

ON MOORE-PENROSE INVERSE OF BLOCK MATRICES AND FULL-RANK FACTORIZATION

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Dedicated to Professor Petar Madić on the occasion of his 75th birthday

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Abstract. We develop a few methods for computing the Moore-Penrose inverse, based on full-rank factorizations which arise from different block decompositions of rectangular matrices. In this way, the paper is a continuation of the previous works given by Noble [5] and Tewarson [10]. We compare the obtained results with the known block representations of the Moore-Penrose inverse. Moreover, efficient block representations of the weighted Moore-Penrose inverse are introduced using the same principles.

1. Introduction

Let $\mathbb{C}^{m \times n}$ be the set of $m \times n$ complex matrices, and $\mathbb{C}_r^{m \times n} = \{X \in \mathbb{C}^{m \times n} : \text{rank}(X) = r\}$. With $A^{|r}$ and $A_{|r}$ we denote the submatrix of A which contains the first r columns of A and the first r rows of A , respectively. Similarly, $A^{r|}$ and $A_{r|}$ denote the last r columns and the last r rows of A , respectively. Finally, $A_{r|}^{|r|}$ denotes the submatrix of A generated by the first r columns and the last r rows of A . The identity matrix of the order k is denoted by I_k , and \mathbb{O} denotes the zero matrix of a convenient size.

Penrose [6], [7] has shown the existence and uniqueness of a solution $X \in \mathbb{C}^{n \times m}$ of the following four equations:

$$(1) AXA = A, \quad (2) XAX = X, \quad (3) (AX)^* = AX, \quad (4) (XA)^* = XA,$$

for any $A \in \mathbb{C}^{m \times n}$. For a sequence \mathcal{S} of elements from the set $\{1, 2, 3, 4\}$, the set of matrices which satisfy the equations represented in \mathcal{S} is denoted by $A\{\mathcal{S}\}$. A matrix from $A\{\mathcal{S}\}$ is called an \mathcal{S} -inverse of A and denoted by $A^{(\mathcal{S})}$.

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We use the following useful expansion for the Moore-Penrose generalized inverse A^\dagger of A , based on the full-rank factorization $A = PQ$ of A [1], [2]:

$$A^\dagger = Q^\dagger P^\dagger = Q^*(QQ^*)^{-1}(P^*P)^{-1}P^* = Q^*(P^*AQ^*)^{-1}P^*.$$

The *weighted Moore-Penrose inverse* is investigated in [3] and [8]. The main results of these papers are:

PROPOSITION 1.1. [8] *Let given positive-definite matrices $M \in \mathbb{C}^{m \times m}$ and $N \in \mathbb{C}^{n \times n}$. For any matrix $A \in \mathbb{C}^{m \times n}$ there exists the unique solution $X = A_{M \circ, \circ N}^\dagger \in A\{1, 2\}$ satisfying*

$$(5) \quad (MAX)^* = MAX \quad (6) \quad (XAN)^* = XAN.$$

Similarly, we use the following notations:

$A_{M \circ, N \circ}^\dagger$ denotes the unique solution of the equations (1), (2), and

$$(7) \quad (MAX)^* = MAX, \quad (8) \quad (NXA)^* = NXA;$$

$A_{\circ M, N \circ}^\dagger$ is the unique solution of the equations (1), (2) and

$$(9) \quad (AXM)^* = AXM, \quad (10) \quad (NXA)^* = NXA;$$

$A_{\circ M, \circ N}^\dagger$ denotes the unique solution of the equations (1), (2) and

$$(11) \quad (AXM)^* = AXM, \quad (12) \quad (XAN)^* = XAN.$$

PROPOSITION 1.2 [8] *The equation (5) is equivalent to $(AXM^{-1})^* = AXM^{-1}$, and (6) can be expressed in the form $(N^{-1}XA)^* = N^{-1}XA$.*

PROPOSITION 1.3 [8] *If $A = PQ$ is a full rank factorization of A , then:*

$$\begin{aligned} A_{M \circ, \circ N}^\dagger &= (QN)^*(Q(QN)^*)^{-1}((MP)^*P)^{-1}(MP)^* \\ &= NQ^*(P^*MANQ^*)^{-1}P^*M. \end{aligned}$$

Using these notations, the following fact can be easily verified.

PROPOSITION 1.4 (a) $A_{M \circ, \circ N}^\dagger = A_{\circ M^{-1}, \circ N}^\dagger = A_{M \circ, N^{-1} \circ}^\dagger = A_{\circ M^{-1}, N^{-1} \circ}^\dagger,$

$$(b) \quad A_{M \circ, N \circ}^\dagger = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^* = A_{M \circ, \circ N^{-1}},$$

$$(c) \quad A_{\circ M, N \circ}^\dagger = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}(M^{-1}P)^*P)^{-1}(M^{-1}P)^* = A_{M^{-1} \circ, \circ N^{-1}},$$

$$(d) \quad A_{\circ M, \circ N}^\dagger = (QN)^*(Q(QN)^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*.$$

From the part (a) of Proposition 1.4 and Proposition 1.2, it is easy to conclude that each of the indices, used in notation of the weighted Moore-Penrose inverse, can be written in one of the following form:

$$\circ M, \circ N \quad M \circ, \circ N \quad \circ M, N \circ \quad M \circ, N \circ$$

For the sake of clarity, we use the notation $\varphi(M, N)$ for an arbitrary of these indices. Following Proposition 1.3 and Proposition 1.4, we conclude the following:

1. $\varphi(M, N) = \circ M, \circ N \Rightarrow A_{\varphi(M, N)}^\dagger = (QN)^*(Q(QN)^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*$;
2. $\varphi(M, N) = M \circ, \circ N \Rightarrow A_{\varphi(M, N)}^\dagger = (QN)^*(Q(QN)^*)^{-1}((MP)^*P)^{-1}(MP)^*$;
3. $\varphi(M, N) = \circ M, N \circ \Rightarrow A_{\varphi(M, N)}^\dagger = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((M^{-1}P)^*P)^{-1}(M^{-1}P)^*$;
4. $\varphi(M, N) = M \circ, N \circ \Rightarrow A_{\varphi(M, N)}^\dagger = (QN^{-1})^*(Q(QN^{-1})^*)^{-1}((MP)^*P)^{-1}(MP)^*$.

The cases 1–4 can be written in the following way:

$$A_{\varphi(M, N)}^\dagger = (QN^{[-1]})^*(Q(QN^{[-1]})^*)^{-1}((M^{[-1]}P)^*P)^{-1}(M^{[-1]}P)^*$$

where $M^{[-1]}$ stands for one of the matrices M or M^{-1} , and $N^{[-1]}$ denotes N or N^{-1} , in view of one of the rules 1–4.

We restate the main block decompositions [4], [11–13]. For a given matrix $A \in \mathbb{C}_r^{m \times n}$ there exist the regular matrices R, G , the permutation matrices E, F and the unitary matrices U, V , such that:

$$(T_1) \quad RAG = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1; \quad (T_2) \quad RAG = \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_2;$$

$$(T_3) \quad RAF = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_3; \quad (T_4) \quad EAG = \begin{bmatrix} I_r & \mathbb{O} \\ K & \mathbb{O} \end{bmatrix} = N_4;$$

$$(T_5) \quad UAG = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1; \quad (T_6) \quad RAV = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1;$$

$$(T_7) \quad UAV = \begin{bmatrix} B & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_2; \quad (T_8) \quad UAF = \begin{bmatrix} B & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_5;$$

$$(T_9) \quad EAV = \begin{bmatrix} B & \mathbb{O} \\ K & \mathbb{O} \end{bmatrix} = N_6;$$

$$(T_{10a}) \quad EAF = \begin{bmatrix} A_{11} & A_{11}T \\ SA_{11} & SA_{11}T \end{bmatrix} = N_7;$$

where the multipliers S, T satisfy $T = A_{11}^{-1}A_{12}$, $S = A_{21}A_{11}^{-1}$;

$$(T_{10b}) \quad EAF = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21}A_{11}^{-1}A_{12} \end{bmatrix} = N_7.$$

(T_{11}) Transformation of similarity for square matrices:

$$RAR^{-1} = RAFF^*R^{-1} = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} F^*R^{-1} = \begin{bmatrix} T_1 & T_2 \\ \mathbb{O} & \mathbb{O} \end{bmatrix}.$$

For the sake of completeness and comparison with our results, we describe known block representations of the Moore-Penrose inverse.

In the beginning, we restate the block representations of the Moore-Penrose inverse from [11], [13]. For $A \in \mathbb{C}_r^{m \times n}$, let

$$R = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad G = [G_1, \quad G_2], \quad U = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix}, \quad V = [V_1, \quad V_2],$$

where R_1, U_1 are the first r rows of R and U , respectively, and G_1, V_1 denote the first r columns of G and V , respectively. Then the Moore-Penrose inverse can be represented in the following way, where the block representations (M_i) correspond to the block decompositions (T_i), $i \in \{1, \dots, 9\}$.

$$(M_1) \quad A^\dagger = G \begin{bmatrix} I_r & -R_1R_2^\dagger \\ -G_2^\dagger G_1 & G_2^\dagger G_1 R_1 R_2^\dagger \end{bmatrix} R = G \begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix} [I_r, \quad -R_1 R_2^\dagger] R \\ = (G_1 - G_2 G_2^\dagger G_1)(R_1 - R_1 R_2^\dagger R_2),$$

$$(M_2) \quad A^\dagger = G \begin{bmatrix} B^{-1} & -B^{-1}R_1R_2^\dagger \\ -G_2^\dagger G_1 B^{-1} & G_2^\dagger G_1 B^{-1}R_1R_2^\dagger \end{bmatrix} R \\ = G \begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix} B^{-1} [I_r, \quad -R_1 R_2^\dagger] R \\ = (G_1 - G_2 G_2^\dagger G_1) B^{-1} (R_1 - R_1 R_2^\dagger R_2),$$

$$(M_3) \quad A^\dagger = F \begin{bmatrix} (I_r + KK^*)^{-1} & -(I_r + KK^*)^{-1}R_1R_2^\dagger \\ K^*(I_r + KK^*)^{-1} & -K^*(I_r + KK^*)^{-1}R_1R_2^\dagger \end{bmatrix} R \\ = F \begin{bmatrix} I_r \\ K^* \end{bmatrix} (I_r + KK^*)^{-1} [I_r, \quad -R_1 R_2^\dagger] R,$$

$$(M_4) \quad A^\dagger = G \begin{bmatrix} (I_r + K^*K)^{-1} & (I_r + K^*K)^{-1}K^* \\ -G_2^\dagger G_1 (I_r + K^*K)^{-1} & -G_2^\dagger G_1 (I_r + K^*K)^{-1}K^* \end{bmatrix} R \\ = G \begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix} (I_r + K^*K)^{-1} [I_r, \quad K^*] E,$$

$$\begin{aligned}
(M_5) \quad A^\dagger &= G \begin{bmatrix} I_r & \mathbb{O} \\ -G_2^\dagger G_1 & \mathbb{O} \end{bmatrix} U = G \begin{bmatrix} I_r \\ -G_2^\dagger G_1 \end{bmatrix} [I_r, \mathbb{O}] U \\
&= (G_1 - G_2 G_2^\dagger G_1) U_1, \\
(M_6) \quad A^\dagger &= V \begin{bmatrix} I_r & -R_1 R_2^\dagger \\ \mathbb{O} & \mathbb{O} \end{bmatrix} R = V \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} [I_r, -R_1 R_2^\dagger] R \\
&= V_1 (R_1 - R_1 R_2^\dagger R_2), \\
(M_7) \quad A^\dagger &= V \begin{bmatrix} B^{-1} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} U = V \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} [B^{-1}, \mathbb{O}] U = V_1 B^{-1} U_1, \\
(M_8) \quad A^\dagger &= F \begin{bmatrix} B^*(BB^* + KK^*)^{-1} & \mathbb{O} \\ K^*(BB^* + KK^*)^{-1} & \mathbb{O} \end{bmatrix} U \\
&= F \begin{bmatrix} B^* \\ K^* \end{bmatrix} (BB^* + KK^*)^{-1} [I_r, \mathbb{O}] U, \\
(M_9) \quad A^\dagger &= V \begin{bmatrix} (B^*B + K^*K)^{-1} B^* & (B^*B + K^*K)^{-1} K^* \\ \mathbb{O} & \mathbb{O} \end{bmatrix} E \\
&= V \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} (B^*B + K^*K)^{-1} [B^*, K^*] E.
\end{aligned}$$

These results are obtained by solving the equations (1)–(4).

Block decomposition (T_{10a}) is investigated in [5], [13], but in two different ways. In [13], the Moore-Penrose inverse is represented by solving the corresponding set of matrix equations. The results in [5] are obtained using a full-rank factorization, implied by the block decomposition (T_{10a}) . The corresponding representation of the Moore-Penrose inverse is:

$$\begin{aligned}
(M_{10a}) \quad A^\dagger &= \begin{bmatrix} I_r \\ T^* \end{bmatrix} \left(A_{11}^* [I_r, S^*] E A F \begin{bmatrix} I_r \\ T^* \end{bmatrix} \right)^{-1} A_{11}^* [I_r, S^*] \\
&= F \begin{bmatrix} I_r \\ T^* \end{bmatrix} (I_r + T T^*)^{-1} A_{11}^{-1} (I_r + S^* S)^{-1} [I_r, S^*].
\end{aligned}$$

The following block representation of the Moore-Penrose inverse was obtained in [10] and [13] for the block decomposition (T_{10b}) :

$$\begin{aligned}
(M_{10b}) \quad A^\dagger &= \begin{bmatrix} I_r \\ (A_{11}^{-1} A_{12})^* \end{bmatrix} (I_r + A_{11}^{-1} A_{12} (A_{11}^{-1} A_{12})^*)^{-1} \\
&\quad \times A_{11}^{-1} (I_r + (A_{21} A_{11}^{-1})^* A_{21} A_{11}^{-1})^{-1} [I_r, (A_{21} A_{11}^{-1})^*].
\end{aligned}$$

Also, in [14] is given the following block representation of the Moore-Penrose inverse, based on the formula $A^\dagger = A^* T A^*$, $T \in A^* A A^* \{1\}$: If $A \in \mathbb{C}_r^{m \times n}$ has the form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, $A_{11} \in \mathbb{C}_r^{r \times r}$, then

$$(M'_{10b}) \quad A^\dagger = [A_{11}, \quad A_{12}]^* K_{11}^* \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}^*,$$

$$\text{where } K_{11} = \left([A_{11}, \quad A_{12}] A^* \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \right)^{-1}.$$

Block decomposition (T_{11}) is investigated in [9], but only for square matrices and the group inverse.

The main idea of this paper is to continue the papers [5] and [10]. In other words, we use the following algorithm: from the presented block decompositions of matrices find the corresponding full-rank factorizations $A = PQ$, and then apply described general representations for A^\dagger and $A^\dagger_{\varphi(M,N)}$. Main advantages of the introduced block representations are their simply derivation, computation and possibility of natural generalization.

2. The Moore-Penrose inverse.

In the following theorem we derive a few representations of the Moore-Penrose inverse, using described block decompositions and full-rank factorizations.

THEOREM 2.1 *The Moore-Penrose inverse of a given matrix $A \in \mathbb{C}_r^{m \times n}$ can be represented as follows, where each block representation (G_i) is derived from the block decomposition (T_i) , $i \in \{1, \dots, 9, 10a, 10b, 11\}$:*

$$(G_1) \quad A^\dagger = (G^{-1}|_r)^* \left((R^{-1|_r})^* A (G^{-1}|_r)^* \right)^{-1} (R^{-1|_r})^*$$

$$= (G^{-1}|_r)^* \left((RR^*)^{-1|_r} (G^*G)^{-1|_r} \right)^{-1} (R^{-1|_r})^*,$$

$$(G_2) \quad A^\dagger = (G^{-1}|_r)^* \left((R^{-1|_r} B)^* A (G^{-1}|_r)^* \right)^{-1} (R^{-1|_r} B)^*$$

$$= (G^{-1}|_r)^* \left(B^* (RR^*)^{-1|_r} B (G^*G)^{-1|_r} \right)^{-1} B^* (R^{-1|_r})^*,$$

$$(G_3) \quad A^\dagger = F \begin{bmatrix} I_r \\ K^* \end{bmatrix} \left((R^{-1|_r})^* A F \begin{bmatrix} I_r \\ K^* \end{bmatrix} \right)^{-1} (R^{-1|_r})^*$$

$$= \left(F|_r + F^{n-r}|_r K^* \right) \left((RR^*)^{-1|_r} (I_r + K K^*) \right)^{-1} (R^{-1|_r})^*,$$

$$(G_4) \quad A^\dagger = (G^{-1}|_r)^* \left([I_r, \quad K^*] E A (G^{-1}|_r)^* \right)^{-1} [I_r, \quad K^*] E$$

$$= (G^{-1}|_r)^* \left((I_r + K^* K) (G^*G)^{-1|_r} \right)^{-1} (E|_r + K^* E_{n-r}),$$

$$(G_5) \quad A^\dagger = (G^{-1}|_r)^* \left(U|_r A (G^{-1}|_r)^* \right)^{-1} U|_r = (G^{-1}|_r)^* \left((G^*G)^{-1|_r} \right)^{-1} U|_r,$$

$$(G_6) \quad A^\dagger = V|_r \left((R^{-1|_r})^* AV|_r \right)^{-1} (R^{-1|_r})^* = V|_r \left((RR^*)^{-1|_r} \right)^{-1} (R^{-1|_r})^*,$$

$$(G_7) \quad A^\dagger = V|_r \left(B^* U|_r AV|_r \right)^{-1} B^* U|_r = V|_r B^{-1} U|_r,$$

$$(G_8) \quad A^\dagger = F \begin{bmatrix} B^* \\ K^* \end{bmatrix} \left(U|_r AF \begin{bmatrix} B^* \\ K^* \end{bmatrix} \right)^{-1} U|_r = F \begin{bmatrix} B^* \\ K^* \end{bmatrix} (BB^* + KK^*)^{-1} U|_r,$$

$$(G_9) \quad A^\dagger = V|_r \left([B^*, K^*] EAV|_r \right)^{-1} [B^*, K^*] E \\ = V|_r (B^* B + K^* K)^{-1} [B^*, K^*] E,$$

$$(G_{10a}) \quad A^\dagger = F \begin{bmatrix} I_r \\ T^* \end{bmatrix} \left(A_{11}^* [I_r, S^*] EAF \begin{bmatrix} I_r \\ T^* \end{bmatrix} \right)^{-1} A_{11}^* [I_r, S^*] E \\ = \left(F|_r + F^{n-r}|_r T^* \right) \left((I_r + S^* S) A_{11} (I_r + TT^*) \right)^{-1} (E|_r + S^* E_{n-r}),$$

$$(G_{10b}) \quad A^\dagger = F \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} \left((A_{11}^*)^{-1} [A_{11}^*, A_{21}^*] EAF \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} \right)^{-1} (A_{11}^*)^{-1} [A_{11}^*, A_{21}^*] E \\ = F \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} (A_{11} A_{11}^* + A_{12} A_{12}^*)^{-1} A_{11} (A_{11}^* A_{11} + A_{21}^* A_{21})^{-1} [A_{11}^*, A_{21}^*] E,$$

$$(G_{11}) \quad A^\dagger = R^* \begin{bmatrix} I_r \\ (T_1^{-1} T_2)^* \end{bmatrix} \left((R^{-1|_r} T_1)^* AR^* \begin{bmatrix} I_r \\ (T_1^{-1} T_2)^* \end{bmatrix} \right)^{-1} (R^{-1|_r} T_1)^* \\ = R^* \begin{bmatrix} I_r \\ (T_1^{-1} T_2)^* \end{bmatrix} \left(T_1^* (RR^*)^{-1|_r} [T_1, T_2] RR^* \begin{bmatrix} I_r \\ (T_1^{-1} T_2)^* \end{bmatrix} \right)^{-1} (R^{-1|_r} T_1)^*.$$

Proof. (G₁) Starting from (T₁), we obtain

$$A = R^{-1} \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} G^{-1} = R^{-1} \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} [I_r, \mathbb{O}] G^{-1},$$

which implies

$$P = R^{-1} \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} = R^{-1|_r}, \quad Q = [I_r, \mathbb{O}] G^{-1} = G^{-1|_r}.$$

Now, we get

$$A^\dagger = Q^* (P^* A Q^*)^{-1} P^* = (G^{-1|_r})^* \left((R^{-1|_r})^* A (G^{-1|_r})^* \right)^{-1} (R^{-1|_r})^* \\ = (G^{-1|_r})^* \left((R^*)^{-1|_r} R^{-1|_r} G^{-1|_r} (G^*)^{-1|_r} \right)^{-1} (R^{-1|_r})^* \\ = (G^{-1|_r})^* \left((RR^*)^{-1|_r} (G^* G)^{-1|_r} \right)^{-1} (R^{-1|_r})^*.$$

The other block representations of the Moore-Penrose inverse can be developed in a similar way.

(G₅) The block decomposition (T₅) implies

$$P = U^* \begin{bmatrix} I_r \\ \mathbb{O} \end{bmatrix} = U^{*|r}, \quad Q = [I_r, \quad \mathbb{O}] G^{-1} = G^{-1}|_r,$$

which means

$$\begin{aligned} A^\dagger &= (G^{-1}|_r)^* \left(U|_r A (G^{-1}|_r)^* \right)^{-1} U|_r \\ &= (G^{-1}|_r)^* \left(U|_r U^{*|r} G^{-1}|_r (G^*)^{-1|r} \right)^{-1} U|_r = (G^{-1}|_r)^* \left((G^* G)^{-1}|_r \right)^{-1} U|_r. \end{aligned}$$

(G₇) It is easy to see that (T₇) implies

$$P = U^* \begin{bmatrix} B \\ \mathbb{O} \end{bmatrix} = U^{*|r} B, \quad Q = [I_r, \quad \mathbb{O}] V^* = V^*|_r.$$

Now,

$$A^\dagger = V|_r \left(B^* U|_r U^{*|r} B V^*|_r V|_r \right)^{-1} B^* U|_r = V|_r (B^* B)^{-1} B^* U|_r = V|_r B^{-1} U|_r.$$

(G_{10a}) From (T_{10a}) we obtain

$$A = E^* \begin{bmatrix} A_{11} & A_{11} T \\ S A_{11} & S A_{11} T \end{bmatrix} F^* = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11} [I_r, \quad T] F^*,$$

which implies, for example, the following full rank factorization of A :

$$P = E^* \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11}, \quad Q = [I_r, \quad T] F^*.$$

Consequently, the Moore-Penrose inverse of A is

$$A^\dagger = F \begin{bmatrix} I_r \\ T^* \end{bmatrix} \left(A_{11}^* [I_r, \quad S^*] E A F \begin{bmatrix} I_r \\ T^* \end{bmatrix} \right)^{-1} A_{11}^* [I_r, \quad S^*] E.$$

This part of the proof can be completed using

$$E A F = \begin{bmatrix} I_r \\ S \end{bmatrix} A_{11} [I_r, \quad T]$$

and

$$F \begin{bmatrix} I_r \\ T^* \end{bmatrix} = F|_r + F^{n-r}|T^*, \quad [I_r, \quad S^*] E = E|_r + S^* E_{n-r}.$$

(G_{10b}) Follows from

$$P = E^* \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} A_{11}^{-1}, \quad Q = [A_{11}, \quad A_{12}] F^*. \quad \square$$

Remark 2.1. (i) A convenient method for finding the matrices S , T and A_{11}^{-1} , required in (T_{10a}), was introduced in [5], and it was based on the following extended Gauss-Jordan transformation:

$$\begin{bmatrix} A_{11} & A_{12} & I \\ A_{21} & A_{22} & \mathbb{O} \end{bmatrix} \longrightarrow \begin{bmatrix} I & T & A_{11}^{-1} \\ \mathbb{O} & \mathbb{O} & -S \end{bmatrix}.$$

(ii) In [10] it was used the following full-rank factorization of the matrix A , represented in the form $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{21} A_{11}^{-1} A_{12} \end{bmatrix}$:

$$P = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix}, \quad Q = [I_r, \quad A_{11}^{-1} A_{12}].$$

3. The weighted Moore-Penrose inverse.

Using the general representation of the weighted Moore-Penrose inverse presented in Proposition 1.3: $A_{M\circ, \circ N}^\dagger = NQ^*(P^*MANQ^*)^{-1}P^*M$, and the algorithm of Theorem 2.1, we obtain the following block representation of the weighted Moore-Penrose inverse $A_{M\circ, \circ N}^\dagger$.

THEOREM 3.1 *The weighted Moore-Penrose inverse $A_{M\circ, \circ N}^\dagger$ of $A \in \mathbb{C}_r^{m \times n}$ possesses the following block representations (Z_i), which correspond to the block decompositions (T_i), $i \in \{1, \dots, 9, 10a, 10b, 11\}$:*

$$(Z_1) \quad N(G^{-1}|_r)^* \left((R^{-1|_r})^* MAN (G^{-1}|_r)^* \right)^{-1} (R^{-1|_r})^* M,$$

$$(Z_2) \quad N(G^{-1}|_r)^* \left((R^{-1|_r} B)^* MAN (G^{-1}|_r)^* \right)^{-1} (R^{-1|_r} B)^* M,$$

$$(Z_3) \quad NF \begin{bmatrix} I_r \\ K^* \end{bmatrix} \left((R^{-1|_r})^* MANF \begin{bmatrix} I_r \\ K^* \end{bmatrix} \right)^{-1} (R^{-1|_r})^* M,$$

$$(Z_4) \quad N(G^{-1}|_r)^* \left([I_r, \quad K^*] EMAN (G^{-1}|_r)^* \right)^{-1} [I_r, \quad K^*] EM,$$

$$(Z_5) \quad N(G^{-1}|_r)^* \left(U|_r MAN (G^{-1}|_r)^* \right)^{-1} U|_r M,$$

$$(Z_6) \quad NV|r \left((R^{-1|r})^* MANV|r \right)^{-1} (R^{-1|r})^* M,$$

$$(Z_7) \quad NV|r (B^*U|rMANV|r)^{-1} B^*U|rM,$$

$$(Z_8) \quad NF \begin{bmatrix} B^* \\ K^* \end{bmatrix} \left(U|rMANF \begin{bmatrix} B^* \\ K^* \end{bmatrix} \right)^{-1} U|rM,$$

$$(Z_9) \quad NV|r ([B^*, K^*]EMANV|r)^{-1} [B^*, K^*]EM,$$

$$(Z_{10a}) \quad NF \begin{bmatrix} I_r \\ T^* \end{bmatrix} \left(A_{11}^*[I_r, S^*]EMANF \begin{bmatrix} I_r \\ T^* \end{bmatrix} \right)^{-1} A_{11}^*[I_r, S^*]EM,$$

$$(Z_{10b}) \quad NF \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} \left((A_{11}^*)^{-1}[A_{11}^*, A_{21}^*]EMANF \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} \right)^{-1} (A_{11}^*)^{-1}[A_{11}^*, A_{21}^*]EM,$$

$$(Z_{11}) \quad NR^* \left[\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix} \right] \left((R^{-1|r}T_1)^* MANR^* \left[\begin{bmatrix} I_r \\ (T_1^{-1}T_2)^* \end{bmatrix} \right] \right)^{-1} (R^{-1|r}T_1)^* M.$$

The following representations can be obtained from the main properties of the weighted Moore-Penrose inverse and Theorem 3.1.

COROLLARY 3.1. *The weighted Moore-Penrose inverse $A_{\varphi(M,N)}^\dagger$ of $A \in \mathbb{C}_r^{m \times n}$ can be represented as follows:*

$$(W_1) \quad N^{[-1]} (G^{-1}|r)^* \left((R^{-1|r})^* M^{[-1]} AN^{[-1]} (G^{-1}|r)^* \right)^{-1} (R^{-1|r})^* M^{[-1]},$$

$$(W_2) \quad N^{[-1]} (G^{-1}|r)^* \left((R^{-1|r}B)^* M^{[-1]} AN^{[-1]} (G^{-1}|r)^* \right)^{-1} (R^{-1|r}B)^* M^{[-1]},$$

$$(W_3) \quad N^{[-1]} F \begin{bmatrix} I_r \\ K^* \end{bmatrix} \left((R^{-1|r})^* M^{[-1]} AN^{[-1]} F \begin{bmatrix} I_r \\ K^* \end{bmatrix} \right)^{-1} (R^{-1|r})^* M^{[-1]},$$

$$(W_4) \quad N^{[-1]} (G^{-1}|r)^* \left([I_r, K^*]EM^{[-1]} AN^{[-1]} (G^{-1}|r)^* \right)^{-1} [I_r, K^*]EM^{[-1]},$$

$$(W_5) \quad N^{[-1]} (G^{-1}|r)^* \left(U|rM^{[-1]} AN^{[-1]} (G^{-1}|r)^* \right)^{-1} U|rM^{[-1]},$$

$$(W_6) \quad N^{[-1]} V|r \left((R^{-1|r})^* M^{[-1]} AN^{[-1]} V|r \right)^{-1} (R^{-1|r})^* M^{[-1]},$$

$$(W_7) \quad N^{[-1]} V|r (B^*U|rM^{[-1]} AN^{[-1]} V|r)^{-1} B^*U|rM^{[-1]},$$

$$\begin{aligned}
(W_8) \quad & N^{[-1]}F \begin{bmatrix} B^* \\ K^* \end{bmatrix} \left(U_{|r} M^{[-1]} A N^{[-1]} F \begin{bmatrix} B^* \\ K^* \end{bmatrix} \right)^{-1} U_{|r} M^{[-1]}, \\
(W_9) \quad & N^{[-1]}V^{|r} \left([B^*, K^*] E M^{[-1]} A N^{[-1]} V^{|r} \right)^{-1} [B^*, K^*] E M^{[-1]}, \\
(W_{10a}) \quad & N^{[-1]}F \begin{bmatrix} I_r \\ T^* \end{bmatrix} \left(A_{11}^* [I_r, S^*] E M^{[-1]} A N^{[-1]} F \begin{bmatrix} I_r \\ T^* \end{bmatrix} \right)^{-1} A_{11}^* [I_r, S^*] E M^{[-1]}, \\
(W_{10b}) \quad & N^{[-1]}F \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} \left((A_{11}^*)^{-1} [A_{11}^*, A_{21}^*] E M^{[-1]} A N^{[-1]} F \begin{bmatrix} A_{11}^* \\ A_{12}^* \end{bmatrix} \right)^{-1} \\
& \quad \times (A_{11}^*)^{-1} [A_{11}^*, A_{21}^*] E M^{[-1]}, \\
(W_{11}) \quad & N^{[-1]}R^* \begin{bmatrix} I_r \\ (T_1^{-1} T_2)^* \end{bmatrix} \left((R^{-1|r} T_1)^* M^{[-1]} A N^{[-1]} R^* \begin{bmatrix} I_r \\ (T_1^{-1} T_2)^* \end{bmatrix} \right)^{-1} (R^{-1|r} T_1)^* M^{[-1]}.
\end{aligned}$$

4. Examples.

Example 4.1. Consider $A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & -1 & -3 \\ 1 & -1 & 0 & 1 \\ 1 & 0 & -1 & -2 \end{bmatrix}$. Using the Gauss-Jordan trans-

formation, we get the following reduced row-echelon form of the matrix A :

$$RAF = R_A = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

The matrix R_A is obtained using the permutation matrix $F = I_4$, and the following regular matrix:

$$R = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Using the method (G_3) we obtain

$$R^{-1|r} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad F \begin{bmatrix} I_r \\ K^* \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -1 \\ -2 & -3 \end{bmatrix},$$

and the following Moore-Penrose inverse of A :

$$A^\dagger = \begin{bmatrix} -\frac{5}{34} & -\frac{3}{17} & \frac{1}{34} & -\frac{1}{34} & \frac{3}{17} & \frac{5}{34} \\ \frac{4}{51} & \frac{13}{102} & -\frac{5}{102} & \frac{5}{102} & -\frac{13}{102} & -\frac{4}{51} \\ \frac{7}{102} & \frac{5}{102} & \frac{1}{51} & -\frac{1}{51} & -\frac{5}{102} & -\frac{7}{102} \\ \frac{1}{17} & -\frac{1}{34} & \frac{3}{34} & -\frac{3}{34} & \frac{1}{34} & -\frac{1}{17} \end{bmatrix}.$$

Example 4.2. For the matrix A used in Example 3.1 we obtain (see [5])

$$A_{11}^{-1} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix}.$$

Using (G_{10a}) we obtain the same Moore-Penrose inverse of A .

Example 4.3. For the matrix $A = \begin{bmatrix} 4 & -1 & 1 & 2 \\ -2 & 2 & 0 & -1 \\ 6 & -3 & 1 & 3 \\ -10 & 4 & -2 & -5 \end{bmatrix}$ we obtain

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ 2 & -1 & 0 & 1 \end{bmatrix}, \quad T_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad T_2 = \begin{bmatrix} -1 & -2 \\ -1 & -3 \end{bmatrix}, \quad F = I_4.$$

Then, one can verify the following:

$$\left(R^{-1|_r} T_1 \right)^* = \begin{bmatrix} -1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix}, \quad R^* \left[\begin{array}{c} I_r \\ (T_1^{-1} T_2)^* \end{array} \right] = \begin{bmatrix} -4 & -6 \\ 1 & 3 \\ -1 & -1 \\ -2 & -3 \end{bmatrix}.$$

Finally, using (G_{11}) , we get

$$A^\dagger = \begin{bmatrix} \frac{8}{81} & \frac{10}{81} & -\frac{2}{81} & -\frac{2}{27} \\ \frac{47}{162} & \frac{79}{162} & -\frac{16}{81} & -\frac{5}{54} \\ \frac{7}{54} & \frac{11}{54} & -\frac{2}{27} & -\frac{1}{18} \\ \frac{4}{81} & \frac{5}{81} & -\frac{1}{81} & -\frac{1}{27} \end{bmatrix}.$$

Example 4.4. Consider $A = \begin{bmatrix} 1 & -5 & 1 & 4 \\ -2 & 7 & 0 & 1 \\ 0 & -3 & 2 & 9 \end{bmatrix}$ and positive definite matrices

$$M = \begin{bmatrix} 5 & -1 & 3 \\ -1 & 2 & -2 \\ 3 & -2 & 3 \end{bmatrix}, \quad N = \begin{bmatrix} 4 & 0 & -1 & 0 \\ 0 & 3 & 2 & 1 \\ -1 & 2 & 5 & -2 \\ 0 & 1 & -2 & 6 \end{bmatrix}.$$

Block decomposition (T_1) can be obtained by applying transformation (T_3) two times:

$$R_1 A F_1 = \begin{bmatrix} I_r & K \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_3, \quad R_2 N_3^T F_2 = \begin{bmatrix} I_r & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{bmatrix} = N_1.$$

Then, the regular matrices R, G can be computed as follows:

$$N_1 = N_1^T = F_2^T N_3 R_2^T = F_2^T R_1 A F_1 R_2^T \Rightarrow R = F_2^T R_1, \quad G = F_1 R_2^T.$$

For given matrix the following can be obtained:

$$N_3 = \begin{bmatrix} 1 & 0 & -\frac{7}{3} & -11 \\ 0 & 1 & -\frac{2}{3} & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad R_1 = \begin{bmatrix} -\frac{7}{3} & -\frac{5}{3} & 0 \\ -\frac{2}{3} & -\frac{1}{3} & 0 \\ -2 & -1 & 1 \end{bmatrix}, \quad F_1 = I_4,$$

$$N_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \frac{7}{3} & \frac{2}{3} & 1 & 0 \\ 11 & 3 & 0 & 1 \end{bmatrix}, \quad F_2 = I_3.$$

From $R = R_1, G = R_2^T$, we get

$$R^{-1|2} = \begin{bmatrix} 1 & -5 \\ -2 & 7 \\ 0 & -3 \end{bmatrix}, \quad G^{-1|2} = \begin{bmatrix} 1 & 0 & -\frac{7}{3} & -11 \\ 0 & 1 & -\frac{2}{3} & -3 \end{bmatrix}.$$

Using formula (Z_1) , we obtain the following representation for the weighted Moore-Penrose inverse of A :

$$A_{M_{\circ}, \circ N}^{\dagger} = \begin{bmatrix} -\frac{8841}{207506} & -\frac{13865}{207506} & -\frac{13865}{207506} \\ \frac{23355}{207506} & \frac{38035}{207506} & -\frac{14947}{207506} \\ \frac{2149}{103753} & \frac{11585}{103753} & -\frac{7035}{103753} \\ \frac{42301}{207506} & \frac{25265}{207506} & \frac{3465}{207506} \end{bmatrix}.$$

Example 4.5. Similarly, block decomposition (T_1) of the matrix A , considered in Example 4.1 can be obtained by transformation (T_3) two times, by means of the following matrices:

$$R_1 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad F_1 = I_4,$$

$$R_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{bmatrix}, \quad F_2 = I_6.$$

From $R = R_1$, $G = R_2^T$, we get

$$R^{-1|2} = \begin{bmatrix} -1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}, \quad G^{-1|2} = \begin{bmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -3 \end{bmatrix}.$$

Using formula (G_1) , we obtain the Moore-Penrose inverse A^\dagger , as in Example 4.1.

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