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DOMAINS OF ATTRACTION OF THE REAL RANDOM VECTOR (X, X^2) AND APPLICATIONS

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Abstract. Many statistics are based on functions of sample moments. Important examples are the sample variance $s^2(n)$, the sample coefficient of variation $SV(n)$, the sample dispersion $SD(n)$ and the non-central t-statistic $t(n)$. The definition of these quantities makes clear that the vector defined by $\left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right)$ plays an important role. In the paper we obtain conditions under which the vector (X, X^2) belongs to a bivariate domain of attraction of a stable law. Applying simple transformations then leads to a full discussion of the asymptotic behaviour of $SV(n)$ and $t(n)$.

1. Introduction

Let $F(x) = P(X \leqslant x)$ and $F_2(x) = P(X^2 \leqslant x)$ denote the distribution function (d.f.) of a real random variable X and X^2 respectively. Let $G(x, y)$ denote the d.f. of the random vector (r.v.) (X, X^2). We find that

$$G(x, y) = P(-\sqrt{y} \leqslant X \leqslant \min(x, \sqrt{y}\,)),\ y \geqslant 0,\ x > -\sqrt{y}.$$

Clearly this relationship can be used to transfer properties from F to G.

Studying the random vector (X, X^2) can be interesting because it is linked to many statistical estimators. To estimate the mean $\mu = E(X)$ and the variance $\sigma^2 = Var(X)$ for example, one uses the sample mean $\overline{X}$ and the sample variance $s_n^2 = \overline{X^2} - \overline{X}^2$ or $s_{n-1}^2 = ns_n^2/(n-1)$. Other related statistical measures are the non-central t-statistic $t(n) = \sqrt{n}\ \overline{X}/s_n$, the coefficient of variation $SV(n) = s_n/\overline{X}$ and the sample dispersion $SD(n) = s_n^2/\overline{X}$. Many asymptotic properties of these statistics are known if the mean μ and the variance σ^2 are finite. On the other hand, if the variance or the mean is not finite, it also makes sense to study asymptotic properties of these quantities. For a recent paper devoted to $t(n)$, we refer to Bentkus et al. (2007) and the references given there. In Albrecher and

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Teugels (2004) and Ladoucette and Teugels (2006), the authors discuss asymptotic properties of $SV(n)$ and $SD(n)$. In the case where $X \geqslant 0$, Omey (2008a) obtained a detailed and complete analysis of $SD(n)$, $SV(n)$ and $t(n)$. It turns out, cf. Section 3, that a key role is played by the real random vector $\left(\sum_{i=1}^{n} X_i, \sum_{i=1}^{n} X_i^2\right)$ and by $T(n)$, where

$$T(n) = \frac{\sum_{i=1}^{n} X_i^2}{(\sum_{i=1}^{n} X_i)^2}$$

In the present paper we remove the restriction that $X \geqslant 0$. To avoid difficulties with the definition of $T(n)$, in section 3 below we assume that $F(x)$ is continuous.

The structure of the paper is as in Omey (2008a), where in each step we remove the restriction that $X \geqslant 0$. In the real case, it turns out that the case where $\mu = E(X) = 0$ needs a special treatment. In section 2 we briefly recall univariate and bivariate domains of attraction and we discuss domains of attraction of the real random vector (X, X^2). In section 3 we use transformations to recover many results concerning $T(n)$, $SV(n)$ and $t(n)$. We finish the paper with some concluding remarks.

We assume the reader is familiar with regular variation. For further use, recall the definition of regular variation: a positive and measurable function $g(x)$ is regularly varying with real index α (notation $g \in RV(\alpha)$) if as $t \to \infty$, $g(ty)/g(t) \to y^\alpha$, $\forall y > 0$. It can be proved that the defining convergence holds locally uniformly (l.u.) with respect to $y > 0$. For this and other properties and applications of $RV(\alpha)$, we refer to Seneta (1976), Geluk and de Haan (1987) and Bingham et.al. (1989). For a recent survey paper, see Jessen and Mikosch (2006).

2. Domains of attraction

In this section we briefly discuss known results about univariate and bivariate domains of attraction.

2.1. Univariate case. Recall that the random variable X belongs to the domain of attraction of a stable law $Y(\alpha)$ with parameter α, $0 < \alpha \leqslant 2$, if there exist positive numbers $a(n)$ and real numbers $c(n)$ so that

$$\frac{S(n) - c(n)}{a(n)} \overset{d}{\Longrightarrow} Y(\alpha). \tag{2.1}$$

Notation: $X \in D(Y(\alpha))$. Here $S(n) = X_1 + X_2 + \cdots + X_n$ is the sequence of partial sums generated by i.i.d. copies of X and $\overset{d}{\Longrightarrow}$ denotes convergence in distribution, i.e., (2.1) is the same as

$$\lim_{n\to\infty} P\Big(\frac{S(n) - c(n)}{a(n)} \leqslant x\Big) = P\big(Y(\alpha) \leqslant x\big).$$

Let $F(x) = P(X \leqslant x)$, $F_{|X|}(x) = P(|X| \leqslant x)$ and let $K_X(x)$ denote the truncated second moment function, i.e.

$$K_X(x) = \int_{-x}^{x} y^2 dF(y) = E\big(X^2 I\{|X| \leqslant x\}\big).$$

The tails are given by $\overline{F}(x) = P(X > x) = 1 - F(x)$ and by $\overline{F}_{|X|}(x) = P(|X| > x) = \overline{F}(x) + F(-x)$. The following result is well known, cf. Feller (1971), Petrov (1995).

THEOREM 2.1. (i) *For* $0<\alpha<2$, *we have* $X \in D(Y(\alpha))$ *iff* $K_X(x) \in RV(2-\alpha)$ *and*

$$\lim_{x\to\infty} \overline{F}(x)/\overline{F}_{|X|}(x) = p, \lim_{x\to\infty} F(-x)/\overline{F}_{|X|}(x) = q,$$

where $0 \leqslant p = 1 - q \leqslant 1$.
(ii) *If* X *is not concentrated in* 1 *point, then* $X \in D(Y(2))$ *iff* $K_X(x) \in RV(0)$.
(iii) *For* $0 < \alpha \leqslant 2$, $K_X(x) \in RV(2-\alpha)$ *holds iff*

$$\lim_{x\to\infty} x^2\overline{F}_{|X|}(x)/K_X(x) = (2-\alpha)/\alpha,$$

and for $0 < \alpha < 2$, *this holds iff* $\overline{F}_{|X|}(x) \in RV(-\alpha)$.

REMARK 2.1. 1) Note that $K_X(x) = K_{|X|}(x)$. It follows that for $0 < \alpha < 2$, $X \in D(Y(\alpha))$ implies that $|X| \in D(Y^*(\alpha))$ for some α-stable law $Y^*(\alpha)$.
2) If $\alpha = 2$, $\overline{F}_{|X|}(x) \in RV(-2)$ implies that $X \in D(Y(2))$ and $|X| \in D(Y^*(2))$.

We have some freedom in choosing the normalizing sequences $\{a(n)\}$ and $\{c(n)\}$. In our paper, we use the normalizing constants by replacing x by n in the functions $a(x)$ and $c(x)$ defined as follows:

- For $\alpha = 2$, $a(x) \in RV(1/2)$ is determined by the asymptotic relation

$$xK_X(a(x))/a^2(x) \to 1. \tag{2.2}$$

If $\mu_2 = E(X^2) < \infty$, then $K_X(x) \to \mu_2$ and $a^2(n) \sim n\mu_2$.

- If $0 < \alpha < 2$, $a(x) \in RV(1/\alpha)$ is determined by the asymptotic relation

$$x\overline{F}_{|X|}(a(x)) \to 1. \tag{2.3}$$

- If $0 < \alpha < 1$, we choose $c(x) = 0$. If $1 < \alpha \leqslant 2$, we have $E|X| < \infty$ and we choose $c(x) = x\mu = xE(X)$.
- If $\alpha = 1$, then $c(x)$ is given by the relation $c(x) = xa(x)E\big(\sin(X/a(x))\big)$, where $a(x)$ is determined by (2.3). If $X \geqslant 0$, we can use $c(x) = xm(a(x))$ where $a(x)$ is given by (2.3) and where $m(x)$ denotes the integrated tail

$$m(x) = \int_0^x \overline{F}(t)\, dt.$$

Note that if $\alpha = 1$ and $E|X| < \infty$, then $c(x)/x \to E(X)$.

2.2. Multivariate case. The random vector (X, Y) belongs to a multivariate domain of attraction of a bivariate stable vector $(Y_1(\alpha), Y_2(\beta))$ if we can find sequences of constants $a(n) > 0, b(n) > 0$ and $c(n), d(n)$ such that

$$\Big(\frac{S_X(n) - c(n)}{a(n)}, \frac{S_Y(n) - d(n)}{b(n)}\Big) \overset{d}{\Longrightarrow} \big(Y_1(\alpha), Y_2(\beta)\big) \tag{2.4}$$

where $S_X(n)$ and $S_Y(n)$ are partial sums of independent copies of (X, Y). Notation $(X, Y) \in D\big(Y_1(\alpha), Y_2(\beta)\big)$. Assuming that $Y_1(\alpha)$ and $Y_2(\beta)$ are nondegenerate, the normalizing constants are determined by the convergence of the marginals in (2.4) and we use the normalizing constants given as before. We denote the d.f. of (X, Y) by $F(x, y)$. In the multivariate case, the following result was initiated by Rvaceva (1962) and further analyzed by Greenwood and Resnick (1979), see also de Haan et al. (1984). For a point process approach we refer to Resnick (1986).

THEOREM 2.2. *Suppose $X \in D(Y_1(\alpha))$, $Y \in D(Y_2(\beta))$ and $0 < \alpha, \beta \leqslant 2$.*
(i) Let $\nu(.)$ denote the Lévy-measure of the bivariate stable law $(Y_1(\alpha), Y_2(\beta))$. If $0<\alpha, \beta<2$, then $(X, Y) \in D(Y_1(\alpha), Y_2(\beta))$ iff for all Borel sets $V \in B(\mathbb{R}^2 \setminus \{(0,0)\})$ with $\nu(\delta V) = 0$ and $\nu(V) < \infty$ we have

$$\lim_{t\to\infty} tP\Big(\Big(\frac{X}{a(t)}, \frac{Y}{b(t)}\Big) \in V\Big) = \nu(V).$$

(ii) If $\alpha = \beta = 2$, then $(X, Y) \in D\big(Y_1(2), Y_2(2)\big)$ iff for each $x, y > 0$ we have

$$\lim_{t\to\infty} \frac{tW\big(a(t)x, b(t)y\big)}{a(t)b(t)} = C$$

where C denotes a constant and where $W(x, y)$ is given by

$$W(x, y) = \int_{|u|\leqslant x} \int_{|v|\leqslant y} uv\, dF(u, v), \quad x, y \geqslant 0.$$

(iii) If $0 < \alpha < 2$ and $\beta = 2$, then $(X, Y) \in D\big(Y_1(\alpha), Y_2(2)\big)$ and $Y_1(\alpha)$ and $Y_2(2)$ are independent.

2.3. The case where $\boldsymbol{X = (X, X^2)}$. We can apply Theorem 2.2 to obtain the following result for the vector (X, X^2).

THEOREM 2.3. (i) *For $0 < \alpha < 2$ we have $(X, X^2) \in D\big(Y_1(\alpha), Y_2(\alpha/2)\big)$ iff $X \in D(Y_1(\alpha))$.*
(ii) For $2 \leqslant \alpha < 4$ we have $(X, X^2) \in D\big(Y_1(2), Y_2(\alpha/2)\big)$ iff $X^2 \in D(Y_2(\alpha/2))$ and also, if $\alpha = 2$, $X \in D(Y_1(2))$. Moreover, $Y_1(2)$ and $Y_2(\alpha/2)$ are independent.
(iii) We have $(X, X^2) \in D\big(Y_1(2), Y_2(2)\big)$ if and only if $X^2 \in D(Y_2(2))$

PROOF. (i) If $(X, X^2) \in D\big(Y_1(\alpha), Y_2(\alpha/2)\big)$ then we automatically have $X \in D(Y_1(\alpha))$. To prove the converse, suppose first that $x, y > 0$. In this case we have

$$tP\Big(\frac{X}{a(t)} > x, \frac{X^2}{a^2(t)} > y\Big) = tP\Big(\frac{X}{a(t)} > \max(x, \sqrt{y})\Big)$$

It follows from Theorem 2.1 (i) and our choice of $a(t)$ that

$$tP\Big(\frac{X}{a(t)} > x, \frac{X^2}{a^2(t)} > y\Big) \to p\big(\max(x, \sqrt{y})\big)^{-\alpha}.$$

For $x < 0 < y$ we have

$$tP\Big(\frac{X}{a(t)} < x, \frac{X^2}{a^2(t)} > y\Big) \to q\big(\min(x, -\sqrt{y})\big)^{-\alpha}.$$

Now the result follows.

(ii) This is case (iii) of Theorem 2.2.

(iii) If $X^2 \in D(Y_2(2))$, then $EX^3 < \infty$, $X \in D(Y_1(2))$ and $a^2(t) \sim tE(X^2)$. For $b(t)$ we have $b^2(t) \sim tK_2(b(t))$ where $K_2(x) = K_{X^2}(x)$. Now consider $W(x, y)$. In our case we have $W(x, y) = E\big(X^3 I\{|X| \leqslant \min\big(x, \sqrt{y}\,\big)\}\big), \quad x, y \geqslant 0$, and it follows that $W\big(a(t)x, b(t)y\big) \to E(X^3)$. On the other hand, using the convention that $1/\infty = 0$, we have

$$\frac{t}{a(t)b(t)} \to \sqrt{\frac{1}{E(X^2)E(X^4)}}.$$

We conclude that

$$\frac{tW(a(t)x, b(t)y)}{a(t)b(t)} \to C = \frac{E(X^3)}{\sqrt{E(X^2)E(X^4)}}. \qquad \square$$

REMARK 2.2. In case (ii) of Theorem 2.3 note that $X^2 \in D(Y_2(\alpha/2))$, $2 < \alpha < 4$ implies that $E(X^2) < \infty$ so that $X \in D(Y_1(2))$.

REMARK 2.3. Theorem 2.3 gives conditions under which

$$\left(\frac{\sum_{i=1}^n X_i - c(n)}{a(n)}, \frac{\sum_{i=1}^n X_i^2 - d(n)}{b(n)}\right) \overset{d}{\Longrightarrow} (Y_1(u), Y_2(v)) \tag{2.5}$$

for some numbers u and v. For further use, in this remark we give the precise form of the normalizing sequences.

(i) If $0 < \alpha < 2$, then in (2.5) we have $u = \alpha$ and $v = \alpha/2$. We can use (2.3) to see that $b(n) = a^2(n)$. Since $\alpha/2 < 1$, we can take $d(n) = 0$ and $c(n)$ according to the different cases of Section 2.1.

(ii) If $2 < \alpha < 4$, then in (2.5) we have $u = 2$ and $v = \alpha/2$. We take $c(n) = n\mu$ and $a(n)$ is determined by (2.2). The sequence $b(n)$ is determined by the relation $n(1 - F_2(b(n))) \to 1$. Finally, we have $d(n) = n\mu_2$.

(iii) If $\alpha = 2$ we have $u = 2$ and $v = 1$. The sequences $a(n), b(n), c(n)$ are determined as in (ii). For $d(n)$ we take (recall that $X^2 \geqslant 0$) $d(n) = nm_2(b(n))$ where $m_2(x) = \int_0^x \overline{F}_2(u)du$.

(iv) In case (iii) of Theorem 2.3, in (2.5) we have $u = v = 2$. Now we take $c(n) = n\mu$ and $d(n) = n\mu_2$. The sequence $a(n)$ is determined by (2.2) and $b(n)$ is determined by $b^2(n) \sim nK_2(b(n))$, where $K_2(x) = E\big(X^4 I\{X^2 \leqslant x\}\big)$.

3. Applications

As mentioned in the introduction, many characteristics in statistics are based on $\big(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\big)$. As examples we mention the sample variance $s^2(n)$, the sample coefficient of variation $SV(n) = s_n/\overline{X}$, and the non-central t-statistic $t(n) = \sqrt{n}\,\overline{X}/s_n$. In this section, we assume that X is a continuous random variable and, as in Ladoucette and Teugels (2006), we define $T(n)$ as follows:

$$T(n) = \frac{\sum_{i=1}^n X_i^2}{\big(\sum_{i=1}^n X_i\big)^2}.$$

Note that $SV^2(n) = nT(n) - 1$ and $t^2(n) = n/(nT(n) - 1)$. Clearly $T(n)$ plays an important role in studying $SV(n)$ and $t(n)$. Without loss of generality, if the mean μ is finite, then we assume that $\mu \geqslant 0$. Otherwise we replace X by $-X$.

3.1. The asymptotic behaviour of $T(n)$. Using the notations of Theorem 2.3 and Remark 2.3, we have the following result. The result completes the proofs of the corresponding results of Albrecher and Teugels (2004) or Ladoucette and Teugels (2006), Ladoucette (2007), Omey (2008a). In Theorem 3.1 below we consider the case where $\alpha \neq 1$. The case where $\alpha = 1$ follows in Remark 3.1 below.

THEOREM 3.1. (i) *Suppose that* $X \in D(Y_1(\alpha))$ *with* $0 < \alpha < 1$ *or with* $1 < \alpha < 2$ *and* $\mu = 0$. *Then*

$$T(n) \overset{d}{\Longrightarrow} \frac{Y_2(\alpha/2)}{Y_1^2(\alpha)}.$$

(ii) *Suppose that* $X \in D(Y_1(\alpha))$ *with* $1 < \alpha < 2$ *and* $\mu \neq 0$. *Then*

$$\frac{n^2}{a^2(n)} T(n) \overset{d}{\Longrightarrow} \frac{1}{\mu^2} Y_2(\alpha/2).$$

(iii) *Suppose that* $X^2 \in D(Y_2(1))$ *and* $X \in D(Y_1(2))$.

(a) *If* $\mu \neq 0$, *then* $\dfrac{n}{b(n)}\Big(nT(n) - \dfrac{d(n)}{n\mu^2}\Big) \overset{d}{\Longrightarrow} \dfrac{1}{\mu^2} Y_2(1)$.

(b) *If* $\mu = 0$, *then* $\dfrac{a^2(n)}{d(n)} T(n) \overset{d}{\Longrightarrow} \dfrac{1}{Y_1^2(2)}$.

(iv) *Suppose that* $X^2 \in D(Y_2(\alpha/2))$ *with* $2 < \alpha < 4$ *and with* $\mu \neq 0$. *Then*

$$\frac{n}{b(n)}\Big(nT(n) - \frac{\mu_2}{\mu^2}\Big) \overset{d}{\Longrightarrow} \frac{1}{\mu^2} Y_2(\alpha/2).$$

(v) *Suppose that* $X^2 \in D(Y_2(2))$ *and* $\mu \neq 0$. *Then*

$$\frac{n}{b(n)}\Big(nT(n) - \frac{\mu_2}{\mu^2}\Big) \overset{d}{\Longrightarrow} Y_3(2)$$

where $Y_3(2) \overset{d}{=} \dfrac{1}{\mu^2} Y_2(2) - 2\dfrac{\mu_2\sqrt{\mu_2}}{\mu^3\sqrt{\mu_4}} Y_1(2)$ *if* $\mu_4 = E(X^4) < \infty$ *and* $Y_3(2) \overset{d}{=} Y_2(2)/\mu^2$ *otherwise.*

(vi) *If* $\mu_2 < \infty$ *and* $\mu = 0$, *then* $T(n) \overset{d}{\Longrightarrow} 1/Y_1^2(2)$.

PROOF. (i) If $0 < \alpha < 1$ we have

$$\left(\frac{\sum_{i=1}^n X_i}{a(n)}, \frac{\sum_{i=1}^n X_i^2}{a^2(n)}\right) \overset{d}{\Longrightarrow} (Y_1(\alpha), Y_2(\alpha/2)). \tag{3.1}$$

Now it follows that

$$P(T(n) \leqslant x) = P\left(\frac{1}{a^2(n)} \sum_{i=1}^n X_i^2 \leqslant x\left(\frac{1}{a(n)} \sum_{i=1}^n X_i\right)^2\right)$$

so that

$$P(T(n) \leqslant x) \to P(Y_2(\alpha/2) \leqslant xY_1^2(\alpha)). \tag{3.2}$$

If $1 < \alpha < 2$ and $\mu = 0$, we still have (3.1) and then again (3.2) follows.

(ii) Since $\mu < \infty$, we have

$$\Big(\frac{\sum_{i=1}^n X_i}{n}, \frac{\sum_{i=1}^n X_i^2}{a^2(n)}\Big) \overset{d}{\Longrightarrow} (\mu, Y_2(\alpha/2))$$

and the result follows as in (i).

(iii) (a) Using the notations

$$A(n) = \frac{(\sum_{i=1}^n X_i)^2 - (n\mu)^2}{na(n)}, B(n) = \frac{\sum_{i=1}^n X_i^2 - d(n)}{b(n)},$$

in this case we have

$$(A(n), B(n)) \overset{d}{\Longrightarrow} (2\mu Y_1(2), Y_2(1)). \tag{3.3}$$

To prove the result, we consider

$$I = P\Big(\frac{d(n)}{b(n)}\Big(\frac{n^2 T(n)}{d(n)} - \frac{1}{\mu^2}\Big) \leqslant x\Big).$$

Using the definition of $T(n)$, we obtain that

$$\begin{aligned} I &= P\Big(\sum_{i=1}^n X_i^2 - \Big(\frac{d(n)}{\mu^2 n^2} + \frac{xb(n)}{n^2}\Big)\Big(\sum_{i=1}^n X_i\Big)^2 \leqslant 0\Big) \\ &= P\Big(B(n)b(n) - \Big(\frac{d(n)}{\mu^2 n^2} + \frac{xb(n)}{n^2}\Big)A(n)na(n) \leqslant xb(n)\mu^2\Big) \\ &= P\Big(B(n) - \Big(\frac{a(n)d(n)}{\mu^2 b(n) n} + \frac{a(n)x}{n}\Big)A(n) \leqslant x\mu^2\Big). \end{aligned}$$

Now recall that $A(n) \overset{d}{\Longrightarrow} 2\mu Y_2(2)$ and observe that $a(x) \in RV(1/2)$. Also note that $b(x) \in RV(1)$ and that $d(x) \in RV(1)$. It follows that $a(x)/x \to 0$ and that $a(x)d(x)/(xb(x)) \to 0$. Using (3.3) we conclude that $I \to P(Y_2(1) \leqslant x\mu^2)$.

(iii)(b) In this case we have

$$\Big(\frac{(\sum_{i=1}^n X_i)^2}{a^2(n)}, \frac{\sum_{i=1}^n X_i^2}{d(n)}\Big) \overset{d}{\Longrightarrow} (Y_1^2(2), 1)$$

and the result follows as in (i).

(iv) In this case we have $d(n) = n\mu_2$ and

$$\big(A(n), B(n)\big) \overset{d}{\Longrightarrow} \big(2\mu Y_1(2), Y_2(\alpha/2)\big) \tag{3.4}$$

where $a^2(n) \sim n\mu_2$ and $b(x) \in RV(2/\alpha)$. Now consider

$$I = P\Big(\frac{n}{b(n)}\Big(nT(n) - \frac{\mu_2}{\mu^2}\Big) \leqslant x\Big).$$

By using the definition of $T(n)$, we find that

$$I = P\Big(\sum_{i=1}^{n} X_i^2 - \Big(\sum_{i=1}^{n} X_i\Big)^2 \Big(\frac{xb(n)}{n^2} + \frac{\mu_2}{n\mu^2}\Big) \leqslant 0\Big)$$
$$= P\Big(B(n)b(n) - \Big(\frac{xb(n)}{n^2} + \frac{\mu_2}{n\mu^2}\Big) na(n)A(n) \leqslant xb(n)\mu^2\Big)$$
$$= P\Big(B(n) - \Big(\frac{a(n)x}{n} + \frac{a(n)\mu_2}{b(n)\mu^2}\Big) A(n) \leqslant \mu^2 x\Big).$$

Since $a(n)/n \to 0$ and $a(n)/b(n) \to 0$, using (3.4), we conclude that $I \to P(Y_2(\alpha/2) - 0 \leqslant \mu^2 x)$ and this proves the result.

(v) Now we have $\big(A(n), B(n)\big) \overset{d}{\Longrightarrow} \big(2\mu Y_1(2), Y_2(2)\big)$ where, cf. (2), $a^2(n) \sim n\mu_2$. As in case (iv) we consider I and again we find that

$$I = P\Big(B(n) - \Big(\frac{a(n)x}{n} + \frac{a(n)\mu_2}{b(n)\mu^2}\Big) A(n) \leqslant \mu^2 x\Big).$$

Now we have to distinghuish between two cases. If $\mu_4 < \infty$, we have $b^2(n) \sim n\mu_4$ and it follows that

$$I \to P\Big(Y_2(2) - \frac{\mu_2\sqrt{\mu_2}}{\mu^2\sqrt{\mu_4}} 2\mu Y_1(2) \leqslant \mu^2 x\Big).$$

In the case where $\mu_4 = \infty$, we have

$$\frac{a^2(n)}{b^2(n)} \sim \frac{n\mu_2}{b^2(n)} \sim \frac{\mu_2}{V_2(b(n))} \to 0.$$

and we find that $I \to P(Y_2(2) \leqslant \mu^2 x)$.

(vi) In this case we have

$$\left(\frac{(\sum_{i=1}^n X_i)^2}{a^2(n)}, \frac{\sum_{i=1}^n X_i^2}{n}\right) \overset{d}{\Longrightarrow} (Y_1^2(2), \mu_2)$$

and $a^2(n) \sim n\mu_2$. Now the result follows. □

REMARK 3.1. 1) If $X^2 \in D(Y_2(\alpha/2))$ with $2 < \alpha \leqslant 4$ and $\mu = 0$, then $\mu_2 < \infty$ and case (vi) applies.

2) In the case where $\alpha = 1$, we have to be more careful. If $\mu = E(X)$ is finite, we have $\sum_{i=1}^n X_i/n \overset{p}{\to} \mu$ and if $\mu \neq 0$, we can proceed as in case (ii) to obtain that

$$\frac{n^2}{a^2(n)} T(n) \overset{d}{\Longrightarrow} \frac{1}{\mu^2} Y_2(1/2).$$

If $\mu = 0$ or if $\mu = \infty$, we assume that either X is symmetric around 0 so that $c(n) = 0$, or that $\sum_{i=1}^n X_i/c(n) \overset{p}{\to} 1$. In the first case, as in case (i), we obtain that

$$T(n) \overset{d}{\Longrightarrow} \frac{Y_2(1/2)}{Y_1^2(1)}.$$

In the second case, we obtain that

$$\frac{c^2(n)}{a^2(n)} T(n) \overset{d}{\Longrightarrow} Y_2(1/2).$$

3.2. The asymptotic behaviour of $SV(n)$. Now we use Theorem 3.1 to obtain the precise asymptotic behaviour of the sample coefficient of variation. Recall that $SV^2(n) = nT(n) - 1$ and that if μ is finite, we assume that $\mu \geqslant 0$. If $\mu > 0$ it follows that for $n \to \infty$, $SV(n) = \sqrt{nT(n) - 1} > 0$. In the result below we use the notation $\sigma^2 = \mu_2 - \mu^2$. In the case where $X \geqslant 0$, the result was obtained in Omey (2008a) and partially in Ladoucette and Teugels (2006).

THEOREM 3.2. (i) *Suppose that* $X \in D(Y_1(\alpha))$ *with* $0 < \alpha < 1$ *or* $X \in D(Y_1(\alpha))$ *with* $1 < \alpha < 2$ *and* $\mu = 0$. *Then*

$$\frac{SV^2(n)}{n} \overset{d}{\Longrightarrow} \frac{Y_2(\alpha/2)}{Y_1^2(\alpha)}.$$

(ii) *Suppose that* $X \in D(Y_1(\alpha))$ *with* $1 < \alpha < 2$ *and* $\mu \neq 0$. *Then*

$$\frac{\sqrt{n}}{a(n)} SV(n) \overset{d}{\Longrightarrow} \frac{1}{\mu}\sqrt{Y_2(\alpha/2)}.$$

(iii) *Suppose that* $X^2 \in D(Y_2(1))$ *and* $X \in D(Y_1(2))$.

(a) *If* $\mu \neq 0$, *then*

$$\frac{n\sqrt{c(n)}}{b(n)}\big(SV(n) - \sqrt{c(n)}\,\big) \overset{d}{\Longrightarrow} \frac{1}{2\mu^2} Y_2(1)$$

where $c(n)$ *is given by* $c(n) = d(n)/n\mu^2 - 1$.

(b) *If* $\mu = 0$, *then*

$$\frac{a^2(n)}{nd(n)} SV^2(n) \overset{d}{\Longrightarrow} \frac{1}{Y_1^2(2)}.$$

(iv) *Suppose that* $X^2 \in D(Y_2(\alpha/2))$ *with* $2 < \alpha < 4$ *and* $\mu \neq 0$. *Then*

$$\frac{n}{b(n)}\Big(SV(n) - \frac{\sigma}{\mu}\Big) \overset{d}{\Longrightarrow} \frac{1}{2\sigma\mu} Y_2(\alpha/2).$$

(v) *Suppose that* $X^2 \in D(Y_2(2))$ *and* $\mu \neq 0$, *then*

$$\frac{n}{b(n)}\Big(SV(n) - \frac{\sigma}{\mu}\Big) \overset{d}{\Longrightarrow} \frac{\mu}{2\sigma} Y_3(2)$$

where $Y_3(2)$ *is given in Theorem* 3.1(v).

(vi) *If* $\mu_2 < \infty$ *and* $\mu = 0$. *Then*

$$\frac{SV^2(n)}{n} \overset{d}{\Longrightarrow} \frac{1}{Y_1^2(2)}$$

PROOF. (i) and (ii) follow immediately from Theorem 3.1 (i),(ii).

(iii) (a) From Theorem 3.1 (iii) we obtain that

$$\frac{n}{b(n)}\big(SV^2(n) - c(n)\big) \overset{d}{\Longrightarrow} \frac{1}{\mu^2} Y_2(1)$$

where $c(n) = d(n)/n\mu^2 - 1$. Since $d(n)/b(n) \to \infty$ and $d(n)/n \to \infty$, we obtain that

$$\frac{SV(n)}{\sqrt{c(n)}} \overset{p}{\longrightarrow} 1.$$

Now observe that

$$\frac{n\sqrt{c(n)}}{b(n)}\big(SV(n)-\sqrt{c(n)}\,\big)=\frac{n\big(SV^2(n)-c(n)\big)}{b(n)}\frac{\sqrt{c(n)}}{SV(n)+\sqrt{c(n)}}\overset{d}{\Longrightarrow}\frac{1}{2\mu^2}Y_2(1)$$

(b) In this case, Theorem 3.1 (vi) gives

$$\frac{a^2(n)}{d(n)}\frac{1+SV^2(n)}{n}\overset{d}{\Longrightarrow}\frac{1}{Y_1^2(2)}.$$

Since $a^2(n)/nd(n)\to 0$, we obtain the result.

(iv) First note that Theorem 3.1(iv) shows that $nT(n)\overset{p}{\longrightarrow}\mu_2/\mu^2$ so that $SV(n)\overset{p}{\longrightarrow}\sigma/\mu$ where $\sigma^2=\mu_2-\mu^2$. Now we have

$$SV(n)-\frac{\sigma}{\mu}=\frac{nT(n)-\mu_2/\mu^2}{\sigma/\mu+SV(n)}.$$

Theorem 3.1(iv) can now be used to obtain the desired result.

(v) Theorem 3.1(v) shows that $nT(n)\overset{p}{\longrightarrow}\mu_2/\mu^2$ so that $SV(n)\overset{p}{\longrightarrow}\sigma/\mu$ where $\sigma^2=\mu_2-\mu^2$. Now we have

$$SV(n)-\frac{\sigma}{\mu}=\frac{nT(n)-\mu_2/\mu^2}{\sigma/\mu+SV(n)}.$$

Theorem 3.1(v) can be used to obtain the desired result.

(vi) Theorem 3.1(vi) shows that

$$\frac{SV^2(n)+1}{n}\overset{d}{\Longrightarrow}\frac{1}{Y_1^2(2)}.$$

Now the result follows. □

Remark 3.2. Also here, the case where $\alpha=1$ can be treated as in Remark 3.1.

3.3. The asymptotic behaviour of $t(n)$. Using the definition of $t(n)$ we see that $t^2(n)=n/SV^2(n)$ and this relation can by used to transfer the asymptotic properties of $SV(n)$ to $t(n)$. Our Theorem 3.3 should be compared to the results of Bentkus et al. (2007). As before, if $\mu<\infty$, we assume that $\mu\geqslant 0$.

Theorem 3.3. (i) *Suppose that $X\in D(Y_1(\alpha))$ with $0<\alpha<1$ or with $1<\alpha<2$ and $\mu=0$. Then*

$$t^2(n)\overset{d}{\Longrightarrow}\frac{Y_1^2(\alpha)}{Y_2(\alpha/2)}.$$

(ii) *Suppose that $X\in D(Y_1(\alpha))$ with $1<\alpha<2$ and $\mu\neq 0$. Then*

$$\frac{a(n)}{n}t(n)\overset{d}{\Longrightarrow}\mu\sqrt{\frac{1}{Y_2(\alpha/2)}}.$$

(iii) *Suppose that $X\in D(Y_1(2))$ and $X^2\in D(Y_2(1))$.*

(a) *If $\mu\neq 0$, then*

$$\frac{nc(n)\sqrt{c(n)}}{b(n)}\left(\frac{1}{\sqrt{c(n)}}-\frac{t(n)}{\sqrt{n}}\right)\overset{d}{\Longrightarrow}\frac{1}{2\mu^2}Y_2(1)$$

where $c(n) = d(n)/n\mu^2 - 1$.

(b) *If* $\mu = 0$, *then*

$$\frac{d(n)}{a^2(n)} t^2(n) \stackrel{d}{\Longrightarrow} Y_1^2(2).$$

(iv) *Suppose that* $X^2 \in D(Y_2(\alpha/2))$ *with* $2 < \alpha < 4$ *and with* $\mu \neq 0$. *Then*

$$\frac{n}{b(n)} \frac{\sigma}{\mu} \left(\frac{\mu}{\sigma} - \frac{t(n)}{\sqrt{n}} \right) \stackrel{d}{\Longrightarrow} \frac{1}{2\sigma^2} Y_2(\alpha/2)$$

(v) *Suppose that* $X^2 \in D(Y_2(2))$ *with* $\mu \neq 0$. *Then*

$$\frac{n}{b(n)} \frac{\sigma}{\mu} \left(\frac{\mu}{\sigma} - \frac{t(n)}{\sqrt{n}} \right) \stackrel{d}{\Longrightarrow} \frac{1}{2} Y_3(2)$$

where $Y_3(2)$ *is given in Theorem* 3.1(v).

(vi) *If* $\mu_2 < \infty$ *and* $\mu = 0$, *then* $t(n) \stackrel{d}{\Longrightarrow} Y_1(2)$.

PROOF. (i), (ii) This follows immediately from Theorem 3.2 (i), (ii).

(iii) (a) To prove this result, recall that $t(n)/\sqrt{n} = 1/SV(n)$. Now we write

$$\frac{1}{\sqrt{c(n)}} - \frac{t(n)}{\sqrt{n}} = \frac{SV(n) - \sqrt{c(n)}}{SV(n)\sqrt{c(n)}}$$

From Theorem 3.2 (iii)(a), we know that $SV(n)/\sqrt{c(n)} \stackrel{p}{\longrightarrow} 1$ and then we obtain that

$$\frac{nc(n)\sqrt{c(n)}}{b(n)} \left(\frac{1}{\sqrt{c(n)}} - \frac{t(n)}{\sqrt{n}} \right) = \frac{n\sqrt{c(n)}}{b(n)} \left(SV(n) - \sqrt{c(n)} \right) \frac{c(n)}{SV(n)\sqrt{c(n)}} \stackrel{d}{\Longrightarrow} \frac{1}{2\mu^2} Y_2(1)$$

(b) Now we use $t^2(n) = n/SV^2(n)$ and Theorem 3.2 (iii)(b)

(iv) To prove this result, as in the proof of (iii)(a) we write

$$\frac{\mu}{\sigma} - \frac{t(n)}{\sqrt{n}} = \frac{SV(n) - \sigma/\mu}{SV(n)\sigma/\mu}$$

and then we obtain

$$\frac{n}{b(n)} \frac{\sigma}{\mu} \left(\frac{\mu}{\sigma} - \frac{t(n)}{\sqrt{n}} \right) = \frac{n}{b(n)} \frac{SV(n) - \sigma/\mu}{SV(n)} \stackrel{d}{\Longrightarrow} \frac{1}{2\sigma^2} Y_2(\alpha/2)$$

(v) Similar as the proof of part (iv).

(vi) If $\mu_2 < \infty$ and $\mu = 0$ we have $s_n^2 \to \mu_2$ and $\sqrt{n}\overline{X}/\sqrt{\mu_2} \stackrel{d}{\Longrightarrow} Y_1(2)$. The result follows. □

4. Concluding remarks

1) The reader can formulate similar asymptotic results for the sample dispersion $SD(n) = s_n^2/\overline{X}$.

2) The case where $\alpha = 1$ needs to be investigated further. In the case of $t(n)$, Bentkus et al. (2007) use the results of Griffin (2002) and obtain the asymptotic behaviour of $A(n)(t(n) - B(n))$ for suitable sequences $A(n)$ and $B(n)$.

3) There are many statistics that use higher sample moments. In a forthcoming paper Omey (2008b) we analyze domains of attraction of the random vector $(X, X^2, \dots, X^k)$ and then apply the results to these statistics.

4) The coefficient of variation and the sample dispersion are widely used measures of variation. For applications in the context of insurance and actuarial risk, we refer to Albrecher and Teugels (2004), Ladoucette (2007) and the references given there. In portfolio theory, the very popular ratio of Sharpe turns out to be given by $1/SV(n)$, cf. Sharpe (1966), Knight and Satchell (2005). The coefficient of variation is also used as a performance measure in queueing systems and in simulation, cf. Krishnamurthy and Suri (2006).

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