)\,dx \right\} \tag{5}$$

$$\det(UB(z)) = \lambda \prod_{j=1}^{q} \left[\frac{|a_j|\,(a_j - z)}{a_j\,(1 - \bar{a}_j z)} \right]^{p_j}, \quad |\lambda| = |\det U| = 1.$$

Thus, $\det\theta \in H^\infty$ and the zeros of $\det\theta$ are exactly the zeros of the Blaschke product $\det(UB)$ (for $\det F(z) \neq 0$, $z \in D$). Since a scalar Blaschke product is determined by its zeros (accounting their multiplicities) up to a constant factor of modulus one, it follows that $\det\theta$ is a Blaschke product if and only if $|\det F(z)| = 1$, $z \in D$, i.e. if and only if $c = 0$ (for $\det F(0) = e^{-c}$). In other words, $\det\theta$ is a Blaschke product if and only if $\theta = UB$ for some Blaschke–Potapov product B and some unitary operator U on H, with $U - I \in S_1$.

It is convenient to extend the notion of Blaschke–Potapov product. If B is a Blaschke–Potapov product and U a unitary operator on H, with $U - I \in S_1$, then let the product UB also be called a *Blaschke–Potapov product*.

Thus, a G function θ is a (finite) Blaschke–Potapov product, in the extended sense, if and only if $\det\theta$ is a (finite) Blaschke product.

Remark 2. If B_1 and B_2 are (finite) Blaschke–Potapov products, then the product B_1B_2 is also a (finite) Blaschke–Potapov product.

The factorization (3) and the notation used in (1) and (2) enable us to introduce the following quantity:

$$R_m(\theta;w):=\int_0^c \left|1-e^{-iy(x)}w\right|^{-m}dx+\sum_{j=1}^q |1-\bar{a}_jw|^{-m}(1-|a_j|)\,p_j, \tag{6}$$

for $\theta \in G$, $|w|\le 1$, $m\in N$, where the function y is canonical.

The integral in (6) can be replaced by an integral with respect to the representing measure of $\det F$, where $\theta = FUB$ (factorization (3)). Namely, if we set

$$\sigma(t):=\sup\{x: x\in[0,c]\wedge(y(x)<t\vee x=0)\},\quad t\in[0,2\pi],$$

then we have

$$\int_0^c v(y(x),z)\,dx=\int_0^{2\pi}\frac{1+e^{-it}z}{1-e^{-it}z}d\sigma(t),\quad z\in D,$$

i.e. $d\sigma$ is the representing measure of $\det F$ (see (5)), and

$$\int_0^c\left|1-e^{-iy(x)}w\right|^{-m}dx=\int_0^{2\pi}\left|1-e^{-it}w\right|^{-m}d\sigma(t). \tag{7}$$

Remark 3. The above considerations imply that $R_m(\theta_1\theta_2;w)=R_m(\theta_1;w)+R_m(\theta_2;w)$, for arbitrary G functions θ_1 and θ_2. Indeed, if $\theta_1\theta_2=\theta$, $\theta_1=F_1B_1$, $\theta_2=F_2B_2$, $\theta=FB$ (F_1, F_2, F – Potapov multiplicative integrals, B_1, B_2, B – Blaschke–Potapov products) and if $d\sigma_1$, $d\sigma_2$, $d\sigma$ are the representing measures of $\det F_1$, $\det F_2$, $\det F$ respectively, then it must be $\det\theta=\det\theta_1.\det\theta_2$ and, consequently, $\det F=\det F_1.\det F_2$, $\det B=\det B_1.\det B_2$ (see (4)), $d\sigma=d\sigma_1+d\sigma_2$. Applying the definition (6) in which the integral is replaced by the integral on the right-hand side of (7), we obtain $R_m(\theta;w)=R_m(\theta_1,w)+R_m(\theta_2;w)$.

Remark 4. If a sequence (θ_n) of G functions converges to a G function θ, in the sense of trace norm, uniformly on compact subsets of D, and if we have for each fixed n a sequence $(B_{n\nu})$ of finite Blaschke–Potapov products, tending to θ_n as $\nu\to\infty$, in the same way, then we can find a sequence (B_n) of finite Blaschke–Potapov products wich converges to θ. This is easy to do: one need only choose B_n to be a $B_{n\nu}$ satisfying $\max\left\{\|F_n(z)-B_{n\nu}(Z)\|_1: |z|\le 1-n^{-1}\right\}<n^{-1}$, for every fixed n.

We shall consider approximation of G functions by finite Blaschke–Potapov products. We will show that any $\theta\in G$ can be approximated in such a way that $R_m(\theta;w)$ is also approximated, for fixed w and m satisfying $R_m(\theta;w)<\infty$.

THEOREM 1. *Let $\theta\in G$, $|w|\le 1$, $m\in N$, and let $R_m(\theta;w)<\infty$. Then there exists a sequence (B_n) of finite Blaschke–Potapov products such that $B_n(z)\to\theta(z)$,*

$n \to \infty$, in the sense of trace norm, uniformly on compact subsets of D, and that $R_m(B_n; w) \to R_m(\theta; w)$, $n \to \infty$.

Proof. Since $\theta \in G$, the factorization (3) holds. It suffices to consider separately the cases $\theta = F$ and $\theta = UB$, for if (B_{ln}) and (B_{2n}) are sequences of finite Blaschke–Potapov products such that $B_{ln}(z) \to F(z)$, $B_{2n}(z) \to UB(z)$, $R_m(B_{ln}; w) \to R_m(F; w)$, $R_m(B_{2n}; w) \to R_m(UB; w)$, as $n \to \infty$, then, by Remarks 2 and 3, $(B_{ln}B_{2n})$ is a sequence of finite Blaschke–Potapov products for which $B_{ln}(z)B_{2n}(z) \to F(z)UB(z) = \theta(z)$ and $R_m(B_{ln}B_{2n}; w) = R_m(B_{ln}; w) + R_m(B_{2n}; w) \to R_m(F; w) + R_m(UB; w) = R_m(\theta; w)$ as $n \to \infty$.

Let $\theta = F$, F a Potapov multiplicative integral. Since the integral products of the form

$$\overset{\curvearrowleft}{\prod}_{j=0}^{k-1} \exp\left\{-v\left(y\left(\xi_j\right), z\right) \Delta E\left(x_j\right)\right\}, \ \xi_j \in \left[x_j, x_{j+1}\right], \ \Delta E\left(x_j\right) = E\left(x_{j+1}\right) - E\left(x_j\right),$$

converge to $F(z)$ as $\max \Delta x_j \to 0$, in the trace norm, uniformly on compact subsets of D, we can choose a sequence of integral products

$$F_n(z) = \overset{\curvearrowleft}{\prod}_{j=0}^{k_n-1} \exp\left\{-v\left(y\left(\xi_{nj}\right), z\right) \Delta E\left(x_{nj}\right)\right\} =: \overset{\curvearrowleft}{\prod}_{j=0}^{k_n-1} f_{nj}(z), \ \ n \in N,$$

such that $F_n(z) \to F(z)$, $n \to \infty$, in the trace norm, uniformly on compact subsets of D, and that

$$\begin{aligned} R_m\left(F_n; w\right) = \sum_{j=0}^{k_n-1} \left|1 - e^{-iy(\xi_{nj})}w\right|^{-m} \Delta x_{nj} \to \\ \int_0^c \left|1 - e^{-iy(x)}w\right|^{-m} dx = R_m(F; w), \ \ n \to \infty, \end{aligned} \tag{8}$$

with $\operatorname{Tr} \Delta E\left(x_{nj}\right) = \Delta x_{nj} < 1$ and $e^{iy(\xi_{nj})} \neq w$, $0 \leq j \leq k_n - 1$, $n \in N$. (The first equality in (8) follows from Remark 3 and from the fact that

$$f_{nj}(z) = \overset{\curvearrowleft}{\int_0^{c_{nj}}} \exp\left\{-v\left(y\left(\xi_{nj}\right), z\right) dE_{nj}(x)\right\},$$

where $c_{nj} := \Delta x_{nj}$ and $E_{nj}(x) := x c_{nj}^{-1} \Delta E\left(x_{nj}\right)$, $0 \leq x \leq c_{nj}$.)

In view of Remark 4, the searching for an appropriate sequence of finite Blaschke–Potapov products for θ can be reduced to the finding of a suitable sequence of finite Blaschke–Potapov products for any F_n, $n \in N$. But according to Remarks 2 and 3, it suffices to find a suitable sequence of finite Blaschke–Potapov products for any f_{nj}, $0 \leq j \leq k_n - 1$, and then multiply them to obtain an appropriate sequence for F_n.

This allows us to assume $\theta(z) = \exp\left\{-v(\eta, z)A\right\}$, where $\eta \in [0, 2\pi]$, $e^{i\eta} \neq w$, and A is a positive operator on H, with $\operatorname{Tr} A < 1$.

As the operator A is nuclear and positive, it is the limit, in the trace norm, of a sequence of finite sums $A_n := \sum_{j=1}^{n} \lambda_j P_j$, where λ_j, $0 < \lambda_j < 1$, are eigenvalues of A, and P_j ($\operatorname{Tr} P_j < \infty$) are the corresponding orthogonal projections. Since the function $v(\eta, \cdot)$ is bounded on compact subsets of D (for $|v(\eta, z)| \leq 2(1-|z|)^{-1}$, $z \in D$), it follows that

$$K_n(z) := \exp\{-v(\eta, z)A_n\} \to \theta(z), \quad n \to \infty, \tag{9}$$

in the trace norm, uniformly on compact subsets of D, and that

$$R_m(K_n; w) = \left|1 - e^{-i\eta}w\right|^{-m} \operatorname{Tr} A_n \to \left|1 - e^{-i\eta}w\right|^{-m} \operatorname{Tr} A = R_m(\theta, w), \ \ n \to \infty. \tag{10}$$

Note that $K_n(z) = \int\limits^{\frown c_n}_0 \exp\{-v(\eta, z) dE_n(x)\}$, where $c_n := \operatorname{Tr} A_n$, and $E_n(x) := xc_n^{-1}A_n$, $0 \leq x \leq c_n$, $n \in N$.

Clearly, we have

$$K_n(z) = \prod_{j=1}^{n} \exp\{-v(\eta, z)\lambda_j P_j\} =: \prod_{j=1}^{n} k_j(Z), \ \ n \in N. \tag{11}$$

Since $K_n \to \theta$ and $R_m(K_n; w) \to R_m(\theta; w)$ as $n \to \infty$ (see (9), (10)), the finding of an appropriate sequence of finite Blaschke–Potapov products for θ reduces to the searching for a suitable sequence for any K_n (by Remark 4) and since each K_n is a finite product of the functions k_j (see (11)), this reduces further to the finding of a sequence for any k_j (by Remarks 2 and 3).

Thus we may assume that θ has the form $\theta(z) = \exp\{-v(\eta, z)\lambda P\}$, $z \in D$, where: $\eta \in [0, 2\pi]$, $e^{i\eta} \neq w$; $0 < \lambda < 1$; $P^* = P$, $P^2 = P$; $\operatorname{Tr} P := p$. Starting with such a θ, set $a_n := \left(1 - \lambda n^{-1}\right) e^{i\eta}$ and $B_n(z) := \left[b\left(P; a_n, z\right)\right]^n$, $n \in N$. Then we have, for $z \in D$ and $n \in N$:

$$\begin{aligned}
&\left\| \exp\left\{-v(\eta, z)\lambda n^{-1} P\right\} - b\left(P; a_n, z\right) \right\|_1 \\
&\quad \leq \left\|\exp\left\{-v(\eta, z)\lambda n^{-1} P\right\} - I + v(\eta, z)\lambda n^{-1} P\right\|_1 \\
&\qquad + \left\| I - v(\eta, z)\lambda n^{-1} P - \left[I + \frac{1 + |a_n| a_n^{-1} z}{1 - \bar{a}_n z}\left(|a_n| - 1\right) P\right]\right\|_1 \\
&\quad \leq \sum_{j=2}^{\infty} \frac{1}{j!} \left\| \left[v(\eta, z)\lambda n^{-1} P\right]^j \right\|_1 \\
&\qquad + \left\| -\frac{1 + e^{-i\eta} z}{1 - e^{-i\eta} z}\lambda n^{-1} P + \frac{1 + e^{-i\eta} z}{1 - \left(1 - \lambda n^{-1}\right) e^{-i\eta} z}\lambda n^{-1} P\right\|_1 \\
&\quad \leq \frac{4}{(1 - |z|)^2}\lambda^2 n^{-2} \exp\left\{\frac{2}{1 - |z|}\lambda n^{-1}\right\} p + \frac{2}{(1 - |z|)^2}\lambda^2 n^{-2} p.
\end{aligned}$$

Since $\|L^n - M^n\|_1 \leq \|L - M\|_1 \|L^{n-1} + L^{n-2}M + \ldots + M^{n-1}\| \leq \|L - M\|_1 \cdot n$ whenever L, $M \in C$, $L - I \in S_1$, $M - I \in S_1$, $LM = ML$, $\|L\| \leq 1$, $\|M\| \leq 1$, it follows

$$\begin{aligned}&\left\| \exp\{-v(\eta,z)\lambda P\} - [b(P;a_n,z)]^n \right\|_1\\ &\leq \left\|\exp\left\{-v(\eta,z)\lambda n^{-1}P\right\} - b\,(P;a_n,z)\right\|_1 \cdot n\\ &\leq \frac{4}{(1-|z|)^2}\lambda^2 n^{-1}\exp\left\{\frac{2}{1-|z|}\lambda n^{-1}\right\}p + \frac{2}{(1-|z|)^2}\lambda^2 n^{-1}p \to 0,\ \ n\to\infty,\end{aligned}$$

uniformly on compact subsets of D. Thus $B_n(z) \to \theta(z)$ as $n \to \infty$, in the trace norm, uniformly on compact subsets of D.

It remains to show $R_m(B_n;w) \to R_m(\theta;w)$, $n \to \infty$. This follows from $R_m(B_n;w) = |1-\bar{a}_n w|^{-m}\, n\,(1-|a_n|)\,p$, $R_m(\theta;w) = \left|1-e^{-i\eta}w\right|^{-m}\lambda p$ and $a_n \to e^{i\eta}$ as $n \to \infty$, with $n\,(1-|a_n|) = \lambda$, $n \in N$.

In the case $\theta(z) = UB(z)$, $z \in D$, the statement follows easily from the nature of convergence of the partial products.

The proof is finished.

Remark 5. The above theorem remains correct if we allow $R_m(\theta;w)$ to be ∞, but we can not get then $R_m(B_n;w) - R_m(\theta;w) \to 0$, $n \to \infty$, i.e. $R_m(B_n;w)$ does not approximate $R_m(\theta;w)$.

Our theorem generalizes the result of Ahern and Clark [**4**] concerning the scalar case dim $H = 1$. Ginzburg [**5**] also considered approximation of bounded operator functions by the finite Blaschke–Potapov products, but without approximation of the quantity R_m.

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