PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 65 (79), 1999, 97–102

ON NONSINGULAR MATRICES AND BOTT PERIODICITY

Zoran Z. Petrović

Communicated by Rade Živaljević

Abstract. We give a K-theoretic proof of the estimate for the maximum dimension of real vector spaces of nonsingular matrices of order n over the field of real, complex numbers and quaternions.

1. Introduction

In [ALP] the authors have discussed the following problem: For a given n, what is the maximum dimension over $\mathbf{R}$ of a vector space of matrices of order n in which every non-zero matrix is nonsingular. Matrices may have their components in $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$, where by $\mathbf{H}$ we denote the skew field of quaternions. In that paper and [ALP1], the authors, using the results from [A1], gave the following answer

THEOREM 1. *If, by $\mathbf{R}(n)$, $\mathbf{C}(n)$, $\mathbf{H}(n)$ we denote the maximum dimension mentioned above, for matrices over* $\mathbf{R}$, $\mathbf{C}$, $\mathbf{H}$ *respectively, then*

$$\mathbf{R}(n) = \rho(n), \quad \mathbf{C}(n) = 2\nu_2(n) + 2, \quad \mathbf{H}(n) = \rho(n/2) + 4.$$

This theorem requires a short remark. First, $\rho(n)$ denotes the Hurwitz–Radon number defined by:

$$\rho(n) = \rho(2^{4a+b}(2m+1)) = 8a + 2^b,$$

where $n = 2^{4a+b}(2m+1)$, $a \geq 0$, $0 \leq b \leq 3$ are integers, while $\nu_2(n)$ is the highest power of 2 dividing n. Also, if n is odd then $\rho(n/2) = 0$. It is mentioned in [ALP1] that it remains interesting to what topological phenomena this algebraic result is related.

AMS Subject Classification (1991): Primary 19M05

Supported by Ministry of Science and Technology of Serbia, grant number 04M03/C

It is the purpose of this paper to give a K-theoretic proof of the following

THEOREM 2. *We have*

$$\mathbf{R}(n) \leq \rho(n), \quad \mathbf{C}(n) \leq \nu_2(n) + 2, \quad \mathbf{H}(n) \leq \rho(n/2) + 4.$$

We use real, complex and symplectic K-theory respectively, and, since symplectic and real K-theory are directly related via the Bott periodicity, this shows how the Bott periodicity is connected to this result. Let us emphasize also, that we use only the additive structure of K-groups (no operations are involved) and therefore the proof is more simple than the proof related to $\rho(n)$ in [A1]. There, the author has proved that the largest possible number of linearly independent vector fields on S^{n-1} is $\rho(n) - 1$ (problem about matrices corresponds to linear vector fields).

In the next section, we give the necessary results concerning K-theory, and then, in the last section, prove Theorem 2.

2. K-theory

The results we need from K-theory concern the K-groups of real projective spaces. Let us first give some notation.

$$\phi(n) := \#\{k \in \mathbf{N} \mid 1 \leq k \leq n, \ \ k \equiv 1, 2, 4, 8 \pmod 8\},$$
$$\psi(n) := \#\{k \in \mathbf{N} \mid 1 \leq k \leq n, \ \ k \equiv 4, 5, 6, 8 \pmod 8\}.$$

Here, $\#A$, for a finite set A, stands for the number of elements in that set.

$$c : \widetilde{\mathrm{KO}}(X) \to \widetilde{\mathrm{KU}}(X),$$
$$q : \widetilde{\mathrm{KU}}(X) \to \widetilde{\mathrm{KSp}}(X),$$
$$c' : \widetilde{\mathrm{KSp}}(X) \to \widetilde{\mathrm{KU}}(X),$$
$$t : \widetilde{\mathrm{KU}}(X) \to \widetilde{\mathrm{KU}}(X),$$

are group homomorphisms induced by complexification, quaternionization, forgetting the quaternion structure, complex conjugation, respectively (see [A2, pp. 27–28]). We have

$$qc' = 2, \quad c'q = 1 + t, \quad tc = c$$

(see [A2]). If ξ_n stands for the canonical (Hopf) line bundle over $\mathbf{R}P^n$, we use the following notation

$$\lambda_n = \xi_n - 1 \in \widetilde{\mathrm{KO}}(\mathbf{R}P^n),$$
$$\mu_n = c\lambda_n \in \widetilde{\mathrm{KU}}(\mathbf{R}P^n),$$
$$\nu_n = q\mu_n \in \widetilde{\mathrm{KSp}}(\mathbf{R}P^n).$$

THEOREM 3. *We have*

a) $\widetilde{\mathrm{KO}}(\mathbf{R}P^n) \cong \mathbf{Z}/2^{\phi(n)}$*, generated by* λ_n*;*

b) $\widetilde{\mathrm{KU}}(\mathbf{R}P^n) \cong \mathbf{Z}/2^{[n/2]}$*, generated by* μ_n*;*

c) $\widetilde{\mathrm{KSp}}(\mathbf{R}P^n) \cong \mathbf{Z}/2^{\psi(n)}$*, generated by* ν_n*.*

Proof. Of course, *a)* and *b)* are very well known (see, e.g. [A1]). For *c)*, one may observe that the isomorphism in question follows from Bott periodicity— $\widetilde{\mathrm{KSp}}^*(X) \cong \widetilde{\mathrm{KO}}^{*-4}(X)$ (see, e.g. [Sw, pp. 214–217]) and the results from [Fu], where the groups $\widetilde{\mathrm{KO}}^*(\mathbf{R}P^n)$ have been calculated. But, rather than trying to identify the generator given in [Fu] with our generator, we give a direct proof of this result.

The second term in the Atiyah–Hirzebruch spectral sequence for symplectic K-theory of $\mathbf{R}P^n$ is

$$E_2^{p,-q} = \tilde{H}^p(\mathbf{R}P^n; \widetilde{\mathrm{KSp}}^{-q})$$

It converges to $\widetilde{\mathrm{KSp}}^*(\mathbf{R}P^n)$. If we look at the terms

$$E_2^{q,-q} = \tilde{H}^q(\mathbf{R}P^n; \widetilde{\mathrm{KSp}}^{-q})$$

which correspond to $\widetilde{\mathrm{KSp}}^0(\mathbf{R}P^n) = \widetilde{\mathrm{KSp}}(\mathbf{R}P^n)$, we find, using the fact that the coefficients for $\widetilde{\mathrm{KSp}}^*$ are given by

$n \pmod 8$	1	2	3	4	5	6	7	8
$\widetilde{\mathrm{KSp}}^{-n}$	0	0	0	$\mathbf{Z}$	$\mathbf{Z}/2$	$\mathbf{Z}/2$	0	$\mathbf{Z}$

(see [Sw, p. 216]), that there are exactly $\psi(n)$ non-zero terms and that they are all $\mathbf{Z}/2$. This means that the order of the group $\widetilde{\mathrm{KSp}}(\mathbf{R}P^n)$ is at most $2^{\psi(n)}$. On the other hand,

$$c'(\nu_n) = c'qc\lambda_n = (1+t)c\lambda_n = (c+tc)\lambda_n = (c+c)\lambda_n = 2c\lambda_n = 2\mu_n.$$

Now, let us suppose first that $n \equiv 2, 3$ or $4 \pmod 8$. Then we have the following table

n	$8k+2$	$8k+3$	$8k+4$
$\psi(n)$	$4k$	$4k$	$4k+1$
$[n/2]$	$4k+1$	$4k+1$	$4k+2$

So, in these cases $\psi(n) = [n/2]-1$. Since, for $c' : \widetilde{\mathrm{KSp}}(\mathbf{R}P^n) \to \widetilde{\mathrm{KU}}(\mathbf{R}P^n)$ we have $c'(\nu_n) = 2\mu_n$, it follows that $\omega(2\mu_n)$ divides $\omega(\nu_n)$, (where for $a \in A$, where A is an abelian group, we denote by $\omega(a)$ the order of that element in that group) and therefore $2^{[n/2]-1} = 2^{\psi(n)}$ divides $\omega(\nu_n)$. Since $\#\widetilde{\mathrm{KSp}}(\mathbf{R}P^n) \leq 2^{\psi(n)}$, we conclude that $\omega(\nu_n) = 2^{\psi(n)}$ and $\widetilde{\mathrm{KSp}}(\mathbf{R}P^n)$ is cyclic, generated by ν_n.

We now show that if the result is true for $\mathbf{R}P^n$, it is also true for $\mathbf{R}P^{n-1}$. Let us look at the Atiyah-Hirzebruch spectral sequences for $\mathbf{R}P^n$ and $\mathbf{R}P^{n-1}$.

One can check easily that the E_2-terms of these spectral sequences of total degree 0 agree except at $E_2^{n,-n}$. Now, since $\mathbf{R}P^{n-1}$ is the $(n-1)$st skeleton of $\mathbf{R}P^n$ and the way Atiyah-Hirzebruch spectral sequence is constructed (using skeletons of the space and for these two space the skeletons are equal, except for the last one), we conclude that, since all differentials relevant to degree zero vanish for $\mathbf{R}P^n$ (if the result is true for $\mathbf{R}P^n$, then every element of total degree zero in E_2 must survive till E_∞), they must also vanish for $\mathbf{R}P^{n-1}$. So, the group $\widetilde{\mathrm{KSp}}(\mathbf{R}P^{n-1})$ has the correct order and we are only left with the group extension problem—is this group really cyclic?

Let us look at the homomorphism i^* induced by the inclusion

$$i : \mathbf{R}P^{n-1} \hookrightarrow \mathbf{R}P^n.$$

We have the following exact sequence

$$\widetilde{\mathrm{KSp}}(\mathbf{R}P^n/\mathbf{R}P^{n-1}) \to \widetilde{\mathrm{KSp}}(\mathbf{R}P^n) \xrightarrow{i^*} \widetilde{\mathrm{KSp}}(\mathbf{R}P^{n-1}).$$

Now, $\mathbf{R}P^n/\mathbf{R}P^{n-1} = S^{n-1}$ and let us make the following table

n	$\widetilde{\mathrm{KSp}}(S^n)$	$\psi(n)$	$\psi(n-1)$
$8k+1$	0	$4k$	$4k$
$8k+2$	0	$4k$	$4k$
$8k+3$	0	$4k$	$4k$
$8k+4$	$\mathbf{Z}$	$4k+1$	$4k$
$8k+5$	$\mathbf{Z}/2$	$4k+2$	$4k+1$
$8k+6$	$\mathbf{Z}/2$	$4k+3$	$4k+2$
$8k+7$	0	$4k+3$	$4k+2$
$8k+8$	$\mathbf{Z}$	$4k+4$	$4k+3$

For n of the form $8k+1$, $8k+2$, $8k+3$ or $8k+7$ we see that our homomorphism is mono and, since $i^*(\nu_n) = i^*(\nu_{n-1})$, we have the result for n of that form. Now, suppose that $n \equiv 4, 5, 6$ or $8 \pmod 8$ and that the group $\widetilde{\mathrm{KSp}}(\mathbf{R}P^{n-1})$ is not cyclic. This means that $\omega(\nu_{n-1}) \le 2^{\psi(n-1)-1}$ and from the isomorphism

$$\widetilde{\mathrm{KSp}}(\mathbf{R}P^n)/\operatorname{Ker}(i^*) \cong \operatorname{Im}(i^*),$$

and the fact that $\operatorname{Im}(i^*)$ is cyclic generated by ν_{n-1}, we get that

$$\#Ker(i^*) = \frac{\#\widetilde{\mathrm{KSp}}(\mathbf{R}P^n)}{\#\operatorname{Im}(i^*)} \ge 2^{\psi(n)-\psi(n-1)+1} \ge 2^2.$$

But, from the spectral sequence for $\mathbf{R}P^n$ we know that $\operatorname{Ker}(i^*)$ is either 0 or $\mathbf{Z}/2$ (everything survives till E_∞ and the filtration in the spectral sequence is given by $F^p\widetilde{\mathrm{KSp}}(X) = \operatorname{Ker}(\widetilde{\mathrm{KSp}}(X) \to \widetilde{\mathrm{KSp}}(X^{p-1}))$; therefore we must have

$F^n\widetilde{\mathrm{KSp}}(\mathbf{R}P^n)/F^{n+1}\widetilde{\mathrm{KSp}}(\mathbf{R}P^n) = \mathrm{Ker}(i^*)/0 = \mathrm{Ker}(i^*))$. Consequently $\omega(\nu_{n-1}) = 2^{\psi(n-1)}$ and this concludes the proof of Theorem 3.

3. Spaces of nonsingular matrices

We now use the results concerning K-groups of projective spaces from the previous section to prove theorem 2. The following construction was introduced for the first time in [S] for the complex case and used in [W], [Mesh], [LY], [P], [R], for the real and complex case. Suppose we have matrices $A_1, \dots, A_r$ of order n over $\mathbf{R}$, $\mathbf{C}$ or $\mathbf{H}$. We define a bundle map

$$n\xi'_{r-1} \xrightarrow{f} \epsilon^n$$

where ξ'_{r-1} stands for the Hopf bundle over $\mathbf{R}P^{r-1}$ in the real case and its complexification (quaternionization) in the complex (quaternion) case (always over $\mathbf{R}P^n$) while ϵ^n stands for the trivial n-dimensional bundle over the same space. This map is defined by

$$f([x]; \lambda_1 x, \dots, \lambda_n x) = ([x]; [\lambda_1, \dots, \lambda_n](A_1 x_1 + \cdots + A_r x_r)),$$

where $[x]$ is the class of $x = (x_1, \dots, x_r) \in S^{r-1}$ in $\mathbf{R}P^{r-1}$ and λ_i, for all i belong to the field in question.

Proof of Theorem 2. For real case, the reader may consult [P] or [R]. We give here the proof for the complex case (which is the easiest of all) and for the quaternion case (which appears to us the most interesting since it gives a link between this result and Bott periodicity via K-theory).

Suppose that $\mathbf{C}(n) \geq 2\nu_2(n) + 3$. The bundle map

$$n\xi'_{2\nu_2(n)+2} \xrightarrow{f} \epsilon^n$$

constructed using as $A_1, \dots, A_{2\nu_2(n)+2}$ matrices of a basis of a space of dimension $\mathbf{C}(n)$, will then be an isomorphism due to the nature of that space (every non-zero matrix in that space is nonsingular). So, in $\widetilde{\mathrm{KU}}(\mathbf{R}P^{2\nu_2(n)+2})$

$$n(\xi'_{2\nu_2(n)+2} - 1) = 0$$

or, since $\xi'_{2\nu_2(n)+2} - 1 = \mu_{2\nu_2(n)+2}$,

$$n\mu = 0. \tag{1}$$

We put μ instead for $\mu_{2\nu_2(n)+2}$ to simplify the expression. On the other hand

$$\widetilde{\mathrm{KU}}(\mathbf{R}P^{2\nu_2(n)+2}) \cong \mathbf{Z}/2^{\nu_2(n)+1}, \tag{2}$$

generated by μ. From (1)

$$2^{\nu_2(n)}(2m+1)\mu = 0$$

and since the group in question is a 2-group, we get

$$2^{\nu_2(n)}\mu = 0$$

and this is false since the order of μ is, by (2) $2^{\nu_2(n)+1}$.

For the quaternion case, let us suppose first that n is odd. We have to prove that $\mathbf{H}(n) \le 4$. Suppose that $\mathbf{H}(n) \ge 5$. As in the previous case, we get

$$n\lambda = 0 \tag{3}$$

in $\widetilde{\mathrm{KSp}}(\mathbf{R}P^4)$. But, $\widetilde{\mathrm{KSp}}(\mathbf{R}P^4) = \mathbf{Z}/2$ amd λ is a generator. Since n is odd, (3) gives that $\lambda = 0$ and that is impossible.

In case n is even, $n = 2l$, we get (assuming that $\mathbf{H}(n) \ge \rho(n/2) + 5$)

$$2l\lambda = 0 \tag{4}$$

in $\widetilde{\mathrm{KSp}}(\mathbf{R}P^{\rho(l)+4}) \cong \mathbf{Z}/2^{\psi(\rho(l)+4)}$. Let us determine the number $\psi(\rho(l)+4)$. If $l = 2^{4a+b}(2m+1)$, where $a \ge 0$, $0 \le b \le 3$, then $\rho(l) = 8a + 2^b$ and we have $\psi(\rho(l)+4) = 4a+b+2$. So the order of λ in $\widetilde{\mathrm{KSp}}(\mathbf{R}P^{\rho(l)+4})$ is 2^{4a+b+2}. But from (4) we get that $2^{4a+b+1}\lambda = 0$ and therefore we must have $\mathbf{H}(n) \le \rho(n/2) + 4$.

References

[A1] J. F. Adams, *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–632.

[A2] J. F. Adams, *Lectures on Lie groups*, Lecture Note in Mathematics, W. A. Benjamin, 1969.

[ALP] J. F. Adams, P. Lax and R. Phillips, *On matrices whose real linear combinations are non-singular*, Proc. Amer. Math. Soc. **16** (1965), 318–322.

[ALP1] J. F. Adams, P. Lax and R. Phillips, *Correction to "On matrices whose real linear combinations are non-singular"*, Proc. Amer. Math. Soc. **17** (1966), 945–947.

[Fu] M. Fujii, *K_O-groups of projective spaces*, Osaka J. Math. **4** (1967), 141–149.

[LY] K. Y. Lam and P. Yiu, *Linear spaces of real matrices of constant rank*, Linear Algebra Appl. **195** (1993), 69–79.

[Mesh] R. Meshulam, *On k-spaces of real matrices*, Linear and Multilinear Algebra **26** (1990), 801–803.

[P] Z. Z. Petrović, *On spaces of matrices satisfying some rank conditions*, Thesis, The Johns Hopkins University, Baltimore, 1996.

[R] E. G. Rees, *Linear spaces of real matrices of given rank*, Contemporary Mathematics **188** (1995), 219–229.

[S] J. Sylvester, *On the dimension of spaces of linear transformations satisfying rank conditions*, Linear Algebra Appl. **78** (1986), 1–10.

[Sw] R. M. Switzer, *Algebraic Topology—Homotopy and Homology*, Springer-Verlag, 1975.

[W] R. Westwick, *Spaces of matrices of fixed rank*, Linear and Multilinear Algebra **20** (1987), 171–174.

Matematički fakultet
Studentski trg 16
11001 Beograd, p.p. 550
Yugoslavia

(Received 08 02 1998)