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ON SOME TRANSFORMS OF TRIGONOMETRIC SERIES

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Abstract. We give a transform of convergent trigonometric series into equivalent convergent series and sufficient conditions for the transformed series to converge faster than the original one.

Introduction. Let

$$\sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x \tag{1}$$

be a convergent real or complex trigonometric series. A method of accelerating the convergence of (1) is given in [1]. It consists of as follows:

Let $r \neq 1$ be a real or complex number and Δ_r^k a linear operator defined by

$$\begin{aligned}\Delta_r(a_n) &= a_{n+1} - r a_n\\ \Delta_r^{k+1}(a_n) &= \Delta_r(\Delta_r^k(a_n)) \quad (k = 1, 2, \ldots).\end{aligned}$$

If $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = r, \quad \lim\limits_{n\to\infty} \frac{\Delta_r^k(a_{n+1})}{\Delta_r^k(a_n)} = r \ (k = 1, 2, \ldots, p)$, then

$$\begin{aligned}\sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x &= \frac{a_1 C_r^1(0)}{1 - 2r\cos\alpha x + r^2} + \sum_{k=1}^{p-1} \frac{\Delta_r^k(a_1) C_r^{k+1}(0)}{(1 - 2r\cos\alpha x + r^2)^{k+1}}\\ &\quad + \frac{1}{(1 - 2r\cos\alpha x + r^2)^p} \sum_{n=1}^{\infty} \Delta_r^p(a_n) \Delta_r^p \cos(\alpha n + \beta)x, \end{aligned} \tag{2}$$

where are $C_r^k(n) = \Delta_r^k \cos(\alpha n + \beta)x \quad (n = 0, 1, 2, \ldots)$.

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A generalization for number series is given in [2] and for power series in [3]. More detailed approach on these issues is given in [6]. In this paper we obtain a generalization of transform (2) for cosine and sine series and give sufficient conditions for the modified transform to converge faster than (1).

For a sequence of real or complex numbers $\{a_n\}_{n=1}^{\infty}$ and a given sequence $\{r_n\}_{n=1}^{\infty}$ we define a linear operator $L_{r_1 \ldots r_p}$ by

$$\begin{aligned} L_{r_1}(a_n) &= a_{n+1} - r_1 a_n \\ L_{r_1 \ldots r_{p+1}}(a_n) &= L_{r_1 \ldots r_p}(a_{n+1}) - r_{p+1} L_{r_1 \ldots r_p}(a_n) \quad (p = 1, 2, \ldots). \end{aligned} \tag{3}$$

In particular, for the sequence $\{\cos(\alpha n + \beta)x\}_{n=1}^{\infty}$ we put

$$C_{r_1 \ldots r_p}(n) = L_{r_1 \ldots r_p} \cos(\alpha n + \beta)x \quad (n = 0, 1, 2, \ldots).$$

For fixed p put

$$E_0 = 1,\ E_1 = \sum_{i=1}^{p} r_i,\ E_2 = \sum_{1 \le i < j \le p} r_i r_j,\ E_3 = \sum_{1 \le i < j < k \le p} r_i r_j r_k,\ \ldots,\ E_p = r_1 r_2 \ldots r_p$$

(where the summation for E_m is performed over all combinations of distinct indices between 1 and p taken m at a time); we note that

$$L_{r_1 \ldots r_p}(a_n) = \sum_{k=0}^{p} (-1)^k E_k a_{n+p-k}. \tag{4}$$

In order to establish the modified transform, we use the following

LEMMA. *Suppose that the coefficients* t_{nm} $(0 \le m \le n)$ *of the infinite triangular matrix* (t_{nm}) *satisfy the following conditions:*

a) $\lim_{n \to \infty} t_{nm} = 0$ *for each fixed* m*;*

b) there exists a constant K *such that* $\sum_{k=0}^{p} |t_{pk}| \le K$ *for each nonnegative* p*.*

Let $\{x_n\}_{n=1}^{\infty}$ *be a sequence and define the sequence* $\{x'_n\}_{n=1}^{\infty}$ *by*

$$x'_n = \sum_{k=0}^{n} t_{nk} x_k \quad (n = 0, 1, 2, \ldots).$$

Then we have: if $\lim_{n \to \infty} x_n = 0$*, then* $\lim_{n \to \infty} x'_n = 0$.

The proof of the lemma is due to Toeplitz [4, p. 325].

1. A modified transform of trigonometric series. The following theorem gives a generalization of transform (2) for trigonometric series (1).

THEOREM 1. *Let* (1) *be a convergent real or complex trigonometric series* ($\alpha \neq 0$), $r_1, ..., r_p$ ($r_j e^{\pm \alpha x i} \neq 1$, $j = 1, ..., p$) *arbitrary real or complex numbers. Then*

$$\sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x = \frac{a_1 C_{r_1}(0)}{1 - 2r_1 \cos \alpha x + r_1^2}$$
$$+ \sum_{k=1}^{p-1} \frac{L_{r_1 \dots r_k}(a_1) C_{r_1 \dots r_{k+1}}(0)}{(1 - 2r_1 \cos \alpha x + r_1^2)...(1 - 2r_{k+1} \cos \alpha x + r_{k+1}^2)}$$
$$+ \left(\prod_{j=1}^{p} (1 - 2r_j \cos \alpha x + r_j^2) \right)^{-1} \sum_{n=1}^{\infty} L_{r_1 \dots r_p}(a_n) L_{r_1 \dots r_p} \cos(\alpha n + \beta)x. \quad (5)$$

Proof. Considering the Euler's formula for $\cos(\alpha n + \beta)x$ we have

$$\sum_{n=1}^{\infty} a_n \cos(\alpha n + \beta)x = \frac{1}{2} \sum_{n=1}^{\infty} a_n e^{(\alpha n + \beta)xi} + \frac{1}{2} \sum_{n=1}^{\infty} a_n e^{-(\alpha n + \beta)xi}.$$

Let $f_1(x) = \sum_{n=1}^{\infty} a_n e^{(\alpha n + \beta)xi}$, $f_2(x) = \sum_{n=1}^{\infty} a_n e^{-(\alpha n + \beta)xi}$. Then

$$f_1(x) = a_1 e^{(\alpha + \beta)xi} + e^{\alpha xi} \Phi_1(x),$$

where $\Phi_1(x) = \sum_{k=1}^{\infty} a_{k+1} e^{(\alpha k + \beta)xi}$. So

$$(1 - r_1 e^{\alpha xi}) \Phi_1(x) = \sum_{k=1}^{\infty} (a_{k+1} - r_1 a_k) e^{(\alpha k + \beta)xi} + r_1 a_1 e^{(\alpha + \beta)xi}.$$

Thus

$$\Phi_1(x) = \frac{r_1 a_1 e^{(\alpha + \beta)xi}}{1 - r_1 e^{\alpha xi}} + \frac{1}{1 - r_1 e^{\alpha xi}} \sum_{k=1}^{\infty} (a_{k+1} - r_1 a_k) e^{(\alpha k + \beta)xi}$$

and hence

$$f_1(x) = a_1 e^{(\alpha + \beta)xi} + e^{\alpha xi} \frac{r_1 a_1 e^{(\alpha + \beta)xi}}{1 - r_1 e^{\alpha xi}} + \frac{e^{\alpha xi}}{1 - r_1 e^{\alpha xi}} \sum_{k=1}^{\infty} (a_{k+1} - r_1 a_k) e^{(\alpha k + \beta)xi}$$
$$= \frac{a_1 e^{(\alpha + \beta)xi}}{1 - r_1 e^{\alpha xi}} + \frac{e^{\alpha xi}}{1 - r_1 e^{\alpha xi}} \sum_{k=1}^{\infty} L_{r_1}(a_k) e^{(\alpha k + \beta)xi}. \quad (6)$$

Applying a similar technique to $\sum_{k=1}^{\infty} L_{r_1}(a_k) e^{(\alpha k + \beta)xi}$, we get

$$\sum_{k=1}^{\infty} L_{r_1}(a_k) e^{(\alpha k + \beta)xi} = L_{r_1}(a_1) e^{(\alpha + \beta)xi} + e^{\alpha xi} \Phi_2(x)$$

(where now $\Phi_2(x) = \sum_{k=1}^{\infty} L_{r_1}(a_{k+1})e^{(\alpha k+\beta)xi}$), where from using (3), we get

$$\begin{aligned}(1-r_2e^{\alpha xi})\Phi_2(x) &= \sum_{k=1}^{\infty} \left(L_{r_1}(a_{k+1}) - r_2 L_{r_1}(a_k)\right) e^{(\alpha k+\beta)xi} + r_2 L_{r_1}(a_1)e^{(\alpha+\beta)xi} \\ &= \sum_{k=1}^{\infty} L_{r_1 r_2}(a_k)e^{(\alpha k+\beta)xi} + r_2 L_{r_1}(a_1)e^{(\alpha+\beta)xi}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=1}^{\infty} L_{r_1}(a_k)e^{(\alpha k+\beta)xi} = L_{r_1}(a_1)e^{(\alpha+\beta)xi} &+ e^{\alpha xi}\frac{r_2 e^{(\alpha+\beta)xi}L_{r_1}(a_1)}{1-r_2e^{\alpha xi}} \\ &+ \frac{e^{\alpha xi}}{1-r_2e^{\alpha xi}}\sum_{k=1}^{\infty} L_{r_1 r_2}(a_k)e^{(\alpha k+\beta)xi}.\end{aligned}$$

Thus, using (6), we obtain

$$\begin{aligned}f_1(x) = \frac{a_1 e^{(\alpha+\beta)xi}}{1-r_1e^{\alpha xi}} &+ \frac{L_{r_1}(a_1)e^{(2\alpha+\beta)xi}}{(1-r_1e^{\alpha xi})(1-r_2e^{\alpha xi})} \\ &+ \frac{e^{2\alpha xi}}{(1-r_1e^{\alpha xi})(1-r_2e^{\alpha xi})}\sum_{k=1}^{\infty} L_{r_1 r_2}(a_k)e^{(\alpha k+\beta)xi}. \qquad (7)\end{aligned}$$

Repeating this process p times we find that

$$f_1(x) = \frac{a_1 e^{(\alpha+\beta)xi}}{1-r_1e^{\alpha xi}} + \sum_{k=1}^{p-1} L_{r_1\ldots r_k}(a_1)\frac{e^{(\alpha(k+1)+\beta)xi}}{(1-r_1e^{\alpha xi})\ldots(1-r_{k+1}e^{\alpha xi})} + R_p^{(1)}(x), \quad (8)$$

where

$$R_p^{(1)}(x) = \frac{1}{(1-r_1e^{\alpha xi})\ldots(1-r_pe^{\alpha xi})}\sum_{n=1}^{\infty} L_{r_1\ldots r_p}(a_n)e^{(\alpha(n+p)+\beta)xi}. \qquad (9)$$

Notice that (6) and (7) are the $p=1$ and $p=2$ cases of (8), respectively. Since $f_2(x) = f_1(-x)$, we have

$$f_2(x) = \frac{a_1 e^{-(\alpha+\beta)xi}}{1-r_1e^{-\alpha xi}} + \sum_{k=1}^{p-1} L_{r_1\ldots r_k}(a_1)\frac{e^{-(\alpha(k+1)+\beta)xi}}{(1-r_1e^{-\alpha xi})\ldots(1-r_{k+1}e^{-\alpha xi})} + R_p^{(2)}(x), \qquad (8')$$

where

$$R_p^{(2)}(x) = \frac{1}{(1-r_1e^{-\alpha xi})\ldots(1-r_pe^{-\alpha xi})}\sum_{n=1}^{\infty} L_{r_1\ldots r_p}(a_n)e^{-(\alpha(n+p)+\beta)xi}. \qquad (9')$$

Multiplying the equations (8) and (8′) by 1/2 and then summing the two together, using (4) for the sequence $\{\cos(\alpha n+\beta)x\}_{n=1}^{\infty}$, we obtain

$$\sum_{n=1}^{\infty} a_n \cos(\alpha n+\beta)x = \frac{a_1 C_{r_1}(0)}{1-2r_1\cos\alpha x + r_1^2} + \sum_{k=1}^{p-1} \frac{L_{r_1\dots r_k}(a_1) C_{r_1\dots r_{k+1}}(0)}{(1-2r_1\cos\alpha x+r_1^2)\dots(1-2r_{k+1}\cos\alpha x + r_{k+1}^2)} + R_p(x), \quad (8'')$$

where

$$R_p(x) = \frac{1}{2}\left(R_p^{(1)}(x)+R_p^{(2)}(x)\right) = \left(\prod_{j=1}^{p}(1-2r_j\cos\alpha x + r_j^2)\right)^{-1} \sum_{n=1}^{\infty} L_{r_1\dots r_p}(a_n) L_{r_1\dots r_p} \cos(\alpha n+\beta)x. \quad (9'')$$

Equalities (8″) and (9″) complete the proof.

In completely analogous way we obtain the similar transform for sine series

$$\sum_{n=1}^{\infty} a_n \sin(\alpha n+\beta)x = \frac{a_1 S_{r_1}(0)}{1-2r_1\cos\alpha x + r_1^2} + \sum_{k=1}^{p-1} \frac{L_{r_1\dots r_k}(a_1) S_{r_1\dots r_{k+1}}(0)}{(1-2r_1\cos\alpha x+r_1^2)\dots(1-2r_{k+1}\cos\alpha x + r_{k+1}^2)} + \left(\prod_{j=1}^{p}(1-2r_j\cos\alpha x + r_j^2)\right)^{-1} \sum_{n=1}^{\infty} L_{r_1\dots r_p}(a_n) L_{r_1\dots r_p} \sin(\alpha n+\beta)x, \quad (10)$$

where $S_{r_1\dots r_p}(n) = L_{r_1\dots r_p}\sin(\alpha n+\beta)x \quad (n=0,1,2,\dots)$.

Remark 1. If $L_{r_1\dots r_p}(a_n)=0$ for some $p\ge 1$ and for n sufficiently large, then (5) and (10) transform trigonometric series into finite sums.

Remark 2. In particular, for $r_1=r_2=\cdots=r_p=r$, we obtain the transform (2).

2. Accelerating convergence of trigonometric series. The following theorem gives a transform of convergent trigonometric series (1) into an equivalent convergent series.

THEOREM 2. *Let* (1) *be a convergent series on* x, $\pi/2 \le |\alpha| x \le 3\pi/2$ *and* $\{r_n\}_{n=1}^{\infty}$ *a sequence of positive real numbers such that for some real* $\lambda > 1$

$$r_n = O\left(n^{-\lambda}\right) \quad \text{or} \quad 1/r_n = O\left(n^{-\lambda}\right). \quad (11)$$

Then

$$\sum_{n=1}^{\infty} a_n \cos(\alpha n+\beta)x = \frac{a_1 C_{r_1}(0)}{1-2r_1\cos\alpha x + r_1^2}$$
$$+\sum_{k=1}^{\infty} \frac{L_{r_1\ldots r_k}(a_1) C_{r_1\ldots r_{k+1}}(0)}{(1-2r_1\cos\alpha x+r_1^2)\ldots(1-2r_{k+1}\cos\alpha x + r_{k+1}^2)}. \tag{12}$$

Proof. In order to prove (12) we need to show that in (8″), $\lim\limits_{p\to\infty} R_p(x) = 0$. Let $r_n(x)$ be the remainder of the convergent series $\sum\limits_{n=1}^{\infty} a_n e^{(\alpha n+\beta)xi}$. Then $\lim\limits_{n\to\infty} r_n(x) = 0$. Thus, using (4) we get

$$R_p^{(1)}(x) = \left(\prod_{j=1}^{p}(1-r_j e^{\alpha x i})\right)^{-1} \sum_{n=1}^{\infty} e^{(\alpha(n+p)+\beta)xi} \sum_{k=0}^{p}(-1)^k E_k a_{n+p-k}$$
$$= \left(\prod_{j=1}^{p}(1-r_j e^{\alpha x i})\right)^{-1} \sum_{k=0}^{p}(-1)^{p-k} e^{(p-k)xi} E_{p-k} r_k(x).$$

In the lemma of Introduction we put

$$t_{pk} = \left(\prod_{j=1}^{p}(1-r_j e^{\alpha x i})\right)^{-1} (-1)^{p-k} e^{(p-k)xi} E_{p-k} \quad (0\le k\le p).$$

We have to show that the conditions of the lemma are satisfied. For each nonnegative integer p we have

$$\|t_{pk}| = \left|\left(\prod_{j=1}^{p}(1-r_j e^{\alpha x i})\right)^{-1} E_{p-k}\right|$$
$$= \left|\sum_{1\le i_1<\ldots<i_{p-k}\le p} \left(\prod_{s=p-k+1}^{p}(1-r_{i_s}e^{\alpha x i})\right)^{-1} \prod_{j=1}^{p-k} \frac{r_{i_j}}{1-r_{i_j}e^{\alpha x i}}\right|.$$

Without lost in generality we may assume that $\alpha > 0$. Then $\pi/2 \le \alpha x \le 3\pi/2$, and since $r_j \ge 0 \quad (j=1,\ldots,p)$, we have

$$\left|1-r_j e^{\alpha x i}\right| > 1, \quad \left|r_j/\left(1-r_j e^{\alpha x i}\right)\right| < 1 \quad (j=1,\ldots,p). \tag{13}$$

Hence

$$|t_{pk}| \le \sum_{1\le i_1<\ldots<i_{p-k}\le p} \prod_{j=1}^{p-k}\left|\frac{r_{i_j}}{1-r_{i_j}e^{\alpha x i}}\right| \le \binom{p}{p-k} M^{p-k} \le p^k M^{p-k},$$

where $M = \max\limits_{1\le j\le p} \left| r_j / \left(1 - r_j e^{\alpha x i}\right) \right| < 1$. Thus $\lim\limits_{p\to\infty} t_{pk} = 0$, so the condition (a) of the lemma holds. Notice that

$$\sum_{k=0}^{p} |t_{pk}| \le \left| \prod_{j=1}^{p} (1 - r_j e^{\alpha x i}) \right|^{-1} \sum_{k=0}^{p} E_{p-k} = \left| \prod_{j=1}^{p} (1 - r_j e^{\alpha x i}) \right|^{-1} \prod_{k=1}^{p} (1 + r_k). \quad (14)$$

From (11) we derive that one of the two series $\sum\limits_{j=1}^{\infty} r_j$ or $\sum\limits_{j=1}^{\infty} 1/r_j$ converges, and hence one of the two infinite products $\prod\limits_{j=1}^{\infty} (1 + r_j)$ or $\prod\limits_{j=1}^{\infty} (1 + 1/r_j)$ converges. Put

$$K = \begin{cases} \prod\limits_{j=1}^{\infty} (1 + r_j), & \text{if } \prod\limits_{j=1}^{\infty} (1 + r_j) \text{ converges} \\ \prod\limits_{j=1}^{\infty} (1 + 1/r_j)\,, & \text{if } \prod\limits_{j=1}^{\infty} (1 + 1/r_j) \text{ converges,} \end{cases}$$

using (14) and (13), we conclude that the condition (b) of the lemma is also satisfied. Thus $\lim\limits_{p\to\infty} R_p^{(1)}(x) = 0$. On the other hand, since

$$\left|1 - r_j e^{\alpha x i}\right| = \left|1 - r_j e^{-\alpha x i}\right| \quad (j = 1, 2, \ldots),$$

the inequalities (13) hold true if we replace x by $-x$. This however, based on (9) and (9′), means that $\lim\limits_{p\to\infty} R_p^{(2)}(x) = 0$. Hence, using (9″), $\lim\limits_{p\to\infty} R_p(x) = 0$. Now (12) follows from (8″).

Obviously, the result of Theorem 2 can be applied for the transform (10) of sine series. Obtained transform, analogous with (12), is

$$\sum_{n=1}^{\infty} a_n \sin(\alpha n + \beta)x = \frac{a_1 S_{r_1}(0)}{1 - 2r_1 \cos \alpha x + r_1^2} + \sum_{k=1}^{\infty} \frac{L_{r_1 \ldots r_k}(a_1) S_{r_1 \ldots r_{k+1}}(0)}{(1 - 2r_1 \cos \alpha x + r_1^2) \ldots (1 - 2r_{k+1} \cos \alpha x + r_{k+1}^2)}. \quad (15)$$

Remark 3. Let $r_j > 0$ $(j = 1, 2, \ldots)$ and $\pi/2 \le |\alpha| x \le 3\pi/2$. Suppose for $p = 1, 2, \ldots$ that $L_{r_1 \ldots r_p}(a_n) \ne 0$ for n sufficiently large and that $\lim\limits_{n\to\infty} \frac{L_{r_1 \ldots r_p}(a_{n+1})}{L_{r_1 \ldots r_p}(a_n)}$ exists. Since, according to (3),

$$\begin{aligned} \frac{L_{r_1}(a_n)}{a_n} &= \frac{a_{n+1}}{a_n} - r_1 \\ \frac{L_{r_1 \ldots r_{p+1}}(a_n)}{L_{r_1 \ldots r_p}(a_n)} &= \frac{L_{r_1 \ldots r_p}(a_{n+1})}{L_{r_1 \ldots r_p}(a_n)} - r_{p+1} \quad (p = 1, 2, \ldots), \end{aligned} \quad (16)$$

if we require the additional condition that the sequence $\{r_n\}_{n=1}^{\infty}$ is chosen so that

$$r_1 = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}, \quad r_{p+1} = \lim_{n\to\infty} \frac{L_{r_1\dots r_p}(a_{n+1})}{L_{r_1\dots r_p}(a_n)} \quad (p = 1, 2, \dots),$$

then the sequences (16) are null-sequences. Whence

$$\lim_{n\to\infty} \frac{L_{r_1\dots r_p}(a_n)}{a_n} = \lim_{n\to\infty} \frac{L_{r_1}(a_n)}{a_n} \frac{L_{r_1 r_2}(a_n)}{L_{r_1}(a_n)} \cdots \frac{L_{r_1\dots r_p}(a_n)}{L_{r_1\dots r_{p-1}}(a_n)} = 0 \quad (p = 1, 2, \dots).$$

Therefore, according to $(8'')$, $(9'')$, (9), $(9')$ and (13), we conclude that the series on the right-hand side of (5) and (10) converge faster than the ones on the left-hand side.

If we require that the sequence $\{r_n\}_{n=1}^{\infty}$ from remark 3 satisfies the condition (11), then the right-hand sides of (12) and (15) converge faster than the left-hand sides.

Example. Let $a_n = 1/(a^n + b^n)$ $(0 < a < b)$. Then

$$r_1 = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{b}, \quad r_{p+1} = \lim_{n\to\infty} \frac{L_{r_1\dots r_p}(a_{n+1})}{L_{r_1\dots r_p}(a_n)} = \frac{a^{p-1}}{b^p} \quad (p = 1, 2, \dots).$$

Obviously, the sequence $\{r_n\}_{n=1}^{\infty}$ satisfies the condition (11) of Theorem 2. In particular, put $a = 2$, $b = 3$, $\alpha = 1$, $\beta = 0$, $x = 3\pi/4$; then in order to calculate the approximate sum of the number series $\sum_{n=1}^{\infty} \frac{1}{2^n+3^n} \cos \frac{3n\pi}{4}$ with error not greater than 10^{-6} we must compute the sum of first 12 terms. Applying the transform (5) of Theorem 1, the same accuracy is obtained by computing the sum of first 7 terms for $p = 1$, 4 terms for $p = 2$, and 2 terms for $p = 3$.

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