PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 87(101) (2010), 75–83 DOI: 10.2298/PIM1001075M

A PROPOSITIONAL p-ADIC PROBABILITY LOGIC

Miloš Milošević

Communicated by Žarko Mijajlović

ABSTRACT. We present the p-adic probability logic $LpPP$ based on the paper [**5**] by A. Khrennikov et al. The logical language contains formulas such as $P_{=s}(\alpha)$ with the intended meaning "the probability of α is equal to s", where α is a propositional formula. We introduce a class of Kripke-like models that combine properties of the usual Kripke models and finitely additive p-adic probabilities. We propose an infinitary axiom system and prove that it is sound and strongly complete with respect to the considered class of models. In the paper the terms finitary and infinitary concern the meta language only, i.e., the logical language is countable, formulas are finite, while only proofs are allowed to be infinite. We analyze decidability of $LpPP$ and provide a procedure which decides satisfiability of a given probability formula.

1. Introduction

The Einsten–Podolsky–Rosen paradox and the empirical violations of Bell's inequality answered negatively the question whether quantum stochastics can be reduced to the classical stochastics and led to the belief that the roots of these paradoxes are in the mathematical foundation of Kolmogorov-style probability theory. Several non-Archimedean approaches are introduced in order to develop a probability theory suitable for applications in mathematical physics.

One of the noteworthy attempts to overcome these obstacles are p-adic valued probabilities. The measure-theoretical p-adic probability is of the utmost importance for this paper and its details can be found in [**4**] and [**5**]. Also, for the basic facts about p-adic numbers an interested reader can consult [**1**] and [**8**].

In this paper, for the p-adic probability logic $LpPP$ are introduced syntax, the corresponding class of models and the infinitary axiomatization, which is proved to be sound and strongly complete with respect to the mentioned class of models. An algorithm which decides satisfiability of a given p-adic probability formula is presented too.

The significance of logic $LpPP$ is comparable to the importance of Hailperin's propositional probability logic and logic LLP_2, exposed in [**2**] and [**6**], respectively, in the classical probability framework.

2010 *Mathematics Subject Classification*: 03B48; 03B42; 03B45.

2. Syntax

Let p be a fixed prime natural number and S be the set of all p-adic integers which are algebraic over $\mathbb{Q}$, i.e., $S = \mathbb{Q}_p^{\mathrm{alg}} \cap \mathbb{Z}_p$. The language of the p-adic probability logic $LpPP$ consists of a denumerable set of propositional variables $\mathrm{Var} = \{r_1, r_2, \dots\}$, logical connectives $\wedge$ and $\neg$, and probability operators $P_{=s}$ for each $s \in S$. The set of all propositional formulas is denoted by For_P and its elements by α, β, γ. Probability operators are applied to propositional formulas and in that way basic probability formulas $P_{=s}\alpha$ are obtained. All Boolean combinations of basic probabilistic formulas form the set For_{pP} with the elements ϕ, ψ, θ. The set For of $LpPP-$formulas is a disjoint union of For_P and For_{pP} and its elements are denoted by Φ, Ψ, Θ.

$\alpha \wedge P_{=s}(\beta),\ P_{=s}(P_{=t}(\gamma)) \notin$ For because the above formation rules do not allow neither mixing of propositional and probabilistic formulas nor nesting of probabilistic operators.

3. Semantics

A class H of subsets of a nonempty set V is an algebra if it contains V and is closed under finite unions and complementation. A finitely additive p-adic probability measure μ is a function $\mu : H \to \mathbb{Q}_p$ with the following properties: $\mu(V) = 1$, $\mu(H_1 \cup H_2) = \mu(H_1) + \mu(H_2)$, for all disjoint sets $H_1, H_2 \in H$, and $\|H\|_\mu = \sup\{\|\mu(A)\|_p \mid A \in V,\ A \subseteq H\} < \infty$. These properties correspond, respectively, to normalization, additivity and boundedness in [**4**] and [**5**].

Semantics to the set of $LpPP$-formulas is given in the possible-world style.

Definition 3.1. An $LpPP$-model is a structure $M = \langle W, v, H, \mu\rangle$ where:

- W is a nonempty set of objects called worlds;
- v associates a valuation of variables $v(w)$ with each world $w \in W$, i.e., $v(w) : \mathrm{Var} \to \{0, 1\}$,
- H is an algebra of subsets of W,
- $\mu : H \to S$ is a finitely additive p-adic probability measure on H.

The class of all $LpPP$-models M with the property that for every $\alpha \in \mathrm{For}_P$, $[\alpha] = \{w \in W \mid w \models \alpha\}$ is a measurable set, i.e., $[\alpha] \in H$, will be denoted by $LpPP_{\mathrm{Meas}}$. This class of models will be in the scope of our research.

Definition 3.2. The satisfiability relation $\models\ \subset LpPP_{\mathrm{Meas}} \times$ For fulfills the following conditions for every $LpPP_{\mathrm{Meas}}$-model $M = \langle W, v, H, \mu\rangle$:

- if $\alpha \in \mathrm{For}_P$, then $M \models \alpha$ if and only if $w \models \alpha$ for each world w,
- if $\alpha \in \mathrm{For}_P$, then $M \models P_{=s}(\alpha)$ if and only if $\mu([\alpha]) = s$,
- if $\phi \in \mathrm{For}_{pP}$, then $M \models \neg\phi$ if and only if $M \not\models \phi$,
- if $\phi, \psi \in \mathrm{For}_{pP}$, then $M \models \phi \wedge \psi$ if and only if $M \models \phi$ and $M \models \psi$.

4. Axioms

The axiomatic system Ax_{LpPP} for $LpPP$ contains the following axiom schemata:

Axiom 1: all the axioms of the classical propositional logic, separately for formulas from For_P and separately for formulas from For_{pP},
Axiom 2: $P_{=1}(\alpha \leftrightarrow \beta) \rightarrow (P_{=s}(\alpha) \rightarrow P_{=s}(\beta))$,
Axiom 3: $P_{=s}(\alpha) \leftrightarrow P_{=1-s}(\neg\alpha)$,
Axiom 4: $(P_{=s_1}(\alpha) \wedge P_{=s_2}(\beta) \wedge P_{=1}\neg(\alpha \wedge \beta)) \rightarrow P_{s_1+s_2}(\alpha \vee \beta)$,
Axiom 5: $P_{=s}(\alpha) \rightarrow \neg P_{=t}(\alpha)$, for $s \neq t$,

and inference rules:

Rule 1: modus ponens, separately for formulas from For_P and separately for formulas from For_{pP},
Rule 2: $\dfrac{\alpha}{P_{=1}(\alpha)}$, $\alpha \in \mathrm{For}_P$,
Rule 3: $\dfrac{\psi \rightarrow \neg P_{=s}(\alpha), \text{ for every } s \in S}{\psi \rightarrow \bot}$.

Axiom 1 and Rule 1 correspond to the classical propositional reasoning. Axioms 2–5 concern the probabilistic part of our system. Axiom 4 corresponds to the finite additivity of measure. Rule 2 is a form of modal necessitation and secures normalization of the measure, while infinitary Rule 3, which first was introduced in [**7**], guarantees that to each formula is attached a probability.

DEFINITION 4.1. $\Phi \in \mathrm{For}$ is deducible from the theory T, which we denote by $T \vdash \Phi$, if there exists a denumerable sequence of formulas $\Phi_0, \Phi_1, \dots, \Phi$ called the proof, such that each member of the sequence is an instance of some axiom schemata or is contained in T, or is obtained from the previous formulas using an inference rule. Formula $\Psi \in \mathrm{For}$ is a theorem (denoted by $\vdash \Psi$) if it is deducible from the empty set.

DEFINITION 4.2. A theory T is consistent if there are at least one formula from For_P and at least one formula from For_{pP} which can not be deduced from T. A theory T is maximal consistent if it is consistent and fulfills the following two conditions:

- for each $\alpha \in \mathrm{For}_P$, if $T \vdash \alpha$, then $\alpha \in T$ and $P_{=1}(\alpha) \in T$,
- for each $\psi \in \mathrm{For}_{pP}$, either $\psi \in T$ or $\neg\psi \in T$.

The set of all formulas which are deducible from T is called the deductive closure of T and denoted by $Cn(T)$. A theory T is deductively closed if $T = Cn(T)$.

5. Soundness and completeness

This section begins with proofs of Soundness and Deduction theorem for the p-adic probability logic $LpPP$.

THEOREM 5.1 (Soundness). *The axiomatic system Ax_{LpPP} is sound with respect to the class of $LpPP_{\mathrm{Meas}}$-models.*

PROOF. The soundness of propositional logic implies that every instance of an axiom schemata for propositional formula holds in every model and that the inference rule R1 preserves validity. We will prove validity of the axiom A4 and the similar reasoning can be applied to the other axioms. Let $M \models P_{=s_1}(\alpha) \wedge P_{=s_2}(\beta) \wedge$

$P_{=1}\neg(\alpha\wedge\beta)$. This holds if and only if $\mu([\alpha]) = s_1$, $\mu([\beta]) = s_2$ and $\mu([\neg(\alpha\wedge\beta)]) = 1$. By the additivity of the measure we conclude that $\mu([\alpha \vee \beta)]) = s_1 + s_2$ meaning $M \vDash P_{=s_1+s_2}(\alpha \vee \beta)$.

Suppose that $\alpha \in \mathrm{For}_P$ is a valid formula. Then for every $LpPP_{\mathrm{Meas}}$-model $M = \langle W, v, H, \mu\rangle$ holds $[\alpha] = W$ and $\mu([\alpha]) = 1$. So, $M \vDash P_{=1}(\alpha)$ and the rule R2 preserves validity. Consider the rule R3 and let $\psi \to \neg P_{=s}(\alpha)$ be valid for every $s \in S$ and let M be an $LpPP_{\mathrm{Meas}}$-model such that $M \nvDash \psi \to \bot$. $M \vDash \psi$ implies $M \vDash \neg P_{=s}(\alpha)$, $s \in S$ which is equivalent to $\mu([\alpha]) \neq s$, for every $s \in S$. In other words, there is no measure attached to $[\alpha]$ and this contradicts $M \in LpPP_{\mathrm{Meas}}$. □

Theorem 5.2 (Deduction theorem). *If T is a theory and $\Phi, \Psi \in \mathrm{For}_P$ or $\Phi, \Psi \in \mathrm{For}_{pP}$, then $T \cup \{\Phi\} \vdash \Psi$ if and only if $T \vdash \Phi \to \Psi$.*

Proof. If $\Phi, \Psi \in \mathrm{For}_P$, then this is the well-known Deduction theorem for propositional logic since there is no rule whose antecedents are formulas from For_{pP} and the consequent is in For_P.

We use the transfinite induction on the length of the proof of Ψ from $T \cup \{\Phi\}$ to prove the implication from left to right for $\Phi, \Psi \in \mathrm{For}_{pP}$. The other direction and the cases when either $\vdash \Psi$ or $\Phi = \Psi$ or Ψ is obtained by Rule 1 can be proved in the same way as in the classical propositional calculus. If Ψ is of the form $\Psi = P_{=1}(\alpha)$, $\alpha \in \mathrm{For}_P$, and Ψ is deduced from $T \cup \{\Phi\}$ by an application of Rule 2, then:

1. $T \cup \{\Phi\} \vdash \alpha$
2. $T \vdash \alpha$, because Φ is not an essential member of any proof for α
3. $T \vdash P_{=1}(\alpha)$, by Rule 2
4. $T \vdash P_{=1}(\alpha) \to (\Phi \to P_{=1}(\alpha))$, since $P_{=1}(\alpha) \to (\Phi \to P_{=1}(\alpha))$ is an instance of the classical propositional tautology $r_1 \to (r_2 \to r_1)$
5. $T \vdash \Phi \to P_{=1}(\alpha)$, by Rule 1 applied on 3 and 4.

If $\Psi = \psi \to \bot$ is obtained from $T \cup \{\Phi\}$ using Rule 3, then:

1. $T \cup \{\Phi\} \vdash \psi \to \neg P_{=s}(\alpha)$, for each $s \in S$
2. $T \vdash \Phi \to (\psi \to \neg P_{=s}(\alpha))$, for each $s \in S$ by the induction hypothesis
3. $T \vdash (\Phi \wedge \psi) \to \neg P_{=s}(\alpha)$, for $s \in S$, using an instance of the classical propositional tautology $(r_1 \to (r_2 \to r_3)) \leftrightarrow ((r_1 \wedge r_2) \to r_3)$
4. $T \vdash (\Phi \wedge \psi) \to \bot$, by the application of Rule 4 on 3.
5. $T \vdash \Phi \to \Psi$ □

In order to prove the completeness theorem, we are going to show that every consistent theory T can be extended to a maximal consistent theory T^*, and use T^* to construct the canonical model. We give the sketches for proofs of the preparatory lemmas.

Lemma 5.1. *For every consistent theory T and every $\alpha \in \mathrm{For}_P$, there exists $s \in S$ such that $T \cup \{P_{=s}(\alpha)\}$ is consistent.*

Proof. Suppose that there is $\phi \in T \cap \mathrm{For}_{pP}$ (if this intersection is empty we set $\phi = \top$). We denote $T \setminus \{\phi\}$ by T_*. If for every $s \in S : T_*, \phi, P_{=s}(\alpha) \vdash \bot$, then by the deduction theorem

$$T_* \vdash \phi \to \neg P_{=s}(\alpha),\ \text{for each s} \in \text{S},$$

and we obtain by Rule 3 $T_* \vdash \phi \to \bot$. Another application of Theorem 5.2 gives $T \vdash \bot$. $\square$

LEMMA 5.2. *Let T be a maximal consistent theory. Then:*

a) *for all $\phi, \psi \in \mathrm{For}_{pP}$, $\phi \vee \psi \in T$ if and only if $\phi \in T$ or $\psi \in T$,*
b) *for all $\Phi, \Psi \in \mathrm{For}$, where either $\Phi, \Psi \in \mathrm{For}_P$ or $\Phi, \Psi \in \mathrm{For}_{pP}$, $\Phi \wedge \Psi \in T$ if and only if $\Phi, \Psi \in T$,*
c) *for each $\Phi \in \mathrm{For}$, if $T \vdash \Phi$, then $\Phi \in T$, i.e., every maximal consistent theory is deductively closed,*
d) *for all formulas Φ, Ψ, where either $\Phi, \Psi \in \mathrm{For}_P$ or $\Phi, \Psi \in \mathrm{For}_{pP}$, if $\Phi, \Phi \to \Psi \in T$, then $\Psi \in T$.*

THEOREM 5.3. *Every consistent theory can be extended to a maximal consistent theory.*

PROOF. Let T be a consistent theory, $Cn_P(T)$ the set of all propositional formulas which are deducible from T, $\alpha_1, \alpha_2, \dots$ an enumeration of all formulas from For_P and $\phi_1, \phi_2, \dots$ an enumeration of all formulas from For_{pP}. We construct a sequence of theories $(T_i)_{i<\omega}$ in the following way:

- $T_0 = T \cup Cn_P(T) \cup \{P_{=1}(\alpha) \mid \alpha \in Cn_P(T)\}$;
- if $T_{2i} \cup \{\phi_i\}$ is consistent then $T_{2i+1} = T_{2i} \cup \{\phi_i\}$, otherwise $T_{2i+1} = T_{2i} \cup \{\neg\phi_i\}$;
- $T_{2i+2} = T_{2i+1} \cup \{P_{=s}(\alpha_i)\}$, for some $s \in S$, such that T_{2i+2} is a consistent theory.

We notice that the existence of s in the step $2i+2$ is secured by Lemma 5.1. $T^* = \bigcup_{j<\omega} T_j$ is a union of consistent theories T_j and it will be proved that T^* is maximal consistent. The first condition of maximality is achieved by constructing T_0, which contains the propositional deductive closure of T and all corresponding formulas of the form $P_{=1}(\alpha)$. Concerning $\phi \in \mathrm{For}_{pP}$, T_{2i+1} contains either $\phi = \phi_i$ or $\neg\phi = \phi_j$, but not both, because otherwise $T_{2\cdot\max\{i,j\}+1}$ would be inconsistent.

We are going to prove that T^* is deductively closed using the transfinite induction on the length of the proof. It will be sufficient to claim its consistency, since T^* does not contain all formulas. In the case of the finite proof for ϕ, there is a T_l such that $T_l \vdash \phi$, and, thus, $\phi \in T^*$.

Suppose that for $\phi = \psi \to \bot$, $T^* \vdash \psi \to \bot$ is obtained using the infinitary rule from $\psi \to \neg P_{=s}(\alpha_i)$, $s \in S$, for some $\alpha_i \in \mathrm{For}_P$, but $\psi \to \bot \notin T^*$. In the step $2i+2$, T_{2i+2} is constructed adding $P_{=t}(\alpha_i)$, where t is a fixed element of S. There is $l \in \omega$ such that $\neg(\psi \to \bot) \in T_l$ and $T_l \vdash \psi$. For some $k > 2i+2, l$, it is fulfilled $P_{=t}(\alpha_i), \psi, \psi \to \neg P_{=t}(\alpha_i) \in T_k$. It implies $T_k \vdash P_{=t}(\alpha_i) \wedge \neg P_{=t}(\alpha_i)$, meaning that T_k is inconsistent. $\square$

COROLLARY 5.1. *Axiom* 3 *is deducible from other axioms and rules of Ax_{LpPP}.*

PROOF. Let Ax_{LpPP}^{-A3} denote the axiomatic system which contains all axioms and rules of Ax_{LpPP} except Axiom 3 and $LpPP^{-A3}$ denote the corresponding logic. One can prove variants of theorems 5.1, 5.2 and 5.3 for $LpPP^{-A3}$ in exactly the same way as above.

Let T^* be a maximal consistent theory in $LpPP^{-A3}$. Suppose there are $s_1, s_2 \in S$ and $\alpha \in \mathrm{For}_P$ such that $P_{=s_1}(\alpha), P_{=s_2}(\neg\alpha) \in T^*$ and $s_2 \neq 1 - s_1$. Since $\alpha \vee \neg\alpha$ is a tautology, it is possible to conclude $P_{=1}(\alpha \vee \neg\alpha) \in T^*$, according to Rule R2, and $P_{=s_1+s_2}(\alpha \vee \neg\alpha) \in T^*$, using an instance of A4 and the deduction theorem. The instance $P_{=1}(\alpha \vee \neg\alpha) \to \neg P_{=s_1+s_2}(\alpha \vee \neg\alpha)$ of A5 and the deduction theorem imply $\neg P_{=s_1+s_2}(\alpha \vee \neg\alpha) \in T^*$. The fact $P_{=s_1+s_2}(\alpha \vee \neg\alpha) \wedge \neg P_{=s_1+s_2}(\alpha \vee \neg\alpha) \in T^*$ contradicts the assumed consistency of T^*. Since there is no maximal consistent theory in $LpPP^{-A3}$ containing $\neg$A3, we conclude $\neg\mathrm{A3} \vdash_{LpPP^{-A3}} \perp$, i.e., $\vdash_{LpPP^{-A3}}$ A3. $\square$

Let T be a consistent theory and its T^* fixed maximal consistent extension. The canonical model $M_T = \langle W, v, H, \mu\rangle$ is defined as follows:

- $W = \{w : \mathrm{Var} \to \mathbf{2} \mid w \vDash Cn_P(T)\}$ and we identify $v(w)$ with w;
- $[\alpha] = \{w \in W : w \vDash \alpha\}$ and $H = \{[\alpha] : \alpha \in \mathrm{For}_P\}$;
- we set $\mu([\alpha]) = s$ iff $P_{=s}(\alpha) \in T^*$.

Theorem 5.4. *Let $M_T = \langle W, v, H, \mu\rangle$ be as above and $\alpha, \beta \in \mathrm{For}_P$. Then, the following hold:*

a) *H is an algebra of subsets of W,*
b) *if $[\alpha] = [\beta]$, then $\mu([\alpha]) = \mu([\beta])$,*
c) *$\mu([\alpha]) = 1 - \mu([\neg\alpha])$, $\mu(\emptyset) = 0$, $\mu(W) = 1$,*
d) *$\mu([\alpha] \cup [\beta]) = \mu([\alpha]) + \mu([\beta])$, for all disjoint $[\alpha]$ and $[\beta]$,*
e) *$\|\mu([\alpha])\|_p \leqslant 1$,*
f) *M_T is an $LpPP_{\mathrm{Meas}}$-model.*

Proof. a) All conditions for H to be an algebra of subsets of W are fulfilled: $W = [\alpha \vee \neg\alpha] \in H$, $[\alpha], [\beta] \in H$ imply that $[\alpha]^c = [\neg\alpha]$, $[\alpha] \cup [\beta] = [\alpha \vee \beta] \in H$.

b) $[\alpha] = [\beta]$ implies $Cn_P(T) \vDash \alpha \leftrightarrow \beta$ and, by the completeness of propositional calculus, $Cn_P(T) \vdash \alpha \leftrightarrow \beta$. Applying Rule 2 we obtain $P_{=1}(\alpha \leftrightarrow \beta)$ and the statement follows by A2.

c)–d) These properties are provided by axioms 3 and 4; $W = [\alpha \vee \neg\alpha]$ and each tautology has a measure 1 by the rule R2.

f) This is a straightforward consequence of parts a)–e). $\square$

Theorem 5.5 (Extended completeness theorem for $LpPP_{\mathrm{Meas}}$). *A theory T is consistent if and only if it has an $LpPP_{\mathrm{Meas}}$-model.*

Proof. The direction from right to left follows from the soundness theorem. For the other direction we consider the canonical model M_T and prove by the induction on complexity of formulas that for each $\Phi \in \mathrm{For}$, $M_T \vDash \Phi$ if and only if $\Phi \in T^*$.

Let $\Phi \in \mathrm{For}_P$. If $\Phi \in Cn_P(T)$, then $M_T \vDash \Phi$ by the definition of M_T. If $M_T \vDash \Phi$ then $Cn_P(T) \vDash \Phi$, and thus, by the completeness of the propositional calculus and the completeness of the propositional theory $Cn_P(T)$, $\Phi \in Cn_P(T)$.

It is an immediate consequence of Definition 2 that $P_{=s}(\alpha) \in T^*$ iff $M_T \vDash P_{=s}(\alpha)$. If $\Phi \in \mathrm{For}_{pP}$ is of the form $\neg\phi$, then $M_T \vDash \neg\phi$ iff $M_T \nvDash \phi$ iff $\phi \notin T^*$ iff

$\neg\phi \in T^*$. For $\Phi = \phi \wedge \psi \in \mathrm{For}_{pP}$, $M_T \vDash \phi \wedge \psi$ iff $M_T \vDash \phi$ and $M_T \vDash \psi$ iff $\phi \in T^*$ and $\psi \in T^*$ iff $\phi \wedge \psi \in T^*$. □

6. Decidability

Firstly, we prove that S is a computable ring, which is one of the crucial facts for the proof that $LpPP$ is decidable.

THEOREM 6.1. *$S = \mathbb{Q}_p^{alg} \cap \mathbb{Z}_p$ is a computable ring, i.e., S is a decidable set, and there are algorithms which for given $s, t \in S$ compute $s + t, s \cdot t, -s$.*

PROOF. We will represent each element $s \in S$ as a pair $(f, B(d, r))$, where f is a polynomial with rational coefficients such that $f(s) = 0$, $d \in \mathbb{Z}$ and $r \in \mathbb{N}$, and $B(d, r) = \{x \in \mathbb{Q}_p \mid \|x - d\|_p < 1/p^r\}$ is a p-adic open ball containing s and no other root of f.

Let $s = (f, B(d_1, r_1))$ and $t = (g, B(d_2, r_2))$ be two elements of S, and $m = \deg(f)$, $n = \deg(g)$. We will sketch an algorithm which decides whether s and t are equal. Without loss of generality suppose that $r_1 \leqslant r_2$. If $\|d_1 - d_2\|_p \geqslant 1/p^{r_1}$, then $B(d_1, r_1)$ and $B(d_2, r_2)$ are disjoint, and thus $s \neq t$. Otherwise, we compute $h = \mathrm{GCD}(f, g)$ and two possible cases may occur:

- $h = 1$ implies $s \neq t$
- for $h \neq 1$ we perform root isolation for h and, if needed, root refinement to obtain isolating intervals of the radius p^{-r_1} (see [**9**] for details on these procedures); we have to check, in the same way as we did it for $B(d_2, r_2)$, if among these isolating intervals exists a subset of $B(d_1, r_1)$; affirmative answer implies $s = t$, and negative leads to the opposite conclusion.

We are going to find a representation for $s + t$. s^k, where $k \in \omega$, is a linear combination of $s^0, \dots, s^{m-1}$, while each power of t is a linear combination of $t^0, \dots, t^{n-1}$. So, $s^k t^l, k, l \in \omega$, and therefore $(s+t)^k$, are linear combinations of $m \cdot n$ elements $s^i t^j$, $0 \leqslant i \leqslant m-1$, $0 \leqslant j \leqslant n-1$:

$$(s+t)^k = c_{0,0}^k s^0 t^0 + \cdots + c_{m-1,n-1}^k s^{m-1} t^{n-1}.$$

We will denote by A a matrix having $m \cdot n + 1$ rows and $m \cdot n$ columns: its i-th row R_i consists of the coefficients $c_{0,0}^{i-1}, \dots, c_{m-1,n-1}^{i-1}$ corresponding to $(s+t)^{i-1}$. Since A has more rows than columns, by solving a system of linear equations, we can find rational numbers $q_1, \dots, q_{m \cdot n+1}$ such that $q_1 R_1 + \cdots + q_{m \cdot n+1} R_{m \cdot n+1} = 0$. Thus, $s + t$ is a zero of the polynomial $h_* = q_1 + q_2 X + \cdots + q_{m \cdot n+1} X^{m \cdot n}$. Bearing in mind that $\|(s+t) - (d_1 + d_2)\|_p \leqslant \max\{\|s - d_1\|_p, \|t - d_2\|_p\}$, it is necessary to perform root isolation for h_*, and possibly root refinement for f and g, and find the isolation interval $\mathcal{O}$ for h_* which contains $d_1 + d_2$. $(h_*, \mathcal{O})$ is a representation for $s + t$.

In order to find a representation for $s \cdot t$ we consider the matrix B having $m \cdot n + 1$ rows and $m \cdot n$ columns whose i-th row is defined by $(s \cdot t)^{i-1}$. $-s$ can be seen as $s \cdot (-1)$. □

THEOREM 6.2. *There is an algorithm which for given $\Phi \in$ For decides its satisfiability.*

Proof. If the propositional formula Φ is not a contradiction, then the theory $\{\Phi\}$ has a model by Theorem 5.5. The decidability procedure for $\Phi \in \mathrm{For}_{pP}$ is based on the following steps and remarks:

– the input is a formula $\Phi \in \mathrm{For}_{pP}$,

– transform Φ into disjunctive normal form $\mathrm{DNF}(\Phi) = \Phi_1 \vee \ldots \vee \Phi_k$ with respect to the basic p-adic probabilistic formulas $P_{=s}(\alpha)$,

– Φ is satisfiable iff at least one disjunct $\Phi_m, m = 1, \ldots, r$, is satisfiable,

– repeat the following procedure for each disjunct of Φ_m until one is satisfied or all are already checked: Φ_m is a conjunction of formulas

$$P_{=s_1}(\alpha_1)^{i_1}, \ldots, P_{=s_n}(\alpha_n)^{i_n}, \quad i_j \in \{0,1\},$$

where $P_{=s_k}(\alpha_k)^0 \equiv \neg P_{=s_k}(\alpha_k)$ and $P_{=s_k}(\alpha_k)^1 \equiv P_{=s_k}(\alpha_k)$; it is necessary to examine if there is any collision with axioms 2, 4 and 5, and the fact deducible by Rule 2 that each tautology has a probability 1; for example a collision with Axiom 5 means that there is a pair of formulas $P_{=s_k}(\alpha_k), P_{=s_l}(\alpha_l)$ such that $s_k \neq s_l$ and $\vdash_P \alpha_k \leftrightarrow \alpha_l$; if no such collision is detected, then the theory $\{P_{=s_1}(\alpha_1)^{i_1}, \ldots, P_{=s_n}(\alpha_n)^{i_n}\}$ is consistent and, by the extended completeness theorem, has a $LpPP_{\mathrm{Meas}}$-model. $\square$

7. Final remarks and future work

Instead of S, the previous results could be easily modified for any recursive ring or field F without ordering compatible with operations. Basic probability formulas of these new logics would be $P_{=s}(\alpha)$, $s \in F$.

In [**3**] is mentioned the following partial ordering on $\mathbb{Z}_p$: for $x = \sum_{i=0}^{+\infty} x_i \cdot p^i$ and $y = \sum_{i=0}^{+\infty} y_i \cdot p^i$ we set $x < y$ if there exists n such that $x_n < y_n$ and $x_k \leqslant y_k$ for all $k > n$. For the purpose of this paper we alter the above ordering as follows: we set that 0 is the minimum, and for $x, y \in S \setminus \{0\}$, $x > y$ iff there exists n such that $x_n < y_n$ and $x_k \leqslant y_k$ for all $k > n$. This ordering has the maximum, namely it is 1, and for integers it holds: $0 < -1 < -2 < -3 < \cdots < 3 < 2 < 1$. Each $s \in S \setminus \mathbb{Z}$ is greater than any negative integer and less than any positive integer, but $s_1, s_2 \in S \setminus \mathbb{Z}$ are not comparable. It makes sense to say that a propositional formula with negative integer measure has a probability close to 0, while a propositional formula with positive integer measure has a probability close to 1. In such a way the zero probability is split to a set of probabilities $[0, 0^+)$, and the probability 1 is split to a set of probabilities $(1^-, 1]$.

Finally, we announce papers on the following topics:

– propositional p-adic probability logic with iterations;
– first-order p-adic probability logic without iterations;
– first-order p-adic probability logic with iterations;
– p-adic probability logic and default reasoning;
– p-adic probability logic which corresponds to the notion of p-adic proportional probability introduced in [**3**]
– p-adic probability logics with basic probability formulas of the form $P(\alpha) \in B(d, r)$, where $B(d, r)$ is an open p-adic ball of rational radius r containing the rational number d.

References

[1] Y. Amice, *Les nombres p-adiques*, Collection SUP, Le mathematicien 14, Presses Universitaires de Frances, Paris, 1975

[2] T. Hailperin, *Probability Logic*, Notre Dame J. Formal Logic **25**(3) (1984), 198–212

[3] A. Khrennikov, *Interpretations of probability and their p-adic extensions*, Theory Probab. Appl. **46**(2) (2001), 256–273

[4] A. Khrennikov, *Towards theory of p-adic valued probabilities*, Studies in Logic, Grammar and Rethoric **14**(27) (2008), 137–154

[5] A. Khrennikov, Sh. Yamada, A. van Rooij, *The measure-theoretical approach to p-adic probability theory*, Ann. Math. Blaise Pascal **6**(1) (1999), 21–32

[6] Z. Ognjanović, M. Rašković, Z. Marković, *Probability logics*; in: Z. Ognjanović (ed.), *Logic in Computer Science*, Zbornik radova 12(20), 2009, 35–111,

[7] M. Rašković, Z. Marković, Z. Ognjanović, *A logic with approwimate conditional probabilities that can model default reasoning*, Int. J. Approx. Reason. **49** (2008), 52–66.

[8] A. Robert, *A Course in p-adic Analysis*, Springer-Verlag, 2000

[9] T. Sturm, V. Weispfenning, *P-adic Root Isolation*, RACSAM, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat. **98**(1-2) (2004), 239–258

Matematički institut SANU
Kneza Mihaila 36
11000 Beograd
Serbia
mionamil@eunet.rs

(Received 12 08 2009)