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A RELATION BETWEEN FOURIER COEFFICIENTS OF HOLOMORPHIC CUSP FORMS AND EXPONENTIAL SUMS

Anne-Maria Ernvall-Hytönen

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ABSTRACT. We consider certain specific exponential sums related to holomorphic cusp forms, give a reformulation for the Lehmer conjecture and prove that certain exponential sums of Fourier coefficients of holomorphic cusp forms contain information on other similar non-overlapping exponential sums. Also, we prove an Omega result for short sums of Fourier coefficients.

1. Introduction

Holomorphic cusp forms can be represented as Fourier series

$$F(z) = \sum_{n=1}^{\infty} a(n) n^{(\kappa-1)/2} e(nz),$$

where $\operatorname{Im} z > 0$, $e(x) = e^{2\pi i x}$, and the numbers $a(n)$ are called normalized Fourier coefficients and κ is the weight of the form; see e.g. [**1**] or [**13**] for an account of the theory of holomorphic modular forms. For properties of exponential sums and related techniques, see [**10**].

It is of interest to consider exponential sums of the normalized Fourier coefficients:

$$A(M, \Delta, \alpha) = \sum_{M \leqslant n \leqslant M+\Delta} a(n)\, e(n\alpha)$$

with $0 < \Delta \leqslant M$ and $\alpha \in \mathbb{R}$. For similar exponential sums involving the divisor function $d(n) = \sum_{d|n} 1$, the notation $D(M, \Delta, \alpha)$ will be used. Wilton's estimate [**17**]

$$\sum_{n \leqslant M} a(n)\, e(n\alpha) \ll M^{1/2} \log M$$

from the year 1929 is a classical result. This estimate is nearly sharp, only the logarithm can be removed and that was done by Jutila in 1987 [**11**]. Therefore,

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moving the focus to short sums was a logical next step. Karppinen and Ernvall-Hytönen [**5**] proved that, for $1 \leqslant \Delta \ll M^{3/4}$,

$$A(M,\Delta,\alpha) \ll \begin{cases} \Delta M^{\varepsilon}, & \text{when } 1 \leqslant \Delta \ll M^{2/5} \\ \Delta^{1/6} M^{1/3+\varepsilon}, & \text{when } M^{2/5} \ll \Delta \ll M^{5/8} \\ \Delta M^{-9/48+\varepsilon}, & \text{when } M^{5/8} \ll \Delta M^{11/16} \\ M^{-1/4}\Delta + M^{1/2-1/32+\varepsilon}, & \text{when } M^{11/16} \ll \Delta M^{3/4}. \end{cases}$$

In this article, we will consider the sum

$$\sum_{M \leqslant n \leqslant M+\Delta} c(n) e\left(\frac{n\sqrt{k}}{\sqrt{M}}\right) w(n),$$

where $c(n)$ is either $a(n)$ or $d(n)$, $k \in \mathbb{N}$, and w is a smooth weight function. In particular, we will show a connection between this sum with $c(n) = a(n)$ and the coefficient $a(k)$. For $k = 1$, such a relation was established in [**5**] for $c(n) = a(n)$ and in [**4**] for $c(n) = d(n)$. We will also show that this sum contains information about similar shifted (not necessarily overlapping) sums.

Also, we will show the Ω-result

$$\sum_{M \leqslant n \leqslant M+c\sqrt{M}} a(n) = \Omega\big(M^{1/4}\big),$$

where the Ω-symbol is to be understood in the following way: $f = \Omega\,(g)$ if $f = o(g)$ does not hold. The question of good Ω-results has been earlier tackled by several mathematicians, Joris [**9**], Redmond [**16**], Corrádi and Katai [**2**], to mention a few. In 1989, Ivić and Hafner [**6**] proved the existence of a positive constant D such that

$$\sum_{n \leqslant M} a(n) n^{(\kappa-1)/2} = \Omega_{\pm}\left(M^{\kappa/2-1/4} \exp\left(D \frac{(\log\log M)^{1/4}}{(\log\log\log M)^{3/4}}\right)\right),$$

where $\Omega_{\pm}$ means the following: $f = \Omega_{\pm}(g)$ if $\limsup f/g > 0$ and $\liminf f/g < 0$. One year later appeared Ivić's paper [**8**] in which he showed that there are $A, B, T_0 > 0$ such that, for $T \geqslant T_0$, every interval $\big[T, T + A\sqrt{T}\,\big]$ contains t_1 and t_2 for which $A(1, t_1 - 1, 0) > Bt_1^{1/4}$ and $A(1, t_2 - 1, 0) < -Bt_2^{1/4}$. Very recently, Ivić [**7**] proved an Ω-result for short sums:

$$A(M,\Delta,1) = \Omega\big(\sqrt{\Delta}\,\big)$$

when $M^{\varepsilon} \leqslant \Delta \leqslant M^{1/2-\varepsilon}$. The result in this article extends this result by treating the “missing" case $\Delta \asymp M^{1/2}$.

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2. Preliminaries

Let us begin with

Definition 2.1. Given $X, Y, Z \in \mathbb{R}$ we write

$$d(X,Y,Z) = \{x \in \mathbb{C} : \ \exists y \in [X,Y] : |x-y| < Z\}.$$

Now we may state a lemma [**12**, Lemma 6] which will be used repeatedly in this article:

LEMMA 2.1. *Let A be a function which is compactly supported in a finite interval $[M_1, M_2]$ and at least $P \geqslant 0$ times differentiable. Assume also that there exist two natural numbers A_0 and A_1 such that for any non-negative integer $\nu \leqslant P$ and for any $x \in [M_1, M_2]$,*

$$A^{(\nu)}(x) \ll A_0 A_1^{-\nu}.$$

Also, let B be a function which is real-valued on $[M_1, M_2]$, and regular throughout the complex domain $d(M_1, M_2, \rho)$; and assume that there exists a quantity B_1 such that

$$0 < B_1 \ll |B'(x)|$$

for any point x in the domain. Then we have

$$\int_{-\infty}^{\infty} A(x)\, e(B(x))\, dx \ll A_0 (A_1 B_1)^{-P} \Big(1 + \frac{A_1}{\varrho}\Big)^P (M_2 - M_1).$$

3. Connecting exponential sums and individual coefficients

The following theorem was proved in [**5**]:

THEOREM 3.1. *Let $M^{1/2+\delta} < \Delta \leqslant \lambda M^{3/4}$, where $0 < \lambda < 1$ is a constant. Let w be a smooth weight function on the interval $[M, M + \Delta]$ which equals 1 on the interval $[a, b] \subset [M, M + \Delta]$ where $a - M = M + \Delta - b = \Delta^{1-\delta}$ with δ a sufficiently small fixed positive real number. Assume further that $\alpha = M^{-1/2}$. Then*

$$\Big| \sum_{M \leqslant n \leqslant M+\Delta} a(n)\, w(n)\, e(\alpha n) \Big| \asymp \Delta M^{-1/4}.$$

The symbol $\asymp$ has to be understood in the following way: $f \asymp g$ if $f = O(g)$ and $g = O(f)$.

However, the following more general theorem holds:

THEOREM 3.2. *Let $M^{1/2+\theta} \ll \Delta \leqslant \lambda M^{3/4}$ and $0 \leqslant T \leqslant M^{3/4}$, where $0 < \lambda \leqslant 1/\sqrt{k}$ is a constant, θ an arbitrarily small fixed positive number, k a positive integer, and let w be a smooth weight function on the interval $[M, M + \Delta]$ such that w is a constant function 1 on the interval $[a, b] \subset [M, M + \Delta]$ where $a - M, M + \Delta - b = \Delta^{1-\delta}$ with $\delta < \frac{2\theta}{1+2\theta}$ a sufficiently small fixed positive real number. Then*

$$\sum_{M+T \leqslant n \leqslant M+T+\Delta} c(n)\, w(n - T)\, e\Big(\frac{\sqrt{k}\, n}{\sqrt{M}}\Big)$$
$$= C c(k) k^{-1/4} \int_{M+T}^{M+T+\Delta} x^{-1/4} w(x - T)\, e\Big(\frac{\sqrt{k}}{\sqrt{M}} x - 2\sqrt{kx}\Big)\, dx + O(1),$$

where $c(n) = a(n)$ or $d(n)$ and C is a constant depending only whether $c(n)$ equals $d(n)$ or $a(n)$ and on the weight of the form.

Notice that the size of the integral is $\asymp M^{-1/4}\Delta$. This can be easily proved using the fact that the exponential part is stationary.

Proof of Theorem 3.2. The proof for $c(n) = a(n)$ with $k = 1$ and $T = 0$ can be found in [**5**] and the proof for both $c(n) = d(n)$ and $c(n) = a(n)$ with $k = 1$ and $T = 0$ can be found in [**4**] and the proof of the above formula is similar. As the case with $c(n) = a(n)$ is easier and similar to the case $c(n) = d(n)$, we are only going to prove the latter case.

Let us first use a Voronoi type summation formula [**10**, Theorem 1.7]

$$D\left(M+T, \Delta, \frac{\sqrt{k}}{\sqrt{M}}\right) = \int_{M+T}^{M+T+\Delta} (\log x + 2\gamma)\, w(x-T)\, e\left(\frac{\sqrt{kx}}{\sqrt{M}}\right) dx$$
$$+ \sum_{n=1}^{\infty} d(n) \int_{M+T}^{M+T+\Delta} \left\{-2\pi Y_0\big(4\pi\sqrt{nx}\,\big) + 4K_0\big(4\pi\sqrt{nx}\,\big)\right\} w(x-T)\, e\left(\frac{\sqrt{kx}}{\sqrt{M}}\right) dx,$$

where Y_0 and K_0 are Bessel functions in the standard notation. The following estimate is well known (see formula (5.16.5) of [**14**])

$$K_0(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}, \quad \text{when } z \to \infty.$$

Therefore, the integral corresponding to the K-function yields

$$\int_{M+T}^{M+T+\Delta} 4K_0\big(4\pi\sqrt{nx}\,\big)\, w(x-T)\, e\left(\frac{\sqrt{kx}}{\sqrt{M}}\right) dx$$
$$\ll \frac{1}{n^{1/4}} \int_{M+T}^{M+T+\Delta} x^{-1/4} e^{-4\pi\sqrt{nx}} dx \ll n^{-3/2}.$$

Hence, the corresponding sums converges to $O(1)$ (as a function of M). Let us now move to the Y-Bessel function. We write it first using Hankel functions [**14**, (5.6.1)]:

$$Y_0(z) = \frac{1}{2i}\big(H_0^{(1)}(z) - H_0^{(2)}(z)\big).$$

The asymptotic expansions for the Hankel functions [**14**, (5.11.5)] give

$$H_0^{(j)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(-1)^{j-1}(z-\pi/4)}\big(1 + c_{1j} z^{-1} + O\big(|z|^{-2}\big)\big). \tag{3.1}$$

The first step to treat these terms is first to integrate and then sum over the O-term:

$$\sum_{n=1}^{\infty} d(n) \int_{M+T}^{M+T+\Delta} (nx)^{-5/4} dx \ll \sum_{n=1}^{\infty} d(n)\, n^{-5/4} \Delta M^{-5/4} \ll 1.$$

Use Lemma 2.1 to treat the integral over the second term in (3.1), except in the case of $n = d$, with the following choices: $M_2 - M_1 = \Delta$, $\varrho = \frac{1}{2}M$, $A_1 = \Delta^{1-\delta}$, $B_1 \asymp \frac{\sqrt{n}}{\sqrt{M}}$ and $A_0 = n^{-3/4} M^{-3/4}$. We obtain

$$\int_{M+T}^{M+T+\Delta} (nx)^{-3/4}\, e\left(\frac{x\sqrt{k}}{\sqrt{M}} \pm 2\sqrt{nx}\right) w(x-T)\, dx$$
$$\ll \Delta^{-P(1-\delta)+1}\, n^{-P/2-3/4} M^{P/2-3/4}$$
$$\ll n^{-P/2-3/4} M^{-P(1/2+\theta)(1-\delta)+P/2-1/4+\theta} \ll n^{-P/2-3/4}.$$

Therefore, the series converges and produces an error term of size $O(1)$. When $n = k$, use integration over the absolute values to obtain the same estimate. Let us now treat the integral corresponding to the first term in the asymptotic Expansion (3.1). When $n \neq k$, we obtain by use of Lemma 2.1 the estimate

$$\begin{aligned} &\ll \Delta^{(1-\delta)(1-P)} n^{-P/2-1/4} M^{P/2-1/4} \\ &\ll M^{-P(1/2+\theta)(1-\delta)+P/2+1/4+\theta} n^{-P/2-1/4} \ll n^{-P/2-1/4}, \end{aligned}$$

when P is sufficiently large. When $n = k$, the first term in the asymptotic expansion for $H_0^{(1)}$ also gives the same estimate. Hence, we have now derived

$$\begin{aligned} D\left(M+T, \Delta, \frac{\sqrt{k}}{\sqrt{M}}\right) = \int_{M+T}^{M+T+\Delta} &\left(2\gamma + \log x - cd(k)k^{-1/4}x^{-1/4}e\left(-2\sqrt{kx}\,\right)\right) \\ &\times w(x-T)\, e\left(\frac{x\sqrt{k}}{\sqrt{M}}\right) dx + O(1), \end{aligned}$$

where c is a constant. Write $q(x) = w(x-T)(\ln x + 2\gamma)$. Now $q^{(P)}(x) \ll \Delta^{(1-\delta)(\varepsilon-P)}$. Using Lemma 2.1 we obtain

$$\begin{aligned} \int_{M+T}^{M+T+\Delta} q(x)\, e\left(\frac{x\sqrt{k}}{\sqrt{M}}\right) dx &\ll \Delta^{(1-\delta)(\varepsilon-P)+1} M^{P/2} \\ &\ll M^{P/2+(1-\delta)(\varepsilon-P)+1/2+\theta} \ll 1. \end{aligned}$$

This proves the theorem. □

As a simple corollary, we obtain

COROLLARY 3.1. *With the assumptions of the previous theorem and supposing, moreover, that* $a(k) = 0$, *we have*

$$\sum_{M \leqslant n \leqslant M+\Delta} a(n)\, w(n)\, e\left(\frac{\sqrt{k}}{\sqrt{M}}\, n\right) = O(1).$$

On the other hand, if $a(k) \neq 0$, *then*

$$\sum_{M \leqslant n \leqslant M+\Delta} a(n)\, w(n)\, e\left(\frac{\sqrt{k}}{\sqrt{M}}\, n\right) \asymp M^{-1/4}\Delta.$$

In other words, the Lehmer conjecture for the eigenfunctions of the Hecke operators is equivalent to the corresponding sums being large.

REMARK 3.1. Notice that if $a(k) \neq 0$, then

$$\left| \sum_{M \leqslant n \leqslant M+\Delta'} a(n)\, e\left(\frac{\sqrt{k}}{\sqrt{M}}\, n\right) \right| \gg M^{-1/4}\Delta$$

for some $\Delta' \in (0, \Delta]$. Otherwise, it would follow from partial summation that the estimate for the smoothed sum would be $o(M^{-1/4}\Delta)$.

THEOREM 3.3. *With the assumptions of Theorem 3.2, the following holds:*

$$\sum_{M\leqslant n\leqslant M+\Delta} c(n)\,w(n)\,e\Big(\frac{\sqrt{k}\,n}{\sqrt{M}}\Big) - \sum_{M+T\leqslant n\leqslant M+T+\Delta} c(n)\,w(n-T)\,e\Big(\frac{\sqrt{k}\,n}{\sqrt{M}}\Big) \ll \frac{T\Delta(T+\Delta)}{M^{7/4}} + \frac{\Delta(\Delta+T)}{M^{5/4}} + 1.$$

PROOF. Using Theorem 3.2, we see that it is sufficient to consider the difference

$$\int_M^{M+\Delta} x^{-1/4} w(x)\,e\Big(\frac{\sqrt{k}}{\sqrt{M}}\,x - 2\sqrt{kx}\Big)dx - \int_M^{M+\Delta} (x+T)^{-1/4} w(x)\,e\Big(\frac{\sqrt{k}(x+T)}{\sqrt{M}}\,x - 2\sqrt{k(x+T)}\Big)dx.$$

We first use the Taylor expansion to treat the terms $x^{-1/4}$ and $(x+T)^{-1/4}$:

$$x^{-1/4} = M^{-1/4} + O\big(M^{-5/4}|x-M|\big).$$

Hence,

$$\begin{aligned}\int_M^{M+\Delta}\Big(\frac{w(x)}{x^{1/4}}\,e\Big(\frac{\sqrt{k}}{\sqrt{M}}x-2\sqrt{kx}\Big) - \frac{w(x)}{(x+T)^{1/4}}\,e\Big(\frac{\sqrt{k}(x+T)}{\sqrt{M}}x-2\sqrt{k(x+T)}\Big)\Big)dx \\ = M^{-1/4}\int_M^{M+\Delta} w(x)\Big(e\Big(\frac{\sqrt{k}x}{\sqrt{M}} - 2\sqrt{kx}\Big) - e\Big(\frac{\sqrt{k}(x+T)}{\sqrt{M}} - 2\sqrt{k(x+T)}\Big)\Big)dx \\ + O\left(\frac{\Delta(\Delta+T)}{M^{5/4}}\right).\end{aligned}$$

Let us now consider the difference

$$\begin{aligned}\Big|e\Big(\frac{\sqrt{k}x}{\sqrt{M}} - 2\sqrt{kx}\Big) - e\Big(\frac{\sqrt{k}(x+T)}{\sqrt{M}} - 2\sqrt{k(x+T)}\Big)\Big| \\ = \Big|e\Big(\frac{\sqrt{k}\,x}{\sqrt{M}} - 2\sqrt{kx} - \frac{\sqrt{k}(x+T)}{\sqrt{M}} + 2\sqrt{k(x+T)}\Big) - 1\Big|.\end{aligned}$$

Since $|e^{iy}-1| \leqslant |y|$, it is sufficient to consider the exponent to obtain an upper bound for the difference of the exponent functions, and thereby for the original integral expression:

$$\Big|\frac{\sqrt{k}}{\sqrt{M}}x - 2\sqrt{kx} - \frac{\sqrt{k}(x+T)}{\sqrt{M}} + 2\sqrt{k(x+T)}\Big| \ll \frac{T(\Delta+T)}{M^{3/2}}.$$

We obtain

$$\int_M^{M+\Delta}\Big(\frac{w(x)}{x^{1/4}}\,e\Big(\frac{\sqrt{k}}{\sqrt{M}}\,x-2\sqrt{kx}\Big) - \frac{w(x)}{(x+T)^{1/4}}\,e\Big(\frac{\sqrt{k}(x+T)}{\sqrt{M}}\,x-2\sqrt{k(x+T)}\Big)\Big)dx \ll \frac{T\Delta(T+\Delta)}{M^{7/4}} + \frac{\Delta(\Delta+T)}{M^{5/4}}. \quad \square$$

4. An Omega-result for short sums of Fourier coefficients

THEOREM 4.1. *Let $c > 0$ be an arbitrary real number. Then*

$$\sum_{M\leqslant n\leqslant M+c\sqrt{M}} a(n) = \Omega\big(M^{1/4}\big).$$

Before proving the theorem, let us prove a lemma:

LEMMA 4.1. *Write $D = Kc^4$, where K is a sufficiently large constant. Write $\|x\|$ to denote the distance from x to the nearest integer. Let b be a sufficiently large constant. Then it is possible to choose an integer $k \in [b^{-1}D, bD]$ such that the following two conditions are satisfied:* (1) $\big\|c\sqrt{k}\big\| > D^{-1/4}$, (2) $a(k) \neq 0$.

PROOF. First, consider the difference

$$c\sqrt{k+1} - c\sqrt{k} = \frac{c}{\sqrt{k}+\sqrt{k+1}} \asymp D^{-1/2}.$$

Therefore, the values $\big\|c\sqrt{k}\big\|$ are somewhat uniformly distributed on the interval $[0,1)$. It is now easy to conclude that only $\asymp D^{3/4}$ of $k \in [b^{-1}D, bD]$ satisfy the condition $\big\|c\sqrt{k}\big\| \leqslant D^{-1/4}$. Since $a(k) \ll k^{\varepsilon}$ by Deligne's estimate [**3**], we obtain

$$\sum_{\substack{b^{-1}D\leqslant k\leqslant bD,\\ \|c\sqrt{k}\|<D^{-1/4}}} |a(k)|^2 \ll D^{3/4+\varepsilon}.$$

The Rankin–Selberg mean value theorem (see e.g. Rankin [**15**]) gives the estimate

$$\sum_{\substack{b^{-1}D\leqslant k\leqslant bD,\\ \|c\sqrt{k}\|>D^{-1/4}}} |a(k)|^2 + O\big(D^{3/4+\varepsilon}\big) \asymp D,$$

which proves the existence of a coefficient satisfying both conditions. □

We may now turn to the proof of the actual theorem.

PROOF OF THEOREM 4.1. Take k as in Lemma 4.1. From the first condition we obtain

$$\left|\sum_{0\leqslant h\leqslant c\sqrt{M}} e\left(\frac{h\sqrt{k}}{\sqrt{M}}\right)\right| = \left|\frac{1 - e\big(\lfloor c\sqrt{M}\rfloor\sqrt{k}M^{-1/2} + \sqrt{k}M^{-1/2}\big)}{1 - e\big(\sqrt{k}M^{-1/2}\big)}\right| \gg M^{1/2},$$

since the denominator is $\asymp M^{-1/2}\sqrt{k} \asymp M^{-1/2}$ as k is a constant, and the nominator is $\asymp 1$ by condition (1) of Lemma 4.1. From Remark 3.1 we know that there exists $\Delta' \leqslant \lambda M^{3/4}$, where $\lambda \in (0,1)$ is a constant, such that

$$\left|\sum_{M\leqslant n\leqslant M+\Delta'} a(n)\, e\left(\frac{\sqrt{k}}{\sqrt{M}}\right)\right| \gg M^{1/2}.$$

Multiplying these two sums together, we obtain

$$M \ll \left|\left(\sum_{0\leqslant h\leqslant c\sqrt{M}} e\left(\frac{h\sqrt{k}}{\sqrt{M}}\right)\right)\left(\sum_{M\leqslant n\leqslant M+\Delta'} a(n)\, e\left(\frac{n\sqrt{k}}{\sqrt{M}}\right)\right)\right|.$$

Change the variable $m = h + n$ and estimate further:

$$\begin{aligned}
&= \left|\sum_{M\leqslant m\leqslant M+\Delta'+c\sqrt{M}} e\left(m\frac{\sqrt{k}}{\sqrt{M}}\right) \sum_{n\in[M,M+\Delta']\cap[m-c\sqrt{M},m]} a(n)\right| \\
&\leqslant \left|\sum_{M+c\sqrt{M}\leqslant m\leqslant M+\Delta'} e\left(\frac{c\sqrt{k}}{\sqrt{M}}\right) \sum_{m-c\sqrt{M}\leqslant n\leqslant m} a(n)\right| \\
&\quad + \left|\sum_{M\leqslant m< M+c\sqrt{M}} e\left(\frac{c\sqrt{k}}{\sqrt{M}}\right) \sum_{n\in[M,M+\Delta']\cap[m-c\sqrt{M},m]} a(n)\right| \\
&\quad + \left|\sum_{M+\Delta'< m\leqslant M+\Delta'+c\sqrt{M}} e\left(\frac{c\sqrt{k}}{\sqrt{M}}\right) \sum_{n\in[M,M+\Delta']\cap[m-c\sqrt{M},m]} a(n)\right|
\end{aligned}$$

We may now use the well-known estimate (see [**10**]) $\sum_{n\leqslant M} a(n) \ll M^{1/3+\varepsilon}$ to treat the second and third term and then use the triangle inequality to the first term to obtain

$$\ll \sum_{M+c\sqrt{M}\leqslant m\leqslant M+\lambda M^{3/4}} \left|\sum_{m-c\sqrt{M}\leqslant n\leqslant m} a(n)\right| + M^{5/6},$$

Therefore the mean of the sums $\left|\sum_{m-c\sqrt{M}\leqslant n\leqslant m} a(n)\right|$ is $\gg M^{1/4}$ and hence, at least one of them has to be $\gg M^{1/4}$. This proves the theorem. $\square$

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Department of Mathematics
University of Turku
Turku
Finland
anmaer@utu.fi

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