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# THE DUAL OF THE BERGMAN SPACE DEFINED ON A HYPERBOLIC PLANE DOMAIN

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**Abstract.** We determine the dual of Bergman space over a plane domain whose boundary has at least three finite boundary points. An important tool in our approach is the existence of the reproducing kernel function with corresponding properties on a hyperbolic plane domain.

We say that a plane domain $D$ is hyperbolic if $D$ has three or more boundary points in $\mathbf{C}$. Suppose that $D$ is a hyperbolic plane domain and let $U$ denote the open unit disk in $C$. This hypothesis implies that there is a universal covering mapping $p: U \to D$ with a Fuchsian covering group $G$ for which $U/G \approxeq D$. The Poincaré density (metric) $r = r_D$ for the region $D$ is defined by the equation $r(p(z))|p'(z)| = r_U(z)$, $z \in U$, where $r_U(z) = (1-|z|^2)^{-1}$ is the hyperbolic density on $U$. To obtain the kernel function, we first form the series:

$$F(z,\zeta) = \sum_{B \in \Gamma} K_U(B(z),\zeta) B'(z)^2$$

defined for $z$ and $\zeta$ in $U$. Then let the kernel function $K = K_D$ be determined by

$$K(p(z),p(\zeta))p'(z)^2\overline{p'(\zeta)^2} = F(z,\zeta) \qquad z,\zeta \in U.$$

For the proof that $K = K_D$ is well defined and satisfies the properties of the next lemma see Lemma 3, p. 80 of [**3**].

Lemma A. *For any hyperbolic plane domain $D$ in $C$, the kernel function $K(z,\zeta)$ defined for all $(z,\zeta)$ in $D \times D$ is holomorphic in $z$ and has the following properties*

$$K(z,\zeta) = \overline{K(\zeta,z)}, \tag{i}$$

$$\iint_D |K(z,\zeta)| dxdy \le \pi r^2(\zeta), \tag{ii}$$

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$$f(z) = \frac{3}{\pi} \iint_D r^{-2}(\zeta) K(z,\zeta) f(\zeta) d\xi d\eta, \tag{iii}$$

*for every integrable holomorphic function $f$ on $D$.*

From now on let $D$ denote a plane domain and let $L^p(D)$, $1 \le p \le +\infty$, be the space of all measurable functions $f$ on $D$ for which

$$\|f\|_p^p = \iint_D |f(z)|^p dx dy < +\infty.$$

Let further $B^p = B^p(D)$ denote the subspace of $L^p$ consisting of holomorphic functions on $D$. For every $\varphi \subset L^p(D)$, $1 \le p \le +\infty$, let $K$ denote the projection defined by:

$$K\varphi(z) = \frac{3}{\pi} \iint_D r^{-2}(\zeta) K(z,\zeta) \varphi(\zeta) d\xi d\eta, \qquad z \in D.$$

By $L_r^\infty$ we denote the space of all measurable functions $\psi$ on $D$ for which

$$\|\psi\|_{\infty,r} = \operatorname*{ess\,sup}_{w \in D} r^{-2}(w)|\psi(w)| < \infty,$$

and by $B_r^\infty$ the subspace of $L_r^\infty$ consisting of holomorphic functions on $D$. If $D = \mathbf{C} \backslash \{a, b, \infty\}$ where $a, b \in \mathbf{C}$, $a \ne b$ and $f \in B^1(D)$, then $f$ must be the identically zero function on $D$. But if the boundary of the plane domain $D$ has at least three finite boundary points $a$, $b$ and $c$, then the function $f(z) = [(z-a)(z-b)(z-c)]^{-1}$ belongs to $B^1(D)$.

THEOREM 1. *Let $D$ be a plane domain, whose boundary has at least three finite boundary points. To each bounded linear functional $\Phi$ on $B^1(D)$, there corresponds a unique $g \in B_r^\infty(D)$ such that*

$$\Phi(f) = \iint_D r^{-2}(w) f(w) \overline{g(w)} du dv, \qquad f \in B^1(D). \tag{1}$$

*Moreover, if $\Phi$ and $g$ are related as in (1), then* $\frac{1}{3}\|g\|_{\infty,r} \le \|\Phi\| \le \|g\|_{\infty,r}$.

*Proof.* Let $\Phi$ be a bounded linear functional on $B^1 = B^1(D)$. By the Hahn–Banach theorem, $\Phi$ can be extended to a bounded linear functional $\tilde{\Phi}$ on $L^1$ so that $\|\tilde{\Phi}\| = \|\Phi\|$.

By Theorem 6.16 of **[4]** there is a unique $\tilde{\psi} \in L^\infty$ such that

$$\tilde{\Phi}(\varphi) = \iint_D \varphi(w) \tilde{\psi}(w) du dv \qquad (\varphi \in L^1)$$

and that $\|\tilde{\Phi}\| = \|\tilde{\psi}\|_\infty$.

Let $\psi(w) = r^2(w)\tilde{\psi}(w)$, so that $\psi \in L_r^\infty(D)$. Let $g = K\psi$. Using again the part (ii) of Lemma A we get:

$$|g(z)| = |(K\psi)(z)| \le \frac{3}{\pi} \iint_D r^{-2}(\zeta) K(z,\zeta) |\psi(\zeta)| d\zeta d\eta \le 3r^2(z) \|\psi\|_{\infty,r},$$

where $z \in D$. Thus $g \in B_r^\infty$ and

$$\|g\|_{\infty,r} \leq 3\|\psi\|_{\infty,r}. \tag{2}$$

For our purpose it is convenient to use the notation:

$$\langle \varphi, \tau \rangle = \iint\limits_D r^{-2}(w)\varphi(w)\overline{\tau(w)}\,dudv,$$

if the right hand side exists. Note that $\tilde{\Phi}(\varphi) = \langle \varphi, \psi \rangle$, $\varphi \in L^1$. Now let $f \in B^1$. Then:

$$\langle f, g \rangle = \langle f, K\psi \rangle = \frac{3}{\pi} \iint\limits_D (r^{-2}(\zeta)f(\zeta) \iint\limits_D r^{-2}(w)\overline{K(\zeta,w)\psi(w)}\,dudv)\,d\xi d\eta.$$

Since $K(z,w) = \overline{K(w,z)}$, using Fubini's theorem and the reproducing property which satisfies $f$, we get $\langle f, g \rangle = \langle f, \psi \rangle$. Hence $\langle f, g \rangle = \tilde{\Phi}(f) = \Phi(f)$.

Let us prove the uniqueness of $g$. Let $g$ and $g_1$ satisfy (1) and let $h = g - g_1$. Then

$$\langle f, h \rangle = 0, \qquad \text{for every } f \in B^1. \tag{3}$$

Using the reproducing property

$$\overline{h(\zeta)} = \overline{(Kh)(\zeta)} = \frac{3}{\pi} \iint\limits_D r^{-2}(w)K(w,\zeta)\overline{h(w)}\,dudv \qquad (\zeta \in D),$$

Fubini's theorem and (3) we can show that

$$\langle \varphi, h \rangle = \langle \varphi, Kh \rangle = \langle K\varphi, h \rangle = 0,$$

for every $\varphi \in L^1$. Now the integral of $h = g - g_1$ over any measurable set $E \subset D$ of finite measure is 0 (as we see by taking $\chi_E$ for $\varphi$) and hence $h \equiv 0$ on $D$. By (2) we have

$$\frac{1}{3}\|g\|_{\infty,r} \leq \|\Phi\| \leq \|g\|_{\infty,r}.$$

Note that, among other things, the space $L_r^\infty$ has an important role in the theory of quasiconformal mapping (see [**2**] and [**3**]).

The dual of $B^1(U)$ was determined in [**1**]. In this case we can describe the dual space with respect to the weighted pairing

$$\langle f, g \rangle_s = \int\limits_U (1 - |w|^2)^s f(w)\overline{g(w)}\,dudv, \qquad \text{for all } s \geq 0.$$

Let $\Lambda_S$, $0 \leq s < +\infty$, denote the space of all measurable functions $g$ on $U$ for which $r^{-s}g$ is a bounded function on $U$ and $H\Lambda_s$ the corresponding subspace consisting of holomorphic functions on $U$.

Proposition 2. *If $\Phi$ is a bounded linear functional on $B^1(U)$, then:* (a) *for every $s > 0$ there exists a $g \in H\Lambda_s$ such that $\Phi(f) = \langle f, g\rangle_s$ for all $f \in B^1$;* (b) *there exists a holomorphic function $G$ which belongs to the Bloch space $\mathcal{B}$ such that*

$$\Phi(f) = \lim_{r\to 1_-} \langle f_r, G\rangle_0, \qquad \text{for all } f \in B^1.$$

Here $f_r$ has the usual meaning defined by $f_r(z) = f(rz)$. In the case $D = U$ Theorem 1 is reduced to the case $s = 2$ of Proposition 2.

*Proof of the part* (a). In the proof of Theorem 1 we showed that there exists $\tilde{\psi} \in L^\infty$ such that $\Phi(f) = \langle f, \tilde{\psi}\rangle_0$ for every $f \in B^1$. Let $\psi = r^s\tilde{\psi}$. If $s > 0$ then Proposition 1.4.10 of [**5**] shows that the operator $T_s$ (see chapter 7 of [**5**] for the definition) is a bounded operator from $\Lambda_s$ into $H\Lambda_s$. Hence $g = T_s\psi$ belongs to $H\Lambda_s$. As in the proof of Theorem 1 we can show that $\langle f, \tilde{\psi}\rangle_0 = \langle f, \psi\rangle_s = \langle f, T_s\psi\rangle_s$ for every $f \in B^1$.

*Proof of the part* (b). Let $G = T_0\tilde{\psi}$. Another application on Proposition 1.4.10 of [**5**] shows that $G \in \mathcal{B}$ (Note that $T_0$ is a bounded operator from $L_0 = L^\infty$ into Bloch space $\mathcal{B}$ which is strictly larger then $L^\infty$, but $T_0$ is not bounded from $L_0 = L^\infty$ into $L^\infty$). Now the assertion (b) follows from the relations

$$\Phi(f_r) = \langle f_r, \tilde{\psi}\rangle_0 = \langle f_r, T_0\tilde{\psi}\rangle_0 = \langle f_r, G\rangle_0, \quad 0 < r < 1, \quad \text{and} \quad \Phi(f) = \lim_{r\to 1_-} \Phi(f_r).$$

It would be interesting to give the appropriate generalizations of Proposition 2 and also of the statements which we used in our proof of Proposition 2, concerning more general domains than the unit disk.

We say that a hyperbolic domain $D$ in $\mathbf{C}$ is strongly hyperbolic if every component of $\partial D$ is different from a point. If $D$ is a hyperbolic domain then $r_D(z) \le \operatorname{dist}(z, \partial D)^{-1}$, for every $z \in D$, where $d(z) = \operatorname{dist}(z, \partial D)$ denotes the distance from $z$ to $\partial D$.

If $D$ is strongly hyperbolic we realized that

$$\frac{1}{4} d_D^{-1}(z) \le r_D(z) \le d_D^{-1}(z). \tag{4}$$

If $D$ is only hyperbolic, then the first inequality in (4) does not hold as the following example shows.

*Example.* If $D = U\backslash\{0\}$, then $r(z) = [|z|\log(1/(|z|)]^{-1}$. Thus $r(z)d(z)$ tends to zero when $D \ni z \to 0$.

Now it is natural to ask whether there exists a version of Theorem 1 and Lemma A with $d$ instead of $r^{-1}$?

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