

ON THE CONVERGENCE OF A MULTICOMPONENT ALTERNATING DIRECTION DIFFERENCE SCHEME

Boško Jovanović

Communicated by Gradimir Milovanović

Abstract. We consider a multicomponent finite-difference scheme (FDS) for solving the heat conduction equation with several variables. Some convergence rate estimates consistent with the smoothness of data are obtained.

We consider the first initial-boundary value problem (IBVP) for the heat conduction equation

$$\begin{aligned} \frac{\partial u}{\partial t} &= \Delta u + f, & (x, t) \in Q = \Omega \times (0, T) = (0, 1)^n \times (0, T), \\ u(x, 0) &= u_0(x), & x \in \Omega, \\ u(x, t) &= 0, & x \in \Gamma = \partial\Omega, \quad t \in (0, T). \end{aligned} \quad (1)$$

We assume that the generalized solution of IBVP (1) belongs to the anisotropic Sobolev space $W_2^{s, s/2}(Q)$, $s \geq 1$ [1]. In this case there exist a trace $u|_{t=t'} \in W_2^{s-1}(\Omega) \subset L_2(\Omega)$. We also assume that the solution u can be oddly extended in space variables outside the domain Ω , preserving the Sobolev class.

Let $\bar{\omega}$ be a uniform mesh in $\bar{\Omega}$ with the step size h . Let us set $\omega = \bar{\omega} \cap \Omega$, $\gamma = \bar{\omega} \setminus \omega$ and $\omega_i = \omega \cup \{x = (x_1, \dots, x_n) \in \gamma \mid x_i = 0\}$. Let $\bar{\theta}$ be a uniform mesh on $[0, T]$ with the step size τ , $\theta = \bar{\theta} \cap (0, T)$, $\theta^- = \theta \cup \{0\}$ and $\theta^+ = \theta \cup \{T\}$. Finally, let $\bar{Q}_{h\tau} = \bar{\omega} \times \bar{\theta}$. For a function v defined on the mesh $\bar{Q}_{h\tau}$ we introduce the finite-difference operators v_{x_i} , $v_{\bar{x}_i}$, v_t and $v_{\bar{t}}$ in a usual manner [2]. Let us denote $v = v(x, t)$ and $\hat{v} = v(x, t + \tau)$.

Let H_h be the set of discrete functions defined on the mesh $\bar{\omega}$, which vanish on γ . Let us denote

$$A_i v = \begin{cases} -v_{x_i \bar{x}_i}, & x \in \omega \\ 0, & x \in \gamma \end{cases} \quad \text{and} \quad Av = \sum_{i=1}^n A_i v.$$

We introduce the following discrete inner product

$$(v, w)_\omega = h^n \sum_{x \in \omega} v(x) w(x)$$

and norms

$$\|v\|_\omega = (v, v)_\omega^{1/2} = \left(h^n \sum_{x \in \omega} v^2(x) \right)^{1/2} \quad \text{and} \quad \|v\|_{\omega_i} = \left(h^n \sum_{x \in \omega_i} v^2(x) \right)^{1/2}.$$

A_i and A are linear, selfadjoint, commutative and positive operators on H_h . Therefore, the "energy" norms

$$\|v\|_{A_i} = (A_i v, v)_\omega^{1/2} = \|v_{x_i}\|_{\omega_i} \quad \text{and} \quad \|v\|_{A_i^{-1}} = (A_i^{-1} v, v)_\omega^{1/2}$$

can be defined.

With T_i and T_t^+ we denote the Steklov averaging operators in space variables x_i and time variable t (see [3])

$$T_i f(x, t) = \frac{1}{h} \int_{x_i-h/2}^{x_i+h/2} f(x_1, \dots, x'_i, \dots, x_n, t) dx'_i,$$

$$T_t^+ f(x, t) = \frac{1}{\tau} \int_t^{t+\tau} f(x_1, \dots, x_n, t') dt'.$$

Finally, C will stand for a generic positive constant, independent of h and τ .

We approximate IBVP (1) with the following alternating direction finite-difference scheme (see [4], [5])

$$(I + \sigma\tau A_i) v_t^i + \sum_{j=1}^n A_j v^j = \tilde{f} \equiv T_1^2 \dots T_n^2 T_t^+ f, \quad t \in \theta^-, \quad (2)$$

$$v^i|_{t=0} = T_1^2 \dots T_n^2 u_0, \quad i = 1, 2, \dots, n$$

where σ is a free weight parameter and $Iv \equiv v$. FDS (2) represents a system of n unknown mesh functions v^i . They can be determined paralelly, contrary to the other variants of the alternating direction method, such as the factorized scheme

$$(I + \sigma\tau A_1) \dots (I + \sigma\tau A_n) v_t + Av = f.$$

The errors defined as $z^i = T_1^2 \dots T_n^2 u - v^i$ satisfy the FDS

$$(I + \sigma\tau A_i) z_t^i + \sum_{j=1}^n A_j z^j = \varphi^i, \quad t \in \theta^-, \quad (3)$$

$$z^i|_{t=0} = 0, \quad i = 1, 2, \dots, n$$

where

$$\begin{aligned}\varphi^i &= \varphi + \Lambda \chi, & \chi &= \sigma \tau T_1^2 \cdots T_n^2 u_t, \\ \varphi &= \sum_{j=1}^n \Lambda_j \eta^j, & \eta^j &= \left(\prod_{l \neq j} T_l^2 \right) (T_j^2 u - T_t^+ u).\end{aligned}$$

To prove the stability and the convergence of the FDS (2) we represent the equation (3) in the matrix form (see also [6])

$$(\mathbf{I} + \sigma \tau \Lambda) \mathbf{z}_t + \mathbf{E} \Lambda \mathbf{z} = \Phi, \quad t \in \theta^-; \quad \mathbf{z}|_{t=0} = \mathbf{0}, \quad (4)$$

where $\mathbf{z} = (z^1, \dots, z^n)^T$, $\Phi = (\varphi^1, \dots, \varphi^n)^T$, $\mathbf{I} = \text{diag}(I, \dots, I)$, $\Lambda = \text{diag}(\Lambda_1, \dots, \Lambda_n)$ and

$$\mathbf{E} = \begin{pmatrix} I & I & \cdots & I \\ I & I & \cdots & I \\ \vdots & \vdots & \ddots & \vdots \\ I & I & \cdots & I \end{pmatrix}.$$

Let us also define the inner product and norms of vector-functions

$$(\mathbf{z}, \mathbf{w}) = \sum_{i=1}^n (z^i, w^i)_\omega, \quad \|\mathbf{z}\| = (\mathbf{z}, \mathbf{z})^{1/2}, \quad \|\mathbf{z}\|_\Lambda = (\Lambda \mathbf{z}, \mathbf{z})^{1/2}.$$

Applying operator Λ to (4) we obtain a FDS in canonical form (see [2])

$$\mathbf{B} \mathbf{z}_t + \mathbf{A} \mathbf{z} = \Psi, \quad (5)$$

where $\mathbf{A} = \Lambda \mathbf{E} \Lambda = \mathbf{A}^* \geq \mathbf{0}$, $\mathbf{B} = \Lambda + \sigma \tau \Lambda^2 = \mathbf{B}^* > \mathbf{0}$ i $\Psi = \Lambda \Phi$. The FDS (5) is stable for

$$\mathbf{B} - 0.5 \tau \mathbf{A} > \mathbf{0}.$$

For $\sigma \geq n/2$ we have

$$\begin{aligned}((\mathbf{B} - 0.5 \tau \mathbf{A}) \mathbf{z}, \mathbf{z}) &= (\Lambda \mathbf{z}, \mathbf{z}) + \sigma \tau (\Lambda \mathbf{z}, \Lambda \mathbf{z}) - 0.5 \tau (\mathbf{E} \Lambda \mathbf{z}, \Lambda \mathbf{z}) \\ &= \sum_{i=1}^n (\Lambda_i z^i, z^i)_\omega + \sigma \tau \sum_{i=1}^n (\Lambda_i z^i, \Lambda_i z^i)_\omega - 0.5 \tau \left\| \sum_{i=1}^n \Lambda_i z^i \right\|_\omega^2 \\ &= \sum_{i=1}^n \|z^i\|_{\Lambda_i}^2 + \tau (\sigma - n/2) \sum_{i=1}^n \|\Lambda_i z^i\|_\omega^2 + 0.5 \tau \sum_{i=2}^n \sum_{j=1}^{i-1} \|\Lambda_i z^i - \Lambda_j z^j\|_\omega^2 \\ &\geq \sum_{i=1}^n \|z^i\|_{\Lambda_i}^2 = \|\mathbf{z}\|_\Lambda^2,\end{aligned}$$

which means that

$$\mathbf{B} - 0.5 \tau \mathbf{A} \geq \Lambda > \mathbf{0},$$

and, consequently, FDS (5) is stable.

Using energy method, multiplying (5) by $2\tau \mathbf{z}_t$, we obtain the a priori estimate

$$\max_{t \in \theta^+} \|\mathbf{z}\|_{\mathbf{A}}^2 + \tau \sum_{t \in \theta^-} \|\mathbf{z}_t\|_{\Lambda}^2 \leq C \tau \sum_{t \in \theta^-} \|\Phi\|_{\Lambda}^2,$$

where $\|\mathbf{z}\|_{\mathbf{A}}^2 = (\mathbf{A} \mathbf{z}, \mathbf{z})$, or, in expanded form

$$\|\mathbf{z}\|_{\mathbf{A}}^2 \equiv \max_{t \in \theta^+} \left\| \sum_{i=1}^n \Lambda_i z^i \right\|_{\omega}^2 + \sum_{i=1}^n \tau \sum_{t \in \theta^-} \|z_t^i\|_{\Lambda_i}^2 \leq C \sum_{i=1}^n \tau \sum_{t \in \theta^-} \|\varphi^i\|_{\Lambda_i}^2. \quad (6)$$

Others standard a priori estimates (see [2]) do not hold because the operators \mathbf{A} and \mathbf{B} do not commute.

Further

$$\|\varphi^i\|_{\Lambda_i} \leq \sum_{j=1}^n \|\eta_{x_j \bar{x}_j x_i}^j\|_{\omega_i} + \|\chi_{x_i \bar{x}_i x_i}\|_{\omega_i}.$$

The value of $\eta_{x_j \bar{x}_j x_i}^j$ in the node $(x, t) \in \omega_i \times \theta^-$ is a bounded linear functional of $u \in W_2^{s, s/2}(e)$, where $e = \prod_{l=1}^n (x_l - 2h, x_l + 2h) \times (t, t + \tau)$ and $s \geq 1$. Moreover, $\eta_{x_j \bar{x}_j x_i}^j$ vanishes on the functions of the form $u = x_1^{\alpha_1} \cdots x_n^{\alpha_n} t^{\beta}$, $\alpha_1 + \dots + \alpha_n + 2\beta \leq 4$. Using the Bramble–Hilbert lemma, in the same manner as in [3], for $\tau \sim h^2$, we obtain

$$|\eta_{x_j \bar{x}_j x_i}^j| \leq C h^{s-4-n/2} |u|_{W_2^{s, s/2}(e)}, \quad 3 \leq s \leq 5.$$

From here, by summation over the mesh, follows

$$\left\{ \tau \sum_{t \in \theta^-} \|\eta_{x_j \bar{x}_j x_i}^j\|_{\omega_i} \right\}^{1/2} \leq C h^{s-3} \|u\|_{W_2^{s, s/2}(Q)}, \quad 3 \leq s \leq 5.$$

In the same manner we can estimate $\chi_{x_i \bar{x}_i x_i}$. From these estimates and the inequality (6) follows

$$\|\mathbf{z}\|_3 \leq C h^{s-3} \|u\|_{W_2^{s, s/2}(Q)}, \quad 3 \leq s \leq 5. \quad (7)$$

Remark. In same cases the assumption $\tau \sim h^2$ can be omitted. For example, using representations

$$\begin{aligned} \eta^j &= \left(\prod_{l=1}^n T_l^2 \right) \frac{1}{\tau} \int_t^{t+\tau} \int_{t'}^t \frac{\partial u(x, t'')}{\partial t} dt'' dt' \\ &+ \left(\prod_{l \neq j} T_l^2 \right) T_t^+ \frac{1}{h} \int_{x_j-h}^{x_j+h} \int_{x_j'}^{x_j} (x_j'' - x_j') \left(1 - \frac{|x_j' - x_j|}{h} \right) \times \\ &\quad \times \frac{\partial^2 u(x_1, \dots, x_j'', \dots, x_n, t)}{\partial x_j^2} dx_j'' dx_j', \\ \chi &= \sigma \tau \left(\prod_{l=1}^n T_l^2 \right) T_t^+ \frac{\partial u}{\partial t} \end{aligned}$$

we directly obtain

$$|\eta_{x_j \bar{x}_j x_i}^j|, |\chi_{x_i \bar{x}_i x_i}| \leq C \frac{h^2 + \tau}{\sqrt{h^n \tau}} \|u\|_{W_2^{5, 5/2}(e)}.$$

From here, similarly as in a previous case, follows

$$\|z\|_3 \leq C (h^2 + \tau) \|u\|_{W_2^{5, 5/2}(Q)}.$$

Another group of convergence rate estimates can be obtained in the following way. Applying $\Lambda_i (I + \sigma \tau \Lambda_i)^{-1}$ to (3), after summation on i we obtain

$$z_t + Az = \psi, \quad t \in \theta^-; \quad z|_{t=0} = 0, \quad (8)$$

where

$$z = \Lambda^{-1} \sum_{i=1}^n \Lambda_i z^i, \quad A = \sum_{i=1}^n \Lambda_i = \sum_{i=1}^n \Lambda_i (I + \sigma \tau \Lambda_i)^{-1}, \quad \psi = \Lambda^{-1} \sum_{i=1}^n \Lambda_i \varphi^i.$$

For $\sigma \geq n/[2(1-\alpha)]$, $0 < \alpha < 1$, we have $0 < \alpha I \leq I - 0.5\tau A \leq I$, so the FDS (8) is absolutely stable.

The operators A and Λ satisfy the relations $A \leq \Lambda$ and $\Lambda^{-1} \leq A^{-1}$. In the case when $\tau \sim h^2$ we also have $\beta \Lambda_i \leq \Lambda_i$, $\beta \Lambda \leq A$, $0 < \beta < 1$. Using these relations, the energy method [2] and the Fourier expansion in t (see [7]) we obtain the a priori estimates

$$\|z\|_0^2 \equiv \tau \sum_{t \in \theta^-} \left\| \frac{\hat{z} + z}{2} \right\|_\omega^2 \leq C \tau \sum_{t \in \theta^-} \|\Lambda^{-1} \psi\|_\omega^2, \quad (9)$$

$$\begin{aligned} \|z\|_1^2 &\equiv \max_{t \in \theta^+} \|z\|_\omega^2 + \tau \sum_{t \in \theta^-} \left\| \frac{\hat{z} + z}{2} \right\|_A^2 + \tau^2 \sum_{t, t' \in \bar{\theta}, t \neq t'} \left\| \frac{z(\cdot, t) - z(\cdot, t')}{t - t'} \right\|_\omega^2 \\ &\leq C \tau \sum_{t \in \theta^-} \|\psi\|_{A^{-1}}^2, \end{aligned} \quad (10)$$

$$\|z\|_2^2 \equiv \max_{t \in \theta^+} \|z\|_A^2 + \tau \sum_{t \in \theta^-} \left\| \Lambda \frac{\hat{z} + z}{2} \right\|_\omega^2 + \tau \sum_{t \in \theta^-} \|z_t\|_\omega^2 \leq C \tau \sum_{t \in \theta^-} \|\psi\|_\omega^2. \quad (11)$$

Further

$$\|\Lambda^{-1} \psi\|_\omega \leq \sum_{j=1}^n \|\eta^j\|_\omega + n \|\chi\|_\omega, \quad (12)$$

$$\|\psi\|_{A^{-1}} \leq \sum_{j=1}^n \left(\|\eta_{x_j}^j\|_{\omega_j} + \|\chi_{x_j}\|_{\omega_j} \right), \quad (13)$$

$$\|\psi\|_\omega \leq \sum_{j=1}^n \left(\|\eta_{x_j \bar{x}_j}^j\|_\omega + \|\chi_{x_j \bar{x}_j}\|_\omega \right). \quad (14)$$

In such a way, the problem of deriving the convergence rate estimate for FDS (8), or (2), is now reduced to estimation of η^j , χ , $\eta_{x_j}^j$, χ_{x_j} , $\eta_{x_j \bar{x}_j}^j$ and $\chi_{x_j \bar{x}_j}$. Using the Bramble–Hilbert lemma, in the same manner as in the previous case, from (9–14) we obtain

$$\|z\|_0 \leq C h^s \|u\|_{W_2^{s, s/2}(Q)}, \quad 1 \leq s \leq 2, \quad (15)$$

$$\|z\|_1 \leq C h^{s-1} \|u\|_{W_2^{s, s/2}(Q)}, \quad 1 \leq s \leq 3, \quad (16)$$

$$\|z\|_2 \leq C h^{s-2} \|u\|_{W_2^{s, s/2}(Q)}, \quad 2 \leq s \leq 4. \quad (17)$$

The convergence rate estimates (7), (15–17) are consistent with the smoothness of data. In such a way, results for standard FDSs for parabolic problems with solutions in the Sobolev classes $W_2^{s, s/2}$ (see [3], [7]) are extended to the new class of multicomponent alternating direction difference schemes. In [4], [5] the convergence of these schemes is proved for the problems with smooth solutions ($u \in C^{2k, k}$).

REFERENCES

- [1] J.L. Lions, E. Magenes, *Boundary-value problems and its applications*, Springer, Berlin 1972.
- [2] А.А. Самарский, *Теория разностных схем*, Наука, Москва 1983.
- [3] B.S. Jovanović, *On the convergence of finite-difference schemes for parabolic equations with variable coefficients*, Numer. Math. **54** (1989), 395–404.
- [4] В.Н. Абрашин, *Об одном варианте метода переменных направлений решения многомерных задач математической физики*, Дифференциальные уравнения **26** (1990), 314–323.
- [5] В.Н. Абрашин, В.А. Муха, *Об одном классе экономичных разностных схем решения многомерных задач математической физики*, Дифференциальные уравнения **28** (1992), 1786–1799.
- [6] П.Н. Вабищевич, *Векторные аддитивные разностные схемы*, Москва, Инст. Математ. Моделир. РАН, 1994, препр. № 2.
- [7] B.S. Jovanović, *The finite-difference method for boundary-value problems with weak solutions*, Posebna izdanja 16, Matematički inastitut, Beograd 1993.

Matematički fakultet
11001 Beograd, p.p. 550
Jugoslavija

(Received 28 04 1994)
(Revised 06 12 1994)