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ON THE CONVERGENCE OF FINITE-DIFFERENCE SCHEME FOR A NONLOCAL ELLIPTIC BOUNDARY VALUE PROBLEM

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ABSTRACT. The finite-difference scheme approximating nonlocal boundary value problem for a second order elliptic equation is studied. A convergence rate estimate in discrete W_2^1-norm is obtained, assuming that the coefficients and the solution to the original problem belongs to Sobolev spaces.

1. Introduction

The generalization of Bitsadze–Samarski problem [**1**] was investigated by many authors (see e.g., [**2**–**6**]). In [**5**] for the Poisson equation is considered a difference scheme, which converges by the rate $O(h^2)$ in the discrete W_2^2-norm to the exact solution from the class $C^4(\bar{\Omega})$.

In the present paper a nonlocal boundary value problem of Bitsadze–Samarski type is considered in a domain $\Omega = (0,1)^2$ for a second order elliptic equation with variable coefficients. The investigation of the corresponding difference scheme is carried out in Sobolev weight space and under assumption that the coefficients and the solution to the original problem belong to Sobolev spaces, the estimate of convergence rate

$$(1) \qquad \|y-u\|_{W_2^1(\omega,r)} \leq ch^{s-1}\|u\|_{W_2^s(\Omega)}, \quad s \in (1;3]$$

is obtained, where $r = r(x_1) = 1 - x_1$, ω is a uniform grid in Ω with the step h.

The main idea is to introduce an auxilary (equivalent to r) weight function $\rho(x_1)$, which gives possibility to state the positive definity of the difference scheme operator, and validity of the first (energetic) fundamental inequality too. The inner product of the indicated type and induced by it norm were used firstly in [**2**] to

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prove uniqueness of classic, and afterwards also difference solution [4] of nonlocal boundary value problems.

2. The problem and its approximation

Let $\Omega = \{(x_1, x_2) : 0 < x_k < 1,\ k = 1, 2\}$ be a unit square with a boundary Γ; $\alpha_1, \alpha_2, \dots, \alpha_m$ arbitrary real numbers; $\xi_1, \xi_2, \dots, \xi_m$ fixed points from $(0, 1)$; note that $0 < \xi_1 < \xi_2 < \dots < \xi_m < 1$, $\xi_0 = 0$, $\xi_{m+1} = 1$.

$$\Gamma_{(i)} = \{(\xi_i, x_2) : 0 < x_2 < 1\},\ i = 1, ..., m + 1,\ \Gamma_* = \Gamma_{(m+1)},\ \Gamma_0 = \Gamma \backslash \Gamma_*.$$

Consider the nonlocal boundary value problem

$$Lu \equiv \sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big) - au = f(x),\ x \in \Omega, \tag{2}$$

$$u(x) = 0,\ x \in \Gamma_0,\quad u(1, x_2) = \sum_{k=1}^{m} \alpha_k u(\xi_k, x_2),\ 0 < x_2 < 1. \tag{3}$$

We assume that the problem (2), (3) with the right-hand side $f \in W_2^{s-2}(\Omega)$, is uniquely solvable in $W_2^s(\Omega)$, $1 < s \le 3$, and the coefficients $a_{1j} = a_{1j}(x_{3-j})$, $a_{2j} = a_{2j}(x)$ $(j = 1, 2)$, and $a = a(x)$ satisfy to the following conditions:

$$\sum_{i,j=1}^{2} a_{ij} t_i t_j \ge \nu_1 (t_1^2 + t_2^2),\quad \nu_1 = \text{const} > 0,\ a \ge 0. \tag{4}$$

$$a_{1j} \in W_p^{s-1}(0;1),\quad p > \max(1/(s-1), 2) \text{ for } s \in (1;2],\quad p = 2 \text{ for } s \in (2;3],$$
$$a_{2j} \in W_q^{s-1}(\Omega),\quad q > 2/(s-1) \text{ for } s \in (1;2],\quad q = 2 \text{ for } s \in (2;3],$$
$$a \in L_{2+\varepsilon}(\Omega),\ 1 < s \le 2,\ \varepsilon > 0,\ a \in W_2^{s-2}(\Omega),\ 2 < s \le 3.$$

Consider the following grid domains in $\bar{\Omega}$:

$$\bar{\omega}_\alpha = \{x_\alpha = i_\alpha h : i_\alpha = 0, 1, \dots, n,\ h = 1/n\},$$
$$\omega_\alpha = \bar{\omega}_\alpha \cap (0, 1),\ \omega_\alpha^+ = \bar{\omega}_\alpha \cap (0; 1],\ \alpha = 1, 2,$$
$$\omega = \omega_1 \times \omega_2,\ \bar{\omega} = \bar{\omega}_1 \times \bar{\omega}_2,\ \gamma_0 = \Gamma_0 \cap \bar{\omega}.$$

For grid functions and difference ratios, we use the standard notation from [7]:

$$v_{x_i} = (v^{(+1_i)} - v)/h,\ v_{\bar{x}_i} = (v - v^{(-1_i)})/h,$$

where $v^{(\pm 1_1)}(x) = v(x_1 \pm h, x_2)$, $v^{(\pm 1_2)}(x) = v(x_1, x_2 \pm h)$. The notations $v^{(\pm 0.5_i)}$, $v^{(+0.5_1, -1_i)}$ will have similar sense. Let

$$\xi_k = (n_k + \theta_k)h,\ 0 \le \theta_k < 1,\ k = 1, 2, \dots, m,$$
$$\omega_{1,k} = \{x_1 : x_1 = ih,\ i = 1, 2, \dots, n_k\}.$$

where n_k are nonnegative integers $0 \le n_1 \le n_2 \le \dots \le n_m < n$, the equality between which will take place if in the corresponding subinterval (between the neighboring points of the grid ω_1) more than one point ξ_k is situated. Suppose

$$h/2 \le 1 - \xi_m - \nu,\ \nu = \text{const} > 0. \tag{5}$$

Let

$$G_k u = (1-\theta_k)\int_0^{\theta_k} t u(n_k h + th, x_2)\, dt + \theta_k \int_{\theta_k}^1 (1-t) u(n_k h + th, x_2)\, dt,$$

$k = 1, 2, \ldots, n_m$. We also need the following averaging operators:

$$S_1^- u = \int_{-1}^0 u(x_1 + th, x_2)\, dt,$$
$$S_1^+ u = \int_0^1 u(x_1 + th, x_2)\, dt,$$
$$T_1 u = \int_{-1}^1 (1 - |t|) u(x_1 + th, x_2)\, dt.$$

The operators S_2^- and T_2 are defined likewise. For these operators the following relations hold:

$$T_k \frac{\partial^2 u}{\partial x_k^2} = u_{\bar{x}_k x_k}, \quad T_k \frac{\partial u}{\partial x_k} = S_k^- u_{x_k}, \quad k = 1, 2,$$

$$G_k \frac{\partial^2 u}{\partial x_1^2} = \frac{1}{h^2}\big((1-\theta_k) u(n_k h, x_2) + \theta_k u(n_k h + h, x_2) - u(\xi_k, x_2)\big), \ k = 1, 2, \ldots, n_m.$$

By $Y_k(x_2)$, $Z_k(x_2)$, $\bar{Z}_k(x_2)$, $\widetilde{Z}(x_2)$ we will denote the expressions

$$Y_k(x_2) = (1-\theta_k) y(n_k h, x_2) + \theta_k y(n_k h + h, x_2) \ \text{ etc.}$$

We approximate the problem (2), (3) by finite-difference scheme

$$Ay \equiv \sum_{i,j=1}^{2} A_{ij} y + ay = -\varphi(x), \ x \in \omega, \quad \varphi = T_1 T_2 f, \tag{6}$$

$$y = 0, \ x \in \gamma_0, \quad y(1, x_2) = \sum_{k=1}^{m} \alpha_k Y_k(x_2), \ x_2 \in \omega_2, \tag{7}$$

where $A_{ij} y = -0.5\big(a_{ij}^{(-0.5_1)} y_{\bar{x}_j}\big)_{x_i} - 0.5\big(a_{ij}^{(+0.5_1)} y_{x_j}\big)_{\bar{x}_i}$.

3. The first fundamental inequality

Let H be the set of all grid functions, defined on $\bar{\omega}$ and vanishing on γ_0, with the inner product and the norm:

$$(y, v)_r = \sum_{\omega} h^2 r(x_1) y(x) v(x), \quad \|y\|_r^2 = (y, y)_r, \ r(x_1) = 1 - x_1.$$

Further, we put

$$\|y]|_r^2 = \sum_{\omega_1^+\times\omega_2} h^2\bar{r}y^2, \quad \|y|]_r^2 = \sum_{\omega_1\times\omega_2^+} h^2ry^2,$$

$$|y|_{1,\omega,r}^2 = \|y_{\bar{x}_1}]|_r^2 + \|y_{\bar{x}_2}|]_r^2, \quad \|y\|_{1,\omega,r}^2 = \|y\|^2 + |y|_{1,\omega,r}^2,$$

$$\|y\|^2 = \sum_{\omega_1^+\times\omega_2} \hbar hy^2, \ \|y\|_0^2 = \sum_{\omega} h^2y^2, \ \|y]|^2 = \sum_{\omega_1^+\times\omega_2} h^2y^2, \quad \|y|]^2 = \sum_{\omega_1\times\omega_2^+} h^2y^2,$$

$$\|y\|_*^2 = \sum_{\omega_2} hy^2, \ \|y|]_*^2 = \sum_{\omega_2^+} hy^2, \ \bar{r} = r + \frac{h}{2}, \ \hbar = h \text{ for } x_1 \in \omega_1, \ \hbar = \frac{h}{2} \text{ for } x_1 = 1.$$

Define the following weight function

$$\rho(x_1) = \begin{cases} \rho_i(x_1), & \xi_i \le x_1 < \xi_{i+1}, \ i = 0,1,2,\dots,m-1, \\ r(x_1), & \xi_m \le x_1 \le 1, \end{cases} \tag{8}$$

where

$$\rho_i(x_1) = r(x_1) - \varkappa \sum_{k=i+1}^{m} \sigma_k r_k(x_1), \quad r_k(x_1) = \xi_k - x_1, \ \varkappa = \sum_{k=1}^{m} |\alpha_k|\sqrt{\xi_k}, \quad \sigma_k = \frac{|\alpha_k|}{\sqrt{\xi_k}}.$$

Suppose that $\varkappa < 1$. Then [**6**]

$$(1-\varkappa^2)r(x_1) \le \rho(x_1) \le r(x_1). \tag{9}$$

In future we will consider that the inner product and norms, involving ρ in index have similar to the expression with index r sense.

In order to use below the results obtained in this section, for a priori estimate of the error of method (when the nonlocal condition will not be homogeneous any more) we will get the estimates for the function $y(x)$ in such form, in which the nonlocal condition still will not be taken into account.

LEMMA 1. *For any $y \in H$ the following estimates are valid:*

$$\|Y_k\|_*^2 \le (\xi_k/\nu)\|y_{\bar{x}_1}]|_r^2, \tag{10}$$

$$\|y\| \le \|y_{\bar{x}_1}]|_r, \tag{11}$$

$$\|y\|_\rho \le (1/\sqrt{8})\|y_{\bar{x}_2}|]_r. \tag{12}$$

PROOF. By virtue of the Cauchy–Buniakovski inequality and via (5) the estimate (10) follows from

$$Y_k(x_2) = \sum_{i=1}^{n_k+1} h\tau_k(i)y_{\bar{x}_1}(ih,x_2), \quad \tau_k(i) = 1 \text{ for } i = 1,2,\dots,n_k, \quad \tau_k(n_k+1) = \theta_k$$

taking into account that $\tau_k^2 \le \tau_k$, $\bar r(ih) > \nu$, $i = 1, 2, \dots, n_m + 1$. Next,

$$\begin{aligned}\|y\|^2 &= \sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{\omega_2} \hbar(i) h^2 y(ih, x_2) y_{\bar x_1}(jh, x_2)\\ &\le \Big(\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{\omega_2} \hbar(i) h^2 y^2(ih, x_2)\Big)^{1/2} \Big(\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{\omega_2} \hbar(i) h^2 y^2_{\bar x_1}(jh, x_2)\Big)^{1/2}\\ &\le \|y\|\, \|y_{\bar x_1}]|_r,\end{aligned}$$

which yields (11).

The estimate (12) follows from (see e.g., [**7**, p.120]) $\|y\|_*^2 \le (1/8)\|y_{\bar x_2}]|_*^2$, noting that $\rho \le r$. □

Denote

$$\Phi(a_{11}, y) = \frac{1}{2}\sum_{\omega_2} h a_{11}(x_2)\Big(\varkappa \sum_{k=1}^{m} \sigma_k Y_k^2(x_2) - y^2(1, x_2)\Big). \tag{13}$$

The following estimate (so called "first fundamental inequality" in terminology used by Ladyzhenskaya) is valid.

LEMMA 2. *If $y(x) \in H$, then*

$$(Ay, y)_\rho \ge c_1 \|y\|^2_{1,\omega,r} + \Phi(a_{11}, y), \; c_1 = \nu_1(1 - \varkappa^2)/2. \tag{14}$$

If moreover, $y(x)$ satisfies the nonlocal condition (7), then in the right-hand side (14) the second term may be neglected.

PROOF. Partial summation yields

$$-2\sum_{\omega_1} h\rho y_{\bar x_1 x_1} y = \sum_{\omega_1^+} h\rho y^2_{\bar x_1} + \sum_{\omega_1^-} h\rho y^2_{x_1} - y^2(1, x_2) - \sum_{\omega_1} h y^2 \rho_{\bar x_1 x_1}. \tag{15}$$

Represent now the weight function ρ in the following way

$$\rho(x_1) = 1 - x_1 - \sum_{k=1}^{m} \varkappa \sigma_k \chi(\xi_k - x_1), \quad \chi(t) = \begin{cases} t, & \text{if } t \ge 0,\\ 0, & \text{if } t < 0.\end{cases}$$

It is not hard to check that $h\chi_{\bar x_1 x_1}(\xi_k - ih) = (1 - \theta_k)\,\delta(n_k, i) + \theta_k\,\delta(n_k + 1, i)$, where $\delta(.,.)$ is the Kronecker delta. Consequently,

$$h\rho_{\bar x_1 x_1}(ih) = -\varkappa \sum_{k=1}^{m} \sigma_k\big((1 - \theta_k)\delta(n_k, i) + \theta_k \delta(n_k + 1, i)\big),$$

and

$$\sum_{\omega_1} h y^2 \rho_{\bar x_1 x_1} = -\sum_{k=1}^{m} \varkappa \sigma_k\big((1 - \theta_k) y^2(n_k h, x_2) + \theta_k y^2(n_k h + h, x_2)\big). \tag{16}$$

Taking into account that

$$\begin{aligned}(1 - \theta_k) y^2(n_k h, x_2) + \theta_k y^2(n_k h + h, x_2) &= Y_k^2(x_2) + h^2 \theta_k (1 - \theta_k) y^2_{x_1}(n_k h, x_2)\\ &\ge Y_k^2(x_2),\end{aligned}$$

from (15), (16) we get

$$(A_{11}y,y)_\rho \ge \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\rho a_{11}y_{\bar x_1}^2 + \frac{1}{2}\sum_{\omega_1^-\times\omega_2} h^2\rho a_{11}y_{x_1}^2 + \Phi(a_{11},y).$$

Furthermore, by using partial summation we have

$$(A_{12}y,y)_\rho = \frac{1}{2}\sum_{\omega_1^+\times\omega_2} h^2\rho a_{12}^{(-0.5_1)}y_{\bar x_1}y_{\bar x_2} + \frac{1}{2}\sum_{\omega_1^-\times\omega_2} h^2\rho a_{12}^{(+0.5_1)}y_{x_1}y_{x_2},$$

$$(A_{2j}y,y)_\rho = \frac{1}{2}\sum_{\omega_1\times\omega_2^+} h^2\rho a_{2j}^{(-0.5_1)}y_{\bar x_j}y_{\bar x_2} + \frac{1}{2}\sum_{\omega_1\times\omega_2^-} h^2\rho a_{2j}^{(+0.5_1)}y_{x_j}y_{x_2},\ j=1,2.$$

Consequently,

$$(Ay,y)_\rho \ge \frac{1}{2}\sum_{\omega^+} h^2\rho \sum_{i,j=1,2}^{2} a_{ij}^{(-0.5_1)}y_{\bar x_i}y_{\bar x_j} + \frac{1}{2}\sum_{\omega^-} h^2\rho \sum_{i,j=1,2}^{2} a_{ij}^{(+0.5_1)}y_{x_i}y_{x_j} + (a,y^2)_\rho + \Phi(a_{11},y).$$

Therefore (4), (9) implies $(Ay,y)_\rho \ge \nu_1(1-\varkappa^2)|y|^2_{1,\omega,r} + \Phi(a_{11},y)$, and taking into account (11), we finally obtain (14).

If $y(x)$ satisfies the nonlocal condition (7) too, then

$$y^2(1,x_2) \le \sum_{k=1}^m \varkappa\sigma_k Y_k^2(x_2)$$

and $\Phi(a_{11},y)\ge 0$. This completes the proof of the lemma. □

Due to Lemma 2 the problem (6), (7) has an unique solution.

4. A priori estimate of error of difference solution

Let u be a solution of the problem (2), (3) and y – a solution of the difference scheme (6), (7). Then for the error $z=y-u$ we obtain the problem

$$Az=\psi,\ x\in\omega,\ z=0,\ x\in\gamma_0,\ z(1,x_2)=\sum_{k=1}^m \alpha_k Z_k + R,\ x_2\in\omega_2, \tag{17}$$

where

$$\psi=\sum_{i,j=1}^{2}(\eta_{ij})_{x_i}+\eta,\quad R=\sum_{k=1}^m\alpha_k R_k,\quad R_k=h^2G_k\frac{\partial^2 u}{\partial x_1^2},\quad \eta=T_1T_2(au)-T_1T_2a\,u$$

$$\eta_{ij}=\frac{1}{2}a_{ij}^{(-0.5_1)}u_{\bar x_j}+\frac{1}{2}a_{ij}^{(+0.5_1,-1_i)}u_{x_j}^{(-1_i)}-S_i^-T_{3-i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big),\quad i,j=1,2.$$

Lemma 3. *The a priori estimate*

$$\|z\|_{1,\omega,r}\le c(\|a_{11}R\|_* + \|\eta_{11}]| + \|\eta_{12}]| + \|\eta_{21}|] + \|\eta_{22}|] + \|\eta\|_0). \tag{18}$$

is valid for the solution of the problem (17).

PROOF. Represent the solution z to the problem (17) as a sum $z = \bar{z} + \tilde{z}$ of solutions of the following two problems:

$$A\bar{z} = 0,\ x \in \omega,\ \bar{z} = 0,\ x \in \gamma_0,\ \bar{z}(1, x_2) = \sum_{k=1}^{m} \alpha_k \bar{Z}_k + R,\ x_2 \in \omega_2, \tag{19}$$

$$A\tilde{z} = \psi,\ x \in \omega,\ \tilde{z} = 0,\ x \in \gamma_0,\ \tilde{z}(1, x_2) = \sum_{k=1}^{m} \alpha_k \tilde{Z}_k,\ x_2 \in \omega_2. \tag{20}$$

It follows from nonlocal condition (19) that

$$-2\Phi(a_{11}, \bar{z}) \le \sum_{\omega_2} h a_{11}(x_2) \left(R^2 + 2R \sum_{k=1}^{m} \alpha_k \bar{Z}_k \right),$$

therefore (10) implies $-2\Phi(a_{11}, \bar{z}) \le \|a_{11}R\|_*^2 + (2\varkappa/\sqrt{\nu})\|a_{11}R\|_* \, \|\bar{z}_{\bar{x}_1}]|_r$, and we obtain

$$-2\Phi(a_{11}, \bar{z}) \le (1 + \varkappa/(\varepsilon_1\sqrt{\nu}))\|a_{11}R\|_*^2 + (\varkappa\varepsilon_1/\sqrt{\nu})\|\bar{z}_{\bar{x}_1}]|^2_{1,\omega,r},\quad \forall \varepsilon_1 > 0. \tag{21}$$

Further, applying Lemma 2 we conclude that $2c_1\|\bar{z}\|^2_{1,\omega,r} + 2\Phi(a_{11}, \bar{z}) \le 0$, the addition of which to (21) (with ε_1 chosen properly) gives

$$\|\bar{z}\|^2_{1,\omega,r} \le c_2\|a_{11}R\|_*^2. \tag{22}$$

On the other hand, applying Lemma 2 to the solution of the problem (20) we come to

$$c_1\|\tilde{z}\|^2_{1,\omega,r} \le \sum_{i,j=1}^{2} \left((\eta_{ij})_{x_i}, \tilde{z}\right)_\rho + (\eta, \tilde{z})_\rho \tag{23}$$

Since

$$\sum_{\omega_{1,k}\times\omega_2} h^2 r_k \eta_{1\beta x_1}\tilde{z} = -\sum_{\omega_{1,k}\times\omega_2} h^2 r_k \eta_{1\beta}\tilde{z}_{\bar{x}_1} + \sum_{i=1}^{n_k}\sum_{j=1}^{i}\sum_{\omega_2} h^3 \tau_k (i+1)\eta_{1\beta}(ih+h, x_2)\tilde{z}_{\bar{x}_1}(jh, x_2)$$

using the Cauchy–Buniakovski inequality we get

$$\left| \sum_{\omega_{1,k}\times\omega_2} h^2 r_k \eta_{1\beta x_1}\tilde{z} \right| \le 2\xi_k \|\tilde{z}_{\bar{x}_1}]|_r \|\eta_{1\beta}]|,$$

noting that $r_k \le \xi_k \bar{r}$. Similarly, $|(\eta_{1\beta x_1}, \tilde{z})_r| \le 2\|\tilde{z}_{\bar{x}_1}]|_r\|\eta_{1\beta}]|$. Therefore from

$$(\eta_{1\beta x_1}, \tilde{z})_\rho = (\eta_{1\beta x_1}, \tilde{z})_r - \sum_{k=1}^{m} \varkappa\sigma_k \sum_{\omega_{1,k}\times\omega_2} h^2 r_k \eta_{1\beta x_1}\tilde{z}$$

it follows

$$|(\eta_{1\beta x_1}, \tilde{z})_\rho| \le 2(1 + \varkappa^2)\|\tilde{z}_{\bar{x}_1}]|_r \|\eta_{1\beta}]|,\quad \beta = 1, 2. \tag{24}$$

Now, it is not difficult to verify that

$$|(\eta_{2\beta x_2}, \widetilde{z})_\rho| \le \|\widetilde{z}_{\bar{x}_2}]|_r \|\eta_{2\beta}|], \quad \beta = 1, 2. \tag{25}$$

The inequality (12) immediately implies that

$$|(\eta, \widetilde{z})_\rho| \le (1/\sqrt{8})\|\eta\|_0 \, \|\widetilde{z}_{\bar{x}_2}|]_r. \tag{26}$$

Substituting (24)–(26) into (23) we finally obtain

$$\|\widetilde{z}\|^2_{1,\omega,r} \le c\bigg(\sum_{\beta=1}^{2}(\|\eta_{1\beta}]| + \|\eta_{2\beta}|]) + \|\eta\|_0\bigg). \tag{27}$$

The inequality (18) follows directly from (22), (27). □

5. Estimate of the convergence rate

In order to estimate the convergence rate of finite-difference scheme (6), (7), it is enough to estimate the norm of error functionals in (18). For this we apply the well-known tecnique (see e.g., [**8**], [**9**]), which uses the generalized Bramble–Hilbert lemma [**10**].

We will show, that

$$\|a_{11}R\|_* \le ch^{s-1}\|u\|_{W_2^s(\Omega)}, \quad s \in (1, 3]. \tag{28}$$

Let $e_k = (n_k h, n_k h + h) \times (x_2 - h/2, x_2 + h/2)$, $\Omega_k = (n_k h, n_k h + h) \times (0, 1)$. We now represent R_k in the form of sum

$$R_k = \Big(h^2 G_k \frac{\partial^2 u}{\partial x_1^2} - h^2 G_k S_2^- \frac{\partial^2 u}{\partial x_1^2}\Big) + h^2 G_k S_2^- \frac{\partial^2 u}{\partial x_1^2} = R'_k + R''_k, \quad k = 1, 2, \dots, n_m.$$

Let us remark, that R'_k is a bounded linear functional of $u \in W_2^s(e_k)$, $s > 1$, which vanishes if u is a second-degree polynomial. Using Bramble–Hilbert lemma we obtain

$$|R'_k| \le ch^{s-1}|u|_{W_2^s(e_k)}, \quad \|R'_k\|_* \le ch^{s-1}|u|_{W_2^s(\Omega)}, \quad s \in (1, 3]. \tag{29}$$

For $s > 1$, R''_k is a bounded linear functional of $u \in W_2^s(e_k)$, which vanishes if u is a first-degree polynomial. Using Bramble–Hilbert lemma we obtain

$$\|R''_k\|_* \le ch^{s-1}|u|_{W_2^s(\Omega)}, \quad s \in (1, 2.5]. \tag{30}$$

In the case $s > 2.5$ we write

$$\|R''_k\|_*^2 \le c\sum_{\omega_2} h^3 \int_{e_k} \Big|\frac{\partial^2 u}{\partial x_1^2}\Big|^2 dx \le ch^3 \Big\|\frac{\partial^2 u}{\partial x_1^2}\Big\|^2_{L_2(\Omega_k)}$$

and since $\partial^2 u/\partial x_1^2 \in W_2^{s-2}(\Omega)$, $s - 2 > 0.5$, we may use an estimate for L_2-norm of a function in a sprit near the boundary in terms of W_2^{s-2}-norm in the domain Ω (see, e.g., [**8**, p. 161], [**11**, p. 47]) :

$$\Big\|\frac{\partial^2 u}{\partial x_1^2}\Big\|_{L_2(\Omega_k)} \le ch^{1/2}\Big\|\frac{\partial^2 u}{\partial x_1^2}\Big\|_{W_2^{s-2}(\Omega)}, \ 0.5 < s - 2 \le 1.$$

This gives $\|R_k''\|_* \le ch^2|u|_{W_2^s(\Omega)}$, $s \in (2.5, 3]$, which along with (29), (30), taking into account continuity of the coefficient a_{11}, proves the inequality (28).

We will show, that

$$\|\eta_{1j}]| \le ch^{s-1}\,\|u\|_{W_2^s(\Omega)}, \quad \|\eta_{2j}|] \le ch^{s-1}\,\|u\|_{W_2^s(\Omega)}, \quad < s \le 3. \tag{31}$$

For $1 < s \le 2$, we represent η_{ij} in the following way

$$\eta_{ij} = \eta_{ij}' + 0.5a_{ij}^{(-0.5_1)}\eta_{ij}''\Big(\frac{\partial u}{\partial x_j}\Big) + 0.5a_{ij}^{(+0.5_1,-1_i)}\eta_{ij}'''\Big(\frac{\partial u}{\partial x_j}\Big), \tag{32}$$

where

$$\eta_{ij}' = 0.5\Big(a_{ij}^{(-0.5_1)} + a_{ij}^{(+0.5_1,-1_i)}\Big)S_i^- T_{3-i}\frac{\partial u}{\partial x_j} - S_i^- T_{3-i}\Big(a_{ij}\frac{\partial u}{\partial x_j}\Big),$$

$$\eta_{ij}''(v) = S_j^- v - S_i^- T_{3-i}v, \quad \eta_{ij}'''(v) = \big(S_j^+ v\big)^{(-1_i)} - S_i^- T_{3-i}v.$$

Let $e = (x_1 - h, x_1 + h) \times (x_2 - h, x_2)$. Since η_{2j}' is a linear functional with respect to a_{2j}, is bounded in $W_q^{s-1}(e)$, $s > 1$ and vanishis if a_{2j} is constant, we have

$$|\eta_{2j}'| \le ch^{s-3-2/q}|a_{2j}|_{W_q^{s-1}(e)}\int_e \Big|\frac{\partial u}{\partial x_j}\Big|\,dx.$$

It is easy to see that

$$\int_e \Big|\frac{\partial u}{\partial x_j}\Big|\,dx \le \Big(\int_e \Big|\frac{\partial u}{\partial x_j}\Big|^{2q/(q-2)}dx\Big)^{(q-2)/(2q)} h^{1+2/q},$$

therefore

$$|\eta_{2j}'| \le ch^{s-2}|a_{2j}|_{W_q^{s-1}(e)}|u|_{W^1_{2q/(q-2)}(e)}$$

and

$$\|\eta_{2j}'|] \le ch^{s-1}\|a_{2j}\|_{W_q^{s-1}(\Omega)}\|u\|_{W^1_{2q/(q-2)}(\Omega)}. \tag{33}$$

Analogously,

$$\|\eta_{1j}']| \le ch^{s-1}\|a_{1j}\|_{W_p^{s-1}(0;1)}\|u\|_{W^1_{2p/(p-2)}(\Omega)}. \tag{34}$$

Since $W^1_{2q/(q-2)}$, $W^1_{2p/(p-2)} \subset W_2^{s-2}$, using (33), (34) and similar estimates for η_{ij}'', η_{ij}''', from (32) we come (31) for $1 < s \le 2$.

In the case $2 < s \le 3$ we write

$$\eta_{i1} = l_1\Big(a_{i1}\frac{\partial u}{\partial x_1}\Big) + 0.5a_{i1}l_2\Big(\frac{\partial u}{\partial x_1}\Big) + 0.5a_{i1}^{(+0.5_1,-1_i)}l_3\Big(\frac{\partial u}{\partial x_1}\Big), \tag{35}$$

$$\eta_{i2} = l_4\Big(a_{i2}\frac{\partial u}{\partial x_2}\Big) + a_{i2}^{(-0.5_i)}l_5\Big(\frac{\partial u}{\partial x_2}\Big) + 0.5u_{\bar{x}_2}l_6(a_{i2}), \tag{36}$$

where

$$l_1(v) = 0.5v^{(-0.5_1)} + 0.5v^{(+0.5_1,-1_i)} - S_i^- T_{3-i}v, \quad l_2(v) = S_1^- v - v^{(-0.5_1)},$$

$$l_3(v) = S_1^+ v - v^{(+0.5_1)}, \quad l_4(v) = v^{(-0.5_i)} - S_i^- T_{3-i}v,$$

$$l_5(v) = 0.5S_2^- v + 0.5S_2^+ v^{(-1_i)} - v^{(-0.5_i)}, \quad l_6(v) = v^{(-0.5_1)} - 2v^{(-0.5_i)} + v^{(+0.5_1,-1_i)}.$$

For $s > 2$, $l_j(v)$ $(j = 1, 2, 3, 4, 5, 6)$ are a bounded linear functionals of $v \in W_2^{s-2}$ and vanish if v is a first-degree polinomial. Hence from (35), (36) we get (31) for $2 < s \leq 3$.

For η the estimate $\|\eta\|_0 \leq ch^{s-1}\,\|u\|_{W_2^s(\Omega)}$, $1 < s \leq 3$ is true.

Finally on the basis of obtained estimates, together with Lemma 3 the following convergence theorem is proved.

THEOREM 1. *The finite-difference scheme* (6), (7) *converges and the convergence rate estimate* (1) *holds.*

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