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AN ASYMPTOTIC FORMULA FOR A SUM INVOLVING ZEROS OF THE RIEMANN ZETA-FUNCTION

Yuichi Kamiya and Masatoshi Suzuki

Abstract. E. Landau gave an interesting asymptotic formula for a sum involving zeros of the Riemann zeta-function. We give an asymptotic formula which can be regarded as a smoothed version of Landau's formula.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta-function. It is important to study non-trivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. Weil's explicit formula is one of useful formulas for the study of ρ. Roughly speaking, it connects certain sums involving ρ with sums involving prime numbers in terms of test functions and those Mellin transforms. We can refer to Lang [**6**] or Patterson [**7**] for the details of Weil's explicit formula.

In this paper, as an application of Weil's explicit formula with a certain test function, we shall study the asymptotic behaviour of a quantity involving ρ, that is,

$$\sum_{\rho} e^{u\rho^2 - v\rho}. \tag{1.1}$$

Some suitable choice of the test function enables us to get asymptotic formulas for (1.1).

Theorem 1.1. (i) *For $v = u$ or $v = 0$ we have*

$$\sum_{\rho} e^{u\rho^2 - v\rho} = \frac{1}{\sqrt{16\pi u}} \log\frac{1}{u} - \frac{\log(16\pi^2) + \mathcal{C}}{\sqrt{16\pi u}} + O(1), \quad u \to +0,$$

where $\mathcal{C}$ is the Euler constant, and the sum $\sum_{\rho}$ runs over all non-trivial zeros ρ counting with multiplicity.

(ii) *For any integer $m \geqslant 2$ we have*

$$\sum_{\rho} e^{u\rho^2 + (\log m)\rho} = -\frac{\Lambda(m)}{\sqrt{4\pi u}} + O(1), \quad u \to +0,$$

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where $\Lambda(m) = \log p$ *if* m *is a power of a prime* p *and* $\Lambda(m) = 0$ *otherwise. The implied constant depends on* m.

(ii)′ *Let* K *be a closed interval contained in* $(-\infty, 0) - \bigcup_m \{-\log m\}$, *where* m *is a power of a prime. Then we have*

$$\sum_\rho e^{u\rho^2 - v\rho} = O(1), \quad u \to +0,$$

uniformly for v *in* K.

(iii) *For any integer* $m \geqslant 2$ *we have*

$$\sum_\rho e^{u\rho^2 - (\log m)\rho} = -\frac{\Lambda(m)}{m\sqrt{4\pi u}} + O(1), \quad u \to +0.$$

The implied constant depends on m.

(iii)′ *Let* K *be a closed interval contained in* $(0, \infty) - \cup_m \{\log m\}$, *where* m *is a power of a prime. Then we have*

$$\sum_\rho e^{u\rho^2 - v\rho} = O(1), \quad u \to +0,$$

uniformly v *in* K.

We can see asymptotic behaviours different from each other for the quantity (1.1) and the difference depends on the choice of v. The first and second terms on the right-hand side of the asymptotic formula in (i) come from the logarithmic derivative of the gamma factor appeared in the functional equation of $\zeta(s)$. On the other hand, the first terms on the right-hand sides of the asymptotic formulas in (ii) and (iii) come from the logarithmic derivative of $\zeta(s)$.

The asymptotic formula in (ii) is related to the results of Landau [**5**], Gonek [**3**] [**4**], and Fujii [**2**]. Landau [**5**] proved that, for fixed $x > 1$,

$$\sum_{0<\gamma\leqslant T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O(\log T)$$

holds. Gonek [**3**] [**4**] gave uniform versions of Landau's result, and Fujii [**2**] gave a refined formula for it under the Riemann Hypothesis. The asymptotic formula in (ii) may be regarded as a smoothed version of Landau's with the measure given by the Gaussian function.

The asymptotic formula in (i) may be regarded as a smoothed version of the asymptotic formula for $N(T)$, number of non-trivial zeros ρ with $0 < \gamma < T$. To see this, let us consider the case $v = u$ in (i) under the Riemann Hypothesis. Then the asypmtotic formula in (i) is

$$\sum_\gamma e^{-u(1/4+\gamma^2)} = \frac{1}{\sqrt{16\pi u}} \log\frac{1}{u} - \frac{\log(16\pi^2) + \mathcal{C}}{\sqrt{16\pi u}} + O(1).$$

The sum on the left-hand side is written as an integral form, and, by integration by parts, it follows that

$$-\int_0^\infty N(T)d\big(e^{-uT^2}\big) = \frac{1}{2\sqrt{16\pi u}}\log\frac{1}{u} - \frac{\log(16\pi^2)+\mathcal{C}}{2\sqrt{16\pi u}} + O(1).$$

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2. An explicit formula for a sum involving zeros of the Riemann zeta-function

In this section we give an explicit formula, which is a variant of Weil's explicit formula.

LEMMA 2.1. *For any positive u and any real v we have*

$$\begin{aligned}\sum_\rho e^{u\rho^2-v\rho} &= e^{u-v} - \frac{\log\pi}{\sqrt{4\pi u}}e^{-v^2/4u} + 1\\ &\quad - \frac{1}{\sqrt{4\pi u}}\sum_{n=2}^\infty \Lambda(n)e^{-(v+\log n)^2/4u} - \frac{1}{\sqrt{4\pi u}}\sum_{n=2}^\infty \frac{\Lambda(n)}{n}e^{-(v-\log n)^2/4u}\\ &\quad + \frac{e^{u/4-v/2}}{2\pi}\int_{-\infty}^\infty \log\Big|\frac{1}{4}+i\frac{t}{2}\Big|\cdot e^{-ut^2+it(u-v)}dt - (E*G_u)(v),\end{aligned}$$

where the functions E and G_u are defined by

$$E(x) = \Big(\frac{1}{e^{2|x|}-1} - \frac{1}{2|x|} + 1\Big)e^{-|x|/2-x/2}, \quad G_u(x) = \frac{1}{\sqrt{4\pi u}}e^{-x^2/4u},$$

and $E*G_u$ means the convolution of E and G_u, that is,

$$(E*G_u)(v) = \int_{-\infty}^\infty E(x)G_u(v-x)dx.$$

PROOF. Since

$$\int_0^\infty \frac{1}{\sqrt{4\pi u}}e^{-(v+\log x)^2/4u}x^s\frac{dx}{x} = e^{us^2-vs},$$

we have

$$\frac{1}{2\pi i}\int_{1+\delta-i\infty}^{1+\delta+i\infty}\frac{\zeta'}{\zeta}(s)e^{us^2-vs}ds = -\sum_{n=2}^\infty \Lambda(n)\frac{1}{\sqrt{4\pi u}}e^{-(v+\log n)^2/4u}.$$

We also have

$$\frac{1}{2\pi i}\int_{1+\delta-i\infty}^{1+\delta+i\infty}\frac{\zeta'}{\zeta}(s)e^{us^2-vs}ds = \frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}\frac{\zeta'}{\zeta}(s)e^{us^2-vs}ds + \sum_\rho e^{u\rho^2-v\rho} - e^{u-v}.$$

The first term on the right-hand side can be expressed in the following form by the functional equation of $\zeta(s)$:

$$\begin{aligned}\frac{1}{2\pi i}\int_{-\delta-i\infty}^{-\delta+i\infty}&\left(-\frac{\zeta'}{\zeta}(1-s)+\log\pi-\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right)-\frac{1}{2}\frac{\Gamma'}{\Gamma}\Big(\frac{1-s}{2}\Big)\right)e^{us^2-vs}ds\\ &=\frac{1}{\sqrt{4\pi u}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n}e^{-(v-\log n)^2/4u}+\frac{\log\pi}{\sqrt{4\pi u}}e^{-v^2/4u}-1\\ &\quad-\frac{1}{4\pi}\int_{-\infty}^{\infty}\left(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+i\frac{t}{2}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}-i\frac{t}{2}\Big)\right)e^{u(1/2+it)^2-v(1/2+it)}dt.\end{aligned}$$

Hence we have

$$\begin{aligned}\sum_{\rho}e^{u\rho^2-v\rho}=&e^{u-v}-\frac{\log\pi}{\sqrt{4\pi u}}e^{-v^2/4u}+1\\ &-\frac{1}{\sqrt{4\pi u}}\sum_{n=2}^{\infty}\Lambda(n)e^{-(v+\log n)^2/4u}-\frac{1}{\sqrt{4\pi u}}\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n}e^{-(v-\log n)^2/4u}\\ &+\frac{e^{u/4-v/2}}{4\pi}\int_{-\infty}^{\infty}\left(\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}+i\frac{t}{2}\Big)+\frac{\Gamma'}{\Gamma}\Big(\frac{1}{4}-i\frac{t}{2}\Big)\right)e^{-ut^2+it(u-v)}dt.\end{aligned}$$

This formula is a special case of Weil's explicit formula (with the test function $\frac{1}{\sqrt{4\pi u}}e^{-(v+\log x)^2/4u}$), but we supply a proof to make the paper self-contained.

Let us denote the last term on the right-hand side by H. By the expression (see, for example, [**1**, p. 28, l. 16])

$$\frac{\Gamma'}{\Gamma}(z)=\log z-\int_0^{\infty}\Big(\frac{1}{e^x-1}-\frac{1}{x}+1\Big)e^{-zx}dx,\quad \operatorname{Re}z>0,$$

we have

$$\begin{aligned}H=&\frac{e^{u/4-v/2}}{4\pi}\int_{-\infty}^{\infty}\left(\log\Big(\frac{1}{4}+i\frac{t}{2}\Big)+\log\Big(\frac{1}{4}-i\frac{t}{2}\Big)\right)e^{-ut^2+it(u-v)}dt\\ &-\frac{e^{u/4-v/2}}{4\pi}\int_0^{\infty}\Big(\frac{1}{e^x-1}-\frac{1}{x}+1\Big)e^{-x/4}\sqrt{\frac{\pi}{u}}\Big(e^{-\frac{(u-v-x/2)^2}{4u}}+e^{-\frac{(u-v+x/2)^2}{4u}}\Big)dx\\ =&\frac{e^{u/4-v/2}}{2\pi}\int_{-\infty}^{\infty}\log\Big|\frac{1}{4}+i\frac{t}{2}\Big|\cdot e^{-ut^2+it(u-v)}dt\\ &-\int_{-\infty}^{\infty}\Big(\frac{1}{e^{2|x|}-1}-\frac{1}{2|x|}+1\Big)e^{-|x|/2-x/2}\frac{1}{\sqrt{4\pi u}}e^{-(v-x)^2/4u}dx.\end{aligned}$$

Hence we obtain the lemma. □

3. Proof of Theorem

To obtain the estimates in the theorem we consider separately each term on the right-hand side of Lemma 2.1.

LEMMA 3.1. *We have* $0\leqslant(E*G_u)(v)\leqslant 1$.

Proof. It is easy to verify that $0 \leqslant E(x) \leqslant 1$. Hence

$$0 \leqslant (E * G_u)(v) \leqslant \int_{-\infty}^{\infty} G_u(x)dx = 1. \qquad \square$$

Here, we remark on the convolution $(E * G_u)(v)$. The assertion of Lemma 2.1 is enough for the proof of the theorem, but we can obtain a more precise behaviour of the convolution. It is not hard to verify that the function E has the property $|E(v-x) - E(v)| \leqslant C|x|$, where C is a positive absolute constant. Hence we have

$$\begin{aligned} |(E * G_u)(v) - E(v)| &\leqslant \int_{-\infty}^{\infty} |E(v-x) - E(v)| G_u(x)dx \\ &\leqslant C \int_{-\infty}^{\infty} |x| G_u(x)dx = C\sqrt{\frac{4u}{\pi}}, \end{aligned}$$

that is, $(E * G_u)(v) = E(v) + O(\sqrt{u})$.

The next lemma is the key for the proof of the theorem.

Lemma 3.2. *For $0 < u < 1$ we have*

$$\int_{-\infty}^{\infty} \log\Big|\frac{1}{4} + i\frac{t}{2}\Big| \cdot e^{-ut^2 + it(u-v)}dt$$
$$= \begin{cases} O\left(\frac{1}{|u-v|^2}\right) & \text{if } v \neq u \text{ and } v \neq 0, \\ \sqrt{\frac{\pi}{4u}} \log \frac{1}{u} - \sqrt{\frac{\pi}{4u}}(4\log 2 + \mathcal{C}) + O(1), & \text{if } v = u \text{ or } v = 0, \end{cases}$$

where the implied constants are absolute.

Proof. Firstly, we consider the case $v \neq u$ and $v \neq 0$. We have

$$\begin{aligned} &\int_{-\infty}^{\infty} \log\Big|\frac{1}{4} + i\frac{t}{2}\Big| \cdot e^{-ut^2 + it(u-v)}dt \qquad (3.1)\\ &= \int_0^{\infty} \log\Big(\frac{1}{16} + \frac{t^2}{4u}\Big) \cdot e^{-t^2} \cos\Big(\frac{t(u-v)}{\sqrt{u}}\Big) \frac{dt}{\sqrt{u}} \\ &= -\frac{1}{u-v} \int_0^{\infty} \frac{d}{dt}\Big(\log\Big(\frac{1}{16} + \frac{t^2}{4u}\Big) \cdot e^{-t^2}\Big) \sin\Big(\frac{t(u-v)}{\sqrt{u}}\Big)dt \\ &= -\frac{\sqrt{u}}{(u-v)^2} \int_0^{\infty} \frac{d^2}{dt^2}\Big(\log\Big(\frac{1}{16} + \frac{t^2}{4u}\Big) \cdot e^{-t^2}\Big) \cos\Big(\frac{t(u-v)}{\sqrt{u}}\Big)dt \\ &= -\frac{\sqrt{u}}{(u-v)^2} \Big\{ \int_0^{\infty} \frac{2(-t^2 + u/4)}{(t^2 + u/4)^2} e^{-t^2} \cos\Big(\frac{t(u-v)}{\sqrt{u}}\Big)dt \\ &\qquad - \int_0^{\infty} \frac{8t^2}{t^2 + \frac{u}{4}} e^{-t^2} \cos\Big(\frac{t(u-v)}{\sqrt{u}}\Big)dt \\ &\qquad + \int_0^{\infty} \log\Big(\frac{1}{16} + \frac{t^2}{4u}\Big) \cdot e^{-t^2}(4t^2 - 2) \cos\Big(\frac{t(u-v)}{\sqrt{u}}\Big)dt \Big\} \\ &= -\frac{\sqrt{u}}{(u-v)^2} \{I_1 + I_2 + I_3\}, \end{aligned}$$

say. As for I_1 and I_2 we easily have

$$|I_1| \leqslant 2\int_0^\infty \frac{1}{t^2+u/4}e^{-t^2}dt \leqslant \frac{8}{u}\int_0^{\sqrt{u}} dt + 2\int_{\sqrt{u}}^\infty \frac{1}{t^2}dt = 10\frac{1}{\sqrt{u}}, \tag{3.2}$$

$$|I_2| \leqslant 8\int_0^\infty e^{-t^2}dt = 4\sqrt{\pi}. \tag{3.3}$$

As for I_3 we have

$$\begin{aligned}|I_3| &\leqslant \int_0^\infty \Big|\log\Big(\frac{1}{16}+\frac{t^2}{4u}\Big)\Big|e^{-t^2}(4t^2+2)dt \\ &\leqslant \log\frac{1}{u}\cdot\int_0^\infty e^{-t^2}(4t^2+2)dt + \int_0^2 \Big|\log\Big(\frac{u}{16}+\frac{t^2}{4}\Big)\Big|e^{-t^2}(4t^2+2)dt \\ &\quad + \int_2^\infty \Big|\log\Big(\frac{u}{16}+\frac{t^2}{4}\Big)\Big|e^{-t^2}(4t^2+2)dt \\ &\ll \log\frac{1}{u} + \int_2^\infty \log t\cdot e^{-t^2}(4t^2+2)dt \ll \log\frac{1}{u}.\end{aligned} \tag{3.4}$$

Substituting (3.2), (3.3), and (3.4) into (3.1), we obtain the first estimate of this lemma.

Next, we consider the case $v=u$. We have

$$\begin{aligned}&\int_{-\infty}^\infty \log\Big|\frac{1}{4}+i\frac{t}{2}\Big|\cdot e^{-ut^2}dt \\ &= \int_0^\infty \log\Big(\frac{1}{16}+\frac{t^2}{4u}\Big)\cdot e^{-t^2}\frac{dt}{\sqrt{u}} \\ &= \frac{\log\frac{1}{u}}{\sqrt{u}}\int_0^\infty e^{-t^2}dt + \frac{1}{\sqrt{u}}\int_0^\infty \log\Big(\frac{u}{16}+\frac{t^2}{4}\Big)\cdot e^{-t^2}dt \\ &= \frac{\sqrt{\pi}}{2}\frac{\log\frac{1}{u}}{\sqrt{u}} + \frac{1}{\sqrt{u}}\int_0^\infty \log\frac{t^2}{4}\cdot e^{-t^2}dt + \frac{1}{\sqrt{u}}\int_0^\infty \log\Big(1+\frac{u}{4t^2}\Big)\cdot e^{-t^2}dt \\ &= \frac{\sqrt{\pi}}{2}\frac{\log\frac{1}{u}}{\sqrt{u}} + \frac{1}{\sqrt{u}}J_1 + \frac{1}{\sqrt{u}}J_2,\end{aligned} \tag{3.5}$$

say. As for J_2 we have

$$\begin{aligned}J_2 &= \int_0^{\sqrt{u}} \log\Big(1+\frac{u}{4t^2}\Big)\cdot e^{-t^2}dt + \int_{\sqrt{u}}^\infty \log\Big(1+\frac{u}{4t^2}\Big)\cdot e^{-t^2}dt \\ &\leqslant \int_0^{\sqrt{u}} \log\Big(1+\frac{u}{4t^2}\Big)dt + \frac{u}{4}\int_{\sqrt{u}}^\infty \frac{1}{t^2}e^{-t^2}dt \\ &= \sqrt{u}\log\frac{5}{4} + \int_0^{\sqrt{u}} \frac{2u}{4t^2+u}dt + \frac{u}{4}\int_{\sqrt{u}}^\infty \frac{1}{t^2}e^{-t^2}dt \\ &\leqslant \sqrt{u}\log\frac{5}{4} + 2\sqrt{u} + \frac{u}{4}\int_{\sqrt{u}}^\infty \frac{1}{t^2}dt \ll \sqrt{u}.\end{aligned} \tag{3.6}$$

For J_1 we have

$$J_1 = \int_0^\infty \log t \cdot e^{-t} \frac{dt}{2\sqrt{t}} - \log 4 \cdot \int_0^\infty e^{-t^2} dt = \frac{1}{2}\Gamma'\Big(\frac{1}{2}\Big) - \sqrt{\pi}\log 2 = -\frac{\sqrt{\pi}}{2}(4\log 2 + \mathcal{C}). \tag{3.7}$$

Substituting (3.6) and (3.7) into (3.5), we obtain the second asymptotic formula in this lemma in the case $v = u$.

Finally, we consider the case $v = 0$. We have

$$\begin{aligned}
&\int_{-\infty}^\infty \log\Big|\frac{1}{4} + i\frac{t}{2}\Big| \cdot e^{-ut^2+itu} dt \\
&= \int_0^\infty \log\Big(\frac{1}{16} + \frac{t^2}{4u}\Big) \cdot e^{-t^2} \cos(\sqrt{u}t) \frac{dt}{\sqrt{u}} \\
&= \frac{\log\frac{1}{u}}{\sqrt{u}} \int_0^\infty e^{-t^2} \cos(\sqrt{u}t) dt + \frac{1}{\sqrt{u}} \int_0^\infty \log\frac{t^2}{4} \cdot e^{-t^2} \cos(\sqrt{u}t) dt \\
&\quad + \frac{1}{\sqrt{u}} \int_0^\infty \log\Big(1 + \frac{u}{4t^2}\Big) \cdot e^{-t^2} \cos(\sqrt{u}t) dt \\
&= \frac{\log\frac{1}{u}}{\sqrt{u}} K_1 + \frac{1}{\sqrt{u}} K_2 + \frac{1}{\sqrt{u}} K_3,
\end{aligned} \tag{3.8}$$

say. As for K_3 we have

$$|K_3| \leqslant J_2 \ll \sqrt{u}. \tag{3.9}$$

For K_1 and K_2 we use

$$\cos(\sqrt{u}t) = 1 + O(ut^2). \tag{3.10}$$

From (3.10) it follows that

$$K_1 = \int_0^\infty e^{-t^2} dt + O\Big(u \int_0^\infty e^{-t^2} t^2 dt\Big) = \frac{\sqrt{\pi}}{2} + O(u), \tag{3.11}$$

$$\begin{aligned}
K_2 &= \int_0^\infty \log\frac{t^2}{4} \cdot e^{-t^2} dt + O\Big(u \int_0^\infty \log\frac{t^2}{4} \cdot e^{-t^2} t^2 dt\Big) \\
&= -\frac{\sqrt{\pi}}{2}(4\log 2 + \mathcal{C}) + O(u).
\end{aligned} \tag{3.12}$$

Substituting (3.9), (3.11), and (3.12) into (3.8), we obtain the second asymptotic formula in this lemma in the case $v = 0$. □

To obtain the theorem we now consider the asymptotic behaviour of the quantity

$$-\frac{1}{\sqrt{4\pi u}} \sum_{n=2}^\infty \Lambda(n) e^{-(v+\log n)^2/4u} - \frac{1}{\sqrt{4\pi u}} \sum_{n=2}^\infty \frac{\Lambda(n)}{n} e^{-(v-\log n)^2/4u} \tag{3.13}$$

in Lemma 2.1. The behaviour of this quantity depends on the choice of v. For the case $v = 0$ and $0 < u < 1$

$$e^{-(\log n)^2/4u} = e^{-(\log n)^2/8u} e^{-(\log n)^2/8u} \leqslant e^{-(\log n)^2/8} e^{-(\log 2)^2/8u},$$

and hence (3.13) is of exponential decay as $u \to +0$. For the case $v = -\log m$, $m \geqslant 2$ is an integer, and $0 < u < 1$ we have

$$\begin{aligned} e^{-\frac{1}{4u}(-\log m + \log n)^2} &\leqslant e^{-\frac{1}{8u}(-\log m + \log n)^2} e^{-\frac{1}{8u}(-\log m + \log(m+1))^2} \\ &\leqslant e^{-\frac{1}{8}(\log n)^2 (1 - \frac{\log m}{\log n})^2} e^{-\frac{1}{8u}(-\log m + \log(m+1))^2}, \quad n \neq m, \end{aligned}$$

and

$$e^{-\frac{1}{4u}(-\log m - \log n)^2} \leqslant e^{-\frac{1}{4u}(\log n)^2} \leqslant e^{-\frac{1}{8}(\log n)^2} e^{-\frac{1}{8u}(\log 2)^2},$$

and hence (3.13) is

$$\begin{aligned} &= -\frac{\Lambda(m)}{\sqrt{4\pi u}} + O\Bigg(\frac{e^{-\frac{1}{8u}(\log 2)^2}}{\sqrt{u}} \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n} e^{-\frac{1}{8}(\log n)^2} \\ &\quad + \frac{e^{-\frac{1}{8u}(-\log m + \log(m+1))^2}}{\sqrt{u}} \Bigg(\sum_{m \neq n = 2}^{m^2} \Lambda(n) + \sum_{n > m^2} \Lambda(n) e^{-\frac{1}{8}(\log n)^2 \frac{1}{4}} \Bigg)\Bigg). \end{aligned}$$

For other v we can similarly consider the asymtotic behaviour of (3.13).

Combining the above arguments and Lemmas 2.1, 3.1, and 3.2, we obtain the assertion of the theorem.

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Yuichi Kamiya
19-4 Nishinobo Daiwa-cho
Okazaki-city Aichi 444-0931
Japan
kamiya-9@m3.catvmics.ne.jp

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Masatoshi Suzuki
Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya 464-8602
Japan
m99009t@math.nagoya-u.ac.jp