PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 87(101) (2010), 1–7 DOI: 10.2298/PIM1001001D

THE KERZMAN–STEIN OPERATOR FOR THE ELLIPSE

Milutin R. Dostanić

Communicated by Stevan Pilipović

Abstract. We give, in the case of ellipse, a simple connection between the spectrum of the Kerzman–Stein operator and the eccentricity of the ellipse.

1. Introduction

Let $\Omega \subset \mathbb{C}$ be a bounded domain with twice differentiable boundary and $\tau = \tau(\xi)$ be the unit tangent vector at a $\xi \in \partial\Omega$, oriented positively with respect to Ω. If $|d\xi|$ is the arclength measure on $\partial\Omega$, then the Kerzman–Stein operator is defined (on $L^2(\partial\Omega)$) by

$$Af(z) = \int\limits_{\partial\Omega} \mathcal{A}(z,\xi)\, f(\xi)\, |d\xi|, \quad \text{where} \quad \mathcal{A}(z,\xi) = \frac{1}{2\pi i}\left(\frac{\tau(\xi)}{\xi - z} - \frac{\overline{\tau(z)}}{\overline{\xi} - \overline{z}}\right).$$

The space $L^2(\partial\Omega)$ is defined using the hermitian inner product

$$(f,g) = \int\limits_{\partial\Omega} f(\xi)\, \overline{g(\xi)}\, |d\xi|.$$

The kernel $\mathcal{A}(\cdot,\cdot)$ is bounded at the diagonal—the apparent singularities cancel each other. Having in mind that $\mathcal{A}$ is bounded on $\partial\Omega$, it follows that A is a Hilbert–Schmidt operator. The operator A can be represented in the form $A = C - C^*$, where $C : L^2(\partial\Omega) \to L^2(\partial\Omega)$ is a (singular) Cauchy operator defined by

$$Cf(z) = \frac{1}{2\pi i}\int\limits_{\partial\Omega} \frac{f(\xi)}{\xi - z}\, d\xi.$$

(Here, integral is interpreted in the sense of Principal Value.)

2010 *Mathematics Subject Classification*: Primary 47G10.
Key words and phrases: Kerzman–Stein operator, eigenvalues, singular values.
Partially supported by MNZZS Grant ON144010.

From this, it follows that $A^* = -A$ and the spectrum of the operator A is discrete and imaginary, except for an accumulation point at zero. The operator A first appeared in the paper of Kerzman and Stein [**3**] where they gave a new method for computing the Riemann map in one dimension. In a later article [**4**], Kerzman formulated some problems concerning the operator A, including the following.

PROBLEM 1.1 (Kerzman 1979). *Relate the spectrum of the Kerzman–Stein operator to the geometry of the domain.*

The first paper giving some results concerning to above problem is [**1**]. Bolt therein proved the following theorem.

THEOREM 1.1. *The Kerzman–Stein operator for an ellipse has eigenvalues* $\pm i\lambda_n$ *where each* $\pm i\lambda_n$ *has multiplicity two. If ellipse is parametrized by* $t \mapsto e^{it} + \rho e^{-it}$ $(0 < \rho < 1)$ *and* $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$*, then* $\lambda_n = \beta_n \rho^{2n-1}(1 + o(1))$*,* $\rho \to 0+$*, where* $0 < \beta_n \leqslant 1$ *and it can be computed explicitly.*

In this paper (in the case of ellipse) we give a connection between the spectrum of the operator A and the eccentricity of ellipse. This connection is expressed through the Hilbert–Schmidt norm of the operator A using a double integral which depends on the eccentricity of an ellipse. This integral is hard for computation and we express it only in the form of a series.

2. Main result

Let $\Delta = \{z : |z| > 1\}$ and $\Omega = \big\{(x, y) : x^2/a^2 + y^2/b^2 < 1\big\}$.

THEOREM 2.1. a) *If* $\lambda_n(A)$ *denotes the eigenvalues of the operator A for ellipse* $\partial\Omega$*, then*

$$\sum_n \lambda_n^2(A) = -\frac{1}{2\pi^2} \int\limits_{\partial\Delta}\int\limits_{\partial\Delta} \frac{|1-\omega z^2|\,|1-\omega\xi^2| - |1-\omega z\xi|^2}{|\xi - z|^2 |1-\omega z\xi|^2}\,|dz|\,|d\xi|,$$

where $\omega = \dfrac{1-\sqrt{1-e^2}}{1+\sqrt{1-e^2}}$ *and e denote the eccentricity of the ellipse* $\partial\Omega$.

b) *For ellipses with small (enough) eccentricity, the eccentricity is uniquely determined by the Hilbert–Schmidt norm of operator A (or by the spectrum of the operator A).*

REMARK 2.1. The Bolt result relates to the asymptotic behavior of the eigenvalues of the Kerzman–Stein operator for the family of ellipses (not only one ellipse) “tending" to unit circle.

PROOF. a) For our proof it is suitable to put $a = \alpha + \beta$, $b = \alpha - \beta$, $\alpha > \beta > 0$. Then $\omega = \beta/\alpha$, and the function $\varphi(\xi) = \alpha\xi + \beta/\xi$ realizes a conformal mapping of Δ onto $\mathbb{C} \setminus \overline{\Omega}$. Let $A_0 : L^2(\partial\Delta) \to L^2(\partial\Delta)$ and $V : L^2(\partial\Omega) \to L^2(\partial\Delta)$ be linear operators defined by

$$A_0 f(z) = \int\limits_{\partial\Delta} K_0(z,\xi) f(\xi)\,|d\xi|, \quad V f(z) = f(\varphi(z))\sqrt{\varphi'(z)}.$$

Here $K_0(z,\xi) = \sqrt{\varphi'(z)}\cdot\overline{\sqrt{\varphi'(\xi)}}\cdot\mathcal{A}\left(\varphi(z),\varphi(\xi)\right)$. Let us denote the eigenvalues of the operator A_0 by $\lambda_n(A_0)$. The operator V is an isometry and we have $\lambda_n(A) = \lambda_n(A_0)$ and so, $\sum_n \lambda_n^2(A) = \sum_n \lambda_n^2(A_0)$. Having on mind that the eigenvalues of the operator A (A_0) is imaginary, we have

$$\sum_n \lambda_n^2(A) = \sum_n \lambda_n^2(A_0) = -\sum_n s_n^2(A_0) = -\int_{\partial\Delta}\int_{\partial\Delta} |K_0(z,\xi)|^2 |dz|\,|d\xi| \tag{2.1}$$

(with $s_n(T)$, we denote the singular values of the operator T; for more information about singular values see [**2**]). If $\xi \in \partial\Delta$ we have

$$\tau(\varphi(\xi)) = \frac{\varphi'(\xi)}{|\varphi'(\xi)|}\frac{d\xi}{|d\xi|} = i\xi\frac{\varphi'(\xi)}{|\varphi'(\xi)|}$$

and then we can transform $K_0(\cdot,\cdot)$ to the following form

$$K_0(z,\xi) = \frac{\xi}{2\pi}\frac{\sqrt{\varphi'(\xi)}\sqrt{\varphi'(z)}}{\varphi(\xi)-\varphi(z)} + \frac{\overline{z}}{2\pi}\overline{\left(\frac{\sqrt{\varphi'(\xi)}\sqrt{\varphi'(z)}}{\varphi(\xi)-\varphi(z)}\right)}. \tag{2.2}$$

Having on mind that for $z,\xi \in \partial\Delta$, we have

$$\frac{\sqrt{\varphi'(\xi)}\sqrt{\varphi'(z)}}{\varphi(\xi)-\varphi(z)} = \frac{\sqrt{\alpha-\beta/z^2}\sqrt{\alpha-\beta/\xi^2}}{\alpha\xi+\beta/\xi-\alpha z-\beta/z} = \frac{\sqrt{1-\omega\overline{z}^2}\sqrt{1-\omega\overline{\xi}^2}}{(\xi-z)(1-\omega\overline{z}\,\overline{\xi})}$$

and from (2.2) we conclude

$$K_0(z,\xi) = \frac{\xi}{2\pi}\frac{\sqrt{1-\omega\overline{z}^2}\sqrt{1-\omega\overline{\xi}^2}}{(\xi-z)(1-\omega\overline{z}\,\overline{\xi})} + \frac{\overline{z}}{2\pi}\frac{\sqrt{1-\omega z^2}\sqrt{1-\omega\xi^2}}{(\overline{\xi}-\overline{z})(1-\omega z\xi)}$$

i.e.,

$$K_0(z,\xi) = \frac{\xi}{2\pi(\xi-z)}\left[\frac{(1-\omega\overline{z}^2)^{\frac12}(1-\omega\overline{\xi}^2)^{\frac12}}{1-\omega\overline{z}\,\overline{\xi}} - \frac{(1-\omega z^2)^{\frac12}(1-\omega\xi^2)^{\frac12}}{1-\omega z\xi}\right] \tag{2.3}$$

(because $z,\xi \in \partial\Delta$). From (2.1) and (2.3) we obtain

$$\sum_n \lambda_n^2(A) = -\frac{1}{4\pi^2}\int_{\partial\Delta}\int_{\partial\Delta}\frac{1}{|\xi-z|^2}|\overline{D}-D|^2\,|dz|\cdot|d\xi|, \tag{2.4}$$

where

$$D = \frac{\sqrt{1-\omega z^2}\sqrt{1-\omega\xi^2}}{(1-\omega z\xi)}.$$

Now, we prove that

$$\int_{\partial\Delta}\int_{\partial\Delta}\frac{1-D^2}{|\xi-z|^2}\,|dz|\,|d\xi| = 0. \tag{2.5}$$

Indeed, we have

$$\int\limits_{\partial\Delta}\int\limits_{\partial\Delta} \frac{1-A^2}{|\xi - z|^2}\,|dz|\,|d\xi| = \int\limits_{\partial\Delta}\int\limits_{\partial\Delta} \frac{\omega(z-\xi)^2}{|\xi - z|^2(1-\omega z\xi)^2}\,|dz|\,|d\xi|$$

$$= \int\limits_{\partial\Delta}\int\limits_{\partial\Delta} \frac{\omega(z-\xi)^2}{(\xi - z)(\overline{\xi} - \overline{z})(1-\omega z\xi)^2}\,\frac{dz}{iz}\,\frac{d\xi}{i\xi}$$

$$= \omega \int\limits_{\partial\Delta}\int\limits_{\partial\Delta} \frac{dz d\xi}{(1-\omega z\xi)^2} = 0.$$

From (2.4) and (2.5) we get

$$\sum_n \lambda_n^2(A) = -\frac{1}{4\pi^2}\int\limits_{\partial\Delta}\int\limits_{\partial\Delta} \frac{2(|D|^2-1)}{|\xi - z|^2}\,|dz|\,|d\xi|$$

i.e.

$$\sum_n \lambda_n^2(A) = -\frac{1}{2\pi^2}\int\limits_{\partial\Delta}\int\limits_{\partial\Delta} \frac{|1-\omega z^2|\,|1-\omega\xi^2| - |1-\omega z\xi|^2}{|\xi - z|^2|1-\omega z\xi|^2}\,|dz|\,|d\xi|. \tag{2.6}$$

b) Let

$$G(\omega) = -\frac{1}{2\pi^2}\int\limits_{\partial\Delta}\int\limits_{\partial\Delta} \frac{|1-\omega z^2|\,|1-\omega\xi^2| - |1-\omega z\xi|^2}{|\xi - z|^2|1-\omega z\xi|^2}\,|dz|\,|d\xi|.$$

Now, we prove that the function G is monotone on some interval $[0,\omega_0]$, $\omega_0 \in (0,1)$. From that, the statement b) of Theorem 2 follows.

Let $|z| = 1$, $0 \leqslant t < 1$. Then we have

$$|1-tz^2| = (1-tz^2)^{\frac{1}{2}}(1-t\,\overline{z}^2)^{\frac{1}{2}} = \sum_{n=0}^{\infty} A_n(z)\,t^n$$

where

$$A_n(z) = \sum_{k=0}^{n} \binom{1/2}{k}(-z^2)^k \binom{1/2}{n-k}(-\overline{z}^2)^{n-k} = (-1)^n z^{-2n}\sum_{k=0}^{\infty}\binom{1/2}{k}\binom{1/2}{n-k} z^{4k}.$$

So, if $|z| = |\xi| = 1$, then we obtain

$$|1-tz^2|\ \ |1-t\,\xi^2| = \sum_{n=0}^{\infty} B_n(z,\xi)t^n \tag{2.7}$$

where $B_n(z,\xi) = \sum_{k=0}^{n} A_k(z)A_{n-k}(\xi)$. If we put $z = \xi$ in (2.7), we get

$$B_n(z,z) = \sum_{k=0}^{n} A_k(z)A_{n-k}(z) = 0 \text{ for } n \geqslant 3.$$

If $|z| = 1$, by differentiation of both sides of the equality

$$(1 - tz^2)^{\frac{1}{2}}(1 - tz^{-2})^{\frac{1}{2}} = \sum_{n=0}^{\infty} A_n(z)t^n$$

with respect to z and multiplying the resulting equality with $(1-tz^2)^{\frac{1}{2}}(1-tz^{-2})^{\frac{1}{2}}$ we get

$$\sum_{n=0}^{\infty} t^n \left(\sum_{k=0}^{n} A_k'(z) A_{n-k}(z) \right) = t\,(z^{-3} - z)$$

i.e., $\sum_{k=0}^{n} A_k'(z)A_{n-k}(z) = 0$ for $n \geqslant 2$. So, we obtained (for $|z| = 1$)

$$\sum_{k=0}^{n} A_k'(z)A_{n-k}(z) = 0, \qquad \sum_{k=0}^{n} A_k(z)A_{n-k}(z) = 0 \quad \text{for } n \geqslant 3. \tag{2.8}$$

From (2.7) we get

$$\begin{aligned} &|1 - \omega z^2|\,|1 - \omega \xi^2| - |1 - \omega z\xi|^2 \\ &= \left(B_1(z,\xi) + \overline{z}\overline{\xi} + z\xi\right)\omega + (B_2(z,\xi) - 1)\omega^2 + \sum_{n=3}^{\infty} B_n(z,\xi)\omega^n, \quad (0 < \omega < 1) \end{aligned} \tag{2.9}$$

By direct calculation we obtain

$$\begin{aligned} B_1(z,\xi) + \overline{z}\overline{\xi} + z\xi &= -\tfrac{1}{2}(z-\xi)^2 - \tfrac{1}{2}\left(\overline{z} - \overline{\xi}\,\right)^2, \\ B_2(z,\xi) - 1 &= -\tfrac{1}{8}(z-\xi)^2(z+\xi)^2\left(1 - \overline{z}^2\overline{\xi}^2\right)^2, \end{aligned} \tag{2.10}$$

where $z, \xi \in \partial\Delta$. From (2.8) it follows that $B_n(z,z) = 0$, $\dfrac{\partial B_n}{\partial \xi}\Big|_{\xi=z} = 0$ for $n \geqslant 3$ and

$$B_n(z,\xi) = \frac{1}{2!}\frac{\partial^2 B_n}{\partial \xi^2}\Big|_{\xi=z}(\xi - z)^2 + \cdots \tag{2.11}$$

From (2.9), (2.10), and (2.11), it follows that the integral on the right-hand side in (2.6) is not singular and so we have

$$G'(\omega) = -\frac{1}{2\pi^2}\int\limits_{\partial\Delta}\int\limits_{\partial\Delta} \frac{1}{|\xi - z|^2}\frac{d}{d\omega}\left(\frac{|1-\omega z^2|\,|1-\omega\xi^2|}{|1-\omega z\xi|^2}\right)|dz|\,|d\xi|.$$

Since

$$\begin{aligned} &\frac{d}{d\omega}\left(\frac{|1-\omega z^2|\,|1-\omega\xi^2|}{|1-\omega z\xi|^2}\right) \\ &= \frac{|1-\omega z^2|\,|1-\omega\xi^2|}{|1-\omega z\xi|^2}\operatorname{Re}\left[\frac{d}{d\omega}\left(\ln(1-\omega z^2) + \ln(1-\omega\xi^2) - 2\ln(1-\omega z\xi)\right)\right] \\ &= -\frac{|1-\omega z^2|\,|1-\omega\xi^2|}{|1-\omega z\xi|^2}\operatorname{Re}\left[\sum_{n=1}^{\infty}\omega^{n-1}(\xi^n - z^n)^2\right], \end{aligned}$$

we get

$$G'(\omega)=\sum_{n=1}^{\infty}\omega^{n-1}\,\mathrm{Re}\Bigg(\frac{1}{2\pi^2}\int\limits_{\partial\Delta}\int\limits_{\partial\Delta}\frac{|1-\omega z^2|\,|1-\omega\xi^2|}{|\xi-z|^2|1-\omega z\xi|^2}\,(\xi^n-z^n)^2|dz|\,|d\xi|\Bigg).$$

Let

$$K_n(\omega)=\frac{1}{2\pi^2}\int\limits_{\partial\Delta}\int\limits_{\partial\Delta}\frac{|1-\omega z^2|\,|1-\omega\xi^2|}{|\xi-z|^2|1-\omega z\xi|^2}\,(\xi^n-z^n)^2|dz|\,|d\xi|.$$

It is easy to see that K_n is real and so we have

$$G'(\omega)=\sum_{n=1}^{\infty}\omega^{n-1}K_n(\omega). \tag{2.12}$$

Using the binomial expansion we obtain (for $|z|=1$)

$$|1-\omega z^2|=\sum_{k\in\mathbb{Z}}c_k(\omega)z^{2k}, \tag{2.13}$$

where

$$c_n(\omega)=\sum_{\substack{k,l\,\geqslant\,0\\ k-l\,=\,n}}\binom{1/2}{k}\binom{1/2}{l}(-\omega)^{k+l}.$$

It is clear that $c_{-n}(\omega)=c_n(\omega)$. If $n\geqslant 0$, we can easily conclude that

$$c_n(\omega)=(-1)^n\omega^n\sum_{\nu=0}^{\infty}\binom{1/2}{\nu}\binom{1/2}{\nu+n}\omega^{2\nu}. \tag{2.14}$$

Since $\left|\binom{1/2}{n}\right|\leqslant L_1n^{-3/2}$, ($L_1$ does not depend on n) from (2.14) we obtain

$$|c_n(\omega)|\leqslant\frac{L_2}{(1+|n|)^{3/2}}\,\omega^{|n|},\quad (0<\omega<1) \tag{2.15}$$

where the constant L_2 does not depend on n and ω. Using (2.13) we calculate K_n. Namely

$$K_1(\omega)=-\frac{2}{1-\omega^2}\sum_{n\in\mathbb{Z}}c_n^2(\omega)\cdot\omega^{|2n+1|}, \tag{2.16}$$

$$\begin{aligned}K_n(\omega)=&-\frac{2}{1-\omega^2}\\ &\times\Bigg[n\sum_{m\in\mathbb{Z}}c_m^2(\omega)\,\omega^{|n+2m|}+2\sum_{r=1}^{n-1}(n-r)\sum_{m\in\mathbb{Z}}c_m(\omega)\,c_{m+r}(\omega)\,\omega^{|n+2m+r|}\Bigg]\end{aligned} \tag{2.17}$$

in the case $n\geqslant 2$. From (2.14) and (2.16) it follows that

$$K_1(\omega)=-\frac{2\omega}{1-\omega^2}(1+o(1)),\quad \omega\to 0+\,. \tag{2.18}$$

If $n\geqslant 2$, from (2.15) and (2.17) we obtain the estimate

$$|K_n(\omega)|\leqslant L_3\,n\,\omega^n\frac{1}{1-\omega^2} \tag{2.19}$$

where the constant L_3 does not depend on n and $\omega \in (0,1)$. From (2.12), (2.18) and (2.19) we conclude that there exists $\omega_0 \in (0,1)$ such that $G'(\omega) < 0$ for $\omega \in (0,\omega_0)$ i.e., the function G is monotone on $[0,\omega_0]$. Theorem 2 is proved. $\square$

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Matematički fakultet
Beogradski univerzitet
Begrad
Serbia
domi@matf.bg.ac.rs

(Received 14 09 2009)
(Revised 04 02 2010)