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REGULARLY VARYING SOLUTIONS OF PERTURBED EULER DIFFERENTIAL EQUATIONS AND RELATED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. An asymptotic analysis in the framework of Karamata regularly varying functions is performed for the solutions of second order linear differential and functional differential equations in the critical case i.e., when condition (1.5) as given below, holds.

1. Introduction

As is witnessed by the recent book [9] theory of regular variation (in the sense of Karamata) has made it possible to develop a new significant aspect of asymptotic analysis of linear and nonlinear ordinary differential equations.

For the reader's convenience we recall here the definition and some basic properties of regularly varying functions. A measurable function $f : [0,\infty) \to (0,\infty)$ is said to be regularly varying of index $\rho \in \mathbf{R}$ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^\rho \quad \text{for all } \lambda > 0.$$

The totality of regularly varying functions of index ρ is denoted by $\mathrm{RV}(\rho)$. We often use the symbol SV to denote $\mathrm{RV}(0)$ and call members of SV slowly varying functions. By definition any function $f(t) \in \mathrm{RV}(\rho)$ can be expressed as $f(t) = t^\rho L(t)$ with $L(t) \in \mathrm{SV}$. One of the most important properties of $\mathrm{RV}(\rho)$-functions is the following representation theorem.

Theorem 1.1. *$f(t) \in \mathrm{RV}(\rho)$ if and only if $f(t)$ is expressed in the form*

$$f(t) = c(t) \exp\left\{ \int_{t_0}^{t} \frac{\delta(s)}{s}\, ds \right\}, \quad t \geqslant t_0, \tag{1.1}$$

for some $t_0 > 0$ and some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty} c(t) = c_0 \in (0,\infty) \quad \text{and} \quad \lim_{t\to\infty} \delta(t) = \rho.$$

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If $c(t) \equiv c_0$ in (1.1), then the function $f(t)$ is called a normalized regularly varying function of index ρ, and the totality of such functions form an important subclass, denoted by n-RV(ρ), of RV(ρ). The symbol n-RV stands for n-RV(0) and its members are referred to as normalized slowly varying functions.

Let $L(t) \in$ SV. What is the asymptotic behavior of $L(t)$ as $t \to \infty$? It may occur that $\lim_{t\to\infty} L(t) = \text{const} > 0$, that is, $L(t)$ is asymptotic to a positive constant as $t \to \infty$. In this case $L(t)$ is said to be a trivial SV-function. Naturally $L(t)$ may exhibit a different behavior: $\lim_{t\to\infty} L(t) = 0$, or $\lim_{t\to\infty} L(t) = \infty$, or otherwise, in which case $L(t)$ is said to be a nontrivial SV-function. Thus it is possible that a nontrivial SV-function may grow to infinity or decay to zero as t tends to ∞, but its behavior at infinity is severely restricted as the following theorem asserts.

THEOREM 1.2. *Let $L(t)$ be a slowly varying function. Then, for any $\varepsilon > 0$,*

$$\lim_{t\to\infty} t^{\varepsilon} L(t) = \infty \quad \text{and} \quad \lim_{t\to\infty} t^{-\varepsilon} L(t) = 0.$$

Useful is the following theorem due to Karamata on the asymptotic behavior of integrals involving slowly varying functions.

THEOREM 1.3. *Let $L(t)$ be a slowly varying function.*

(i) *If $\gamma > -1$, then* $\displaystyle\int_{t_0}^{t} s^{\gamma} L(s)\, ds \sim \frac{t^{\gamma+1}}{\gamma+1} L(t) \quad \text{as} \quad t \to \infty.$

(ii) *If $\gamma < -1$, then* $\displaystyle\int_{t}^{\infty} s^{\gamma} L(s)\, ds \sim -\frac{t^{\gamma+1}}{\gamma+1} L(t) \quad \text{as} \quad t \to \infty.$

Here the symbol $\sim$ is used to denote the asymptotic equivalence:

$$F(t) \sim G(t) \Longleftrightarrow \lim_{t\to\infty} \frac{F(t)}{G(t)} = 1.$$

For a comprehensive exposition of theory of regular variation and its applications to various branches of mathematical analysis the reader is referred to the book [1].

We are now in a position to state an example of typical results on the study of asymptotic behavior of solutions of the linear ordinary differential equation

$$x'' + q(t)x = 0 \tag{A}$$

in the framework of regular variation.

THEOREM 1.4. (Howard and Marić [4]) *Suppose that $q(t)$ is a continuous function which is integrable on $[a, \infty)$. Let a constant $c \in (-\infty, \frac{1}{4})$ be given and let λ_0 and λ_1, $\lambda_0 < \lambda_1$, be the real roots of the quadratic equation*

$$\lambda^2 - \lambda + c = 0. \tag{1.2}$$

Then, equation (A) *has a fundamental set of solutions $\{x_0(t), x_1(t)\}$ such that $x_0(t) \in$ n-RV(λ_0), $x_1(t) \in$ n-RV(λ_1) if and only if*

$$\lim_{t\to\infty} t \int_t^{\infty} q(s)\, ds = c. \tag{1.3}$$

After the publication of [9] attempts have been made by the present authors [6, 7, 8] to establish the existence of regularly varying solutions to functional differential equations with deviating arguments of the form

$$x''(t) + q(t)x(g(t)) = 0, \tag{B}$$

where $q(t)$ and $g(t)$ are continuous on $[a,\infty)$, and $g(t)$ is increasing. There holds

Theorem 1.5. [8] *Let c, λ_0 and λ_1 be as in Theorem 1.4 with c replaced by $-c$. Suppose that $q(t)$ is eventually negative and that $g(t)$ satisfies $g(t) < t$, $g(t) \to \infty$, as $t \to \infty$ and $g(t)/t \to 1$, as $t \to \infty$.*

Then, equation (B) has two solutions $x_0(t) \in$ n-RV(λ_0) and $x_1(t) \in$ n-RV(λ_1) if and only if (1.3) is satisfied.

We note that if in particular $c = 0$, in which case $\lambda_0 = 0$ and $\lambda_1 = 1$, the conclusion of Theorem 1.5 holds if the last condition on $q(t)$ is replaced by $\limsup_{t\to\infty} t/g(t) < \infty$.

It should be noticed that the constant c in Theorem 1.4 is not allowed to exceed 1/4. In fact, let $q(t)$ be eventually positive, and suppose that $c > 1/4$. Then, the roots of (1.2) are imaginary and so equation (A) is oscillatory by the well-known oscillation criterion of Hille [2]. A question then arises: What happens if $c = 1/4$? This is a critical case in the sense that equation (A) may or may not be oscillatory as is shown by the equation

$$x'' + q(t)x = 0, \quad q(t) = \frac{1}{4t^2} + \frac{d}{(t\log t)^2}, \tag{1.4}$$

where d is a positive constant. It is clear that $q(t)$ satisfies (1.3) with $c = 1/4$ and it is known that (1.4) is oscillatory or nonoscillatory according as $d > 1/4$ or $d \leqslant 1/4$. Therefore, additional conditions are needed to ensure oscillation or non-oscillation of equation (A) in case $c = 1/4$; see e.g., the books [3] and [10]. A criterion for non-oscillation of (A) in the language of regular variation, applicable to this case, has been given by Howard and Marić in [4].

Theorem 1.6. [4] *Let $q(t)$ be continuous and integrable on $[a,\infty)$. Suppose that*

$$\lim_{t\to\infty} t\int_t^\infty q(s)\,ds = \frac{1}{4}. \tag{1.5}$$

Put $\phi(t) = t\int_t^\infty q(s)\,ds - \frac{1}{4}$, and let the integral $\int^\infty |\phi(t)|/t\,dt$ converge. Suppose moreover that

$$\int^\infty \frac{\psi(t)}{t}dt < \infty, \quad \textit{where} \quad \psi(t) = \int_t^\infty \frac{|\phi(s)|}{s}\,ds.$$

Then equation (A) possesses a fundamental set of solutions $\{x_0(t), x_1(t)\}$ such that

$$x_0(t) = t^{1/2}L_0(t), \quad L_0(t) \in \text{n-RV}, \quad \lim_{t\to\infty} L_0(t) \in (0,\infty),$$
$$x_1(t) = t^{1/2}\log t\,L_1(t), \quad L_1(t) \in \text{n-RV}, \quad \lim_{t\to\infty} L_1(t) \in (0,\infty).$$

The objective of this paper is first (in Section 2) to make a more detailed analysis of regularly varying solutions of equation (A) under the condition (1.5), obtaining a generalized version of Theorem 1.6, and then (in Section 3) to make use of the results for (A) to establish the existence of similar solutions for the functional differential equation (B) with $q(t)$ satisfying (1.5). In what follows the function $q(t)$ is assumed to be integrable over $[a,\infty)$ unless stated otherwise.

In analyzing the non-oscillation and asymptotic behavior of pertinent equations in the critical case a crucial role is played by the concept of generalized regularly varying functions which was introduced by Jaroš and Kusano [5].

A measurable function $f:[0,\infty)\to(0,\infty)$ is said to be a (generalized) regularly varying function of index ρ with respect to $\log t$ if $f(t)$ is expressed as $f(t)=g(\log t)$ for some regularly varying function $g(\tau)$ of index ρ in the sense of Karamata. We use the symbol $\mathrm{RV}_{\log t}(\rho)$ to denote the set of all regularly varying functions of index ρ with respect to $\log t$. As before, $\mathrm{SV}_{\log t}$ stands for $\mathrm{RV}_{\log t}(0)$, and members of $\mathrm{SV}_{\log t}$ are called (generalized) slowly varying functions with respect to $\log t$. If, in the representation $f(t)=g(\log t)$, $g(\tau)$ is in n-RV(ρ) (or in n-RV), then $f(t)$ is termed a normalized regularly varying function of index ρ (or a normalized slowly varying function) with respect to $\log t$. The set of all normalized RV(ρ)- or SV-functions with respect to $\log t$ will be denoted by n-$\mathrm{RV}_{\log t}(\rho)$ or n-$\mathrm{RV}_{\log t}$.

Since the composition of a slowly varying function and $\log t$ is clearly slowly varying, we see that $\mathrm{SV}_{\log t}\subset\mathrm{SV}$ and $\mathrm{RV}_{\log t}(\rho)\subset\mathrm{RV}(\rho)$. It should be noted, however, that both inclusions are proper. For example, $\log t\in\mathrm{SV}$ but $\log t\notin\mathrm{SV}_{\log t}$ because $\log t\in\mathrm{RV}_{\log t}(1)$.

The representation theorem for RV(ρ)-functions combined with the above definition provides a characterization for the class $\mathrm{RV}_{\log t}(\rho)$.

THEOREM 1.7. *$f(t)\in\mathrm{RV}_{\log t}(\rho)$ if and only if $f(t)$ is expressed in the form*

$$f(t)=\exp\left\{\int_{t_0}^{t}\frac{\delta(s)}{s\log s}\,ds\right\},\quad t\geqslant t_0, \tag{1.6}$$

for some $t_0>1$ and some measurable functions $c(t)$ and $\delta(t)$ such that

$$\lim_{t\to\infty}c(t)=c_0\in(0,\infty)\quad\textit{and}\quad\lim_{t\to\infty}\delta(t)=\rho.$$

We note that $f(t)\in$ n-$\mathrm{RV}_{\log t}(\rho)$ if and only if $c(t)\equiv c_0$ in the representation formula (1.6).

2. Regularly varying solutions of (A) in the critical case

We begin by considering equation (A) with $q(t)$ satisfying condition (1.5). In this case equation (A) may well be called a perturbed Euler differential equation.

First observe that a regularly varying solution of (A), if any, must have the regularity index 1/2. This is an immediate consequence of the following lemma.

LEMMA 2.1. *If (A) has an n-RV(ρ)-solution, then it holds that*

$$\lim_{t\to\infty}t\int_t^\infty q(s)\,ds=\rho(1-\rho). \tag{2.1}$$

PROOF. Let $x(t) \in$ n-RV(ρ) be a solution of (A) on $[t_0, \infty)$. Define $u(t) = x'(t)/x(t)$ and $v(t) = tu(t)$. It is known that $u(t)$ satisfies the Riccati equation

$$u' + u^2 + q(t) = 0, \quad t \geqslant t_0. \tag{2.2}$$

On the other hand, using the representation theorem (Theorem 1.1), we see that $x(t) \in$ n-RV(ρ) satisfies

$$\lim_{t\to\infty} v(t) = \lim_{t\to\infty} \frac{tx'(t)}{x(t)} = \rho,$$

which implies in particular that $u(t) \to 0$ as $t \to \infty$ and that $u(t)^2 = v(t)^2/t^2$ is integrable on $[t_0, \infty)$. From (2.2) it then follows that $q(t)$ is integrable on $[t_0, \infty)$, and integration of (2.2) from t to ∞ yields

$$u(t) = \int_t^\infty u(s)^2 ds + \int_t^\infty q(s)\, ds, \quad t \geqslant t_0,$$

which, written for $v(t)$, takes the form

$$v(t) = t\int_t^\infty \frac{v(s)^2}{s^2}\, ds + t\int_t^\infty q(s)\, ds, \quad t \geqslant t_0. \tag{2.3}$$

Letting $t \to \infty$ in (2.3), we conclude that (2.1) holds as desired. □

Our task therefore is to seek RV(1/2)-solutions of equation (A) under condition (1.5). We are based on the simple fact that the change of variables $x = t^{1/2}y$ transforms (A) into the differential equation

$$(ty')' + t\Big(q(t) - \frac{1}{4t^2}\Big)y = 0. \tag{2.4}$$

Thus, to obtain a solution of (A) belonging to RV(1/2) of (A) it suffices to verify the existence of an SV-solution $y(t)$ of (2.4) and form the function $x(t) = t^{1/2}y(t)$. As is shown in [5], the class of generalized regularly varying functions with respect to $\log t$, which is smaller than the classical regularly varying functions, is a well-suited framework for the asymptotic analysis of (2.4). So, our attention will be directed to the existence of $\mathrm{RV}_{\log t}(\rho)$-solutions of differential equations of the form

$$(ty')' + p(t)y = 0, \tag{2.5}$$

where $p(t)$ is continuous and integrable on $[a, \infty)$.

Basic to the subsequent development is the following existence theorem.

THEOREM 2.1. *Let $d \in (-\infty, \frac{1}{4})$ be a given constant and let μ_0 and μ_1, $\mu_0 < \mu_1$, be the real roots of the quadratic equation*

$$\mu^2 - \mu + d = 0. \tag{2.6}$$

Then, equation (2.5) has a fundamental set of solutions $\{y_0(t), y_1(t)\}$ such that

$$y_0(t) \in \text{n-RV}_{\log t}(\mu_0), \quad y_1(t) \in \text{n-RV}_{\log t}(\mu_1) \tag{2.7}$$

if and only if

$$\lim_{t\to\infty} \log t \int_t^\infty p(s)\, ds = d. \tag{2.8}$$

PROOF. (The “only if" part) Suppose that (2.5) possesses a pair of solutions on $[t_0,\infty)$ satisfying (2.7). Let $y(t)$ stands for $y_i(t)$, $i=0$ or 1. Put

$$u(t)=\frac{ty'(t)}{y(t)},\quad v(t)=u(t)\log t. \tag{2.9}$$

Using the representation for $y(t)$

$$y(t)=y(t_0)\exp\Big\{\int_{t_0}^t \frac{\delta(s)}{s\log s}\,ds\Big\},\quad \lim_{t\to\infty}\delta(t)=\mu,$$

where $\mu=\mu_i$, $i=0$ or 1, we see that

$$\lim_{t\to\infty}v(t)=\lim_{t\to\infty}u(t)\log t=\lim_{t\to\infty}\delta(t)=\mu,$$

which implies that $u(t)\to 0$ as $t\to\infty$ and $u(t)^2/t=v(t)^2/t(\log t)^2$ is integrable on $[t_0,\infty)$. Consequently, integrating from t to ∞ the Riccati equation

$$u'+\frac{u^2}{t}+p(t)=0,\quad t\geqslant t_0, \tag{2.10}$$

satisfied by $u(t)$, we obtain

$$u(t)=\int_t^\infty \frac{u(s)^2}{s}\,ds+\int_t^\infty p(s)\,ds,$$

which, in view of (2.9), can be transformed into

$$v(t)=\log t\int_t^\infty \frac{v(s)^2}{s(\log s)^2}\,ds+\log t\int_t^\infty p(s)\,ds,\quad t\geqslant t_0.$$

Passing to the limit as $t\to\infty$, we conclude that

$$\lim_{t\to\infty}\log t\int_t^\infty p(s)\,ds=\mu-\mu^2=d,$$

ensuring the truth of (2.8).

(The “if" part) Suppose that (2.8) is satisfied. First we will be concerned with the existence of a solution of (2.5) belonging to n-$\mathrm{RV}_{\log t}(\mu_0)$. Note that $\mu_0<1/2$. Put

$$P(t)=\log t\int_t^\infty p(s)\,ds-d.$$

Let l be a positive constant such that

$$\frac{2(\mu_0+2)}{1-2\mu_0}l<1\quad\text{if}\quad \mu_0>0,\qquad \frac{2|\mu_0|+2}{1+2|\mu_0|}l<1\quad\text{if}\quad \mu_0<0$$

and choose $t_0\geqslant a$ so that $|P(t)|\leqslant l^2$, $t\geqslant t_0$. Define

$$V=\{v\in C_0[t_0,\infty):|v(t)|\leqslant l,\ \ t\geqslant t_0\},$$

where $C_0[t_0,\infty)$ denotes the set of all continuous functions on $[t_0,\infty)$ that tend to 0 as $t\to\infty$. Clearly, $C_0[t_0,\infty)$ is a Banach space with the norm $\|v\|_0=\sup_{t\geqslant t_0}|v(t)|$.

Consider the integral operator $\mathcal{F}$ defined by

$$\mathcal{F}v(t)=(\log t)^{1-2\mu_0}\int_t^\infty \frac{2\mu_0P(s)+(v(s)+P(s))^2}{s(\log s)^{2-2\mu_0}}\,ds,\quad t\geqslant t_0.$$

It can be shown easily that $v \in V$ implies $|\mathcal{F}v(t)| \leqslant l,\ t \geqslant t_0$, and that $v_1, v_2 \in V$ implies $\|\mathcal{F}v_1 - \mathcal{F}v_2\|_0 \leqslant 4l/(1-2\mu_0)\cdot\|v_1 - v_2\|_0$. This shows that $\mathcal{F}$ is a contraction mapping on V, and so there exists a unique $v \in V$ such that $v = \mathcal{F}v$, that is, $v = v(t)$ is a solution of the integral equation

$$v(t) = (\log t)^{1-2\mu_0} \int_t^\infty \frac{2\mu_0 P(s) + (v(s) + P(s))^2}{s(\log s)^{2-2\mu_0}}\, ds, \quad t \geqslant t_0. \tag{2.11}$$

With this $v(t)$ we construct the function

$$y_0(t) = \exp\Big\{\int_{t_0}^t \frac{\mu_0 + v(s) + P(s)}{s \log s}\, ds\Big\}, \quad t \geqslant t_0,$$

and claim that $y_0(t)$ is a solution of equation (2.5) in $\mathrm{RV}_{\log t}(\mu_0)$. That $y_0(t) \in \mathrm{RV}_{\log t}(\mu_0)$ is a consequence of Theorem 1.7 since $\mu_0 + v(t) + P(t) \to \mu_0$ as $t \to \infty$. To show that $y_0(t)$ is a solution of (2.5) it suffices to verify that the function $u(t) = (\mu_0 + v(t) + P(t))/\log t$ satisfies the Riccati equation (2.10) on $[t_0, \infty)$. It is a matter of elementary calculation to see that (2.10) can then be transformed into

$$\Big(\frac{v(t)}{(\log t)^{1-2\mu_0}}\Big)' + \frac{2\mu_0 P(t) + (v(t) + P(t))^2}{t(\log t)^{2-2\mu_0}} = 0,$$

But this is the differential equation that follows from differentiation of the integral equation (2.11). This establishes the existence of an $\mathrm{RV}_{\log t}(\mu_0)$-solution of equation (2.5). □

Up to this point the smaller root μ_0 of (2.6) has been tacitly assumed to be nonzero. We remark here that if $\mu_0 = 0$, which occurs in the case $d = 0$, the construction of the solution $y_0(t)$ of equation (2.5) becomes slightly simpler, and proceeds as follows. We let a constant $0 < l < 1/4$ be fixed, choose $T > a$ so that $|P(t)| \leqslant l$ for $t \geqslant T$, solve, with the help of the contraction mapping principle, the integral equation

$$v(t) = \log t \int_t^\infty \frac{(v(s) + P(s))^2}{s(\log s)^2}\, ds, \quad t \geqslant T,$$

in the set $V = \{v \in C_0[T, \infty) : 0 \leqslant v(t) \leqslant l,\ \ t \geqslant T\}$, and finally form the function

$$y_0(t) = \exp\Big\{\int_T^t \frac{v(s) + P(s)}{s \log s}\, ds\Big\}, \quad t \geqslant T.$$

Then $y_0(t)$ is shown to be an $\mathrm{SV}_{\log t}$-solution of (2.5).

Next, we turn to the construction of an $\mathrm{RV}_{\log t}(\mu_1)$-solution of (2.5). We note that $\mu_1 > 1/2$. Let m be a positive constant such that

$$m < \frac{1}{4} \quad \text{and} \quad \frac{2(\mu_1 + 2)}{2\mu_1 - 1} m \leqslant 1.$$

Choose $t_1 \geqslant a$ so that $|P(t)| \leqslant m^2$, $t \geqslant t_1$, and consider the set W and the integral operator $\mathcal{G}$ defined by $W = \{w \in C_0[t_1, \infty) : |w(t)| \leqslant m,\ \ t \geqslant t_1\}$, and

$$\mathcal{G}w(t) = (\log t)^{1-2\mu_1} \int_{t_1}^t s^{-1} (\log s)^{2\mu_1 - 2} \big\{2\mu_1 P(s) + (w(s) - P(s))^2\big\}\, ds, \ \ t \geqslant t_1.$$

It is straightforward to check that $w \in W$ implies $|\mathcal{G}w(t)| \leqslant m, \quad t \geqslant t_1$, and that $w_1, w_2 \in W$ implies $\|\mathcal{G}w_1 - \mathcal{G}w_2\|_0 \leqslant 4m/(2\mu_1 - 1)\cdot\|w_1 - w_2\|_0$. This shows that $\mathcal{G}$ is a contraction on W, so that there exists a $w \in W$ such that $w = \mathcal{G}w$, which is equivalent to the integral equation

(2.12)

$$w(t) = (\log t)^{1-2\mu_1} \int_{t_1}^{t} s^{-1}(\log s)^{2\mu_1-2}\{2\mu_1 P(s) + (w(s) - P(s))^2\}\, ds, \quad t \geqslant t_1.$$

Let $y_1(t)$ be defined with this $w(t)$ by

$$y_1(t) = \exp\left\{\int_{t_1}^{t} \frac{\mu_1 - w(s) + P(s)}{s \log s}\, ds\right\}, \quad t \geqslant t_1.$$

On the one hand, $y_1(t) \in \mathrm{RV}_{\log t}(\mu_1)$ since $-w(t) + P(t) \to 0$ as $t \to \infty$, and on the other, $y_1(t)$ is a solution of equation (2.5) since $u(t) = (\mu_1 - w(t) + P(t))/\log t$ satisfies the Riccati equation (2.10) on $[t_1, \infty)$. In fact, substituting $u(t)$ for (2.10) yields

$$((\log t)^{2\mu_1-1} w(t))' = (\log t)^{2\mu_1-2}\{2\mu_1 P(t) + (w(t) - P(t))^2\},$$

which follows from direct differentiation of (2.12). Thus condition (2.8) is also sufficient for equation (2.5) to possess an $\mathrm{RV}_{\log t}(\mu_1)$-solution. This completes the proof of Theorem 2.1.

The main result of this section is obtained as a corollary to Theorem 2.1.

THEOREM 2.2. *Let d, μ_0 and μ_1 be as in Theorem 2.1and suppose the function $t(q(t) - 1/4t^2)$ is integrable over (t, ∞). Then equation* (A) *possesses a fundamental set of solutions $\{x_0(t), x_1(t)\}$ such that*

$$x_i(t) = t^{1/2} y_i(t), \quad y_i(t) \in \text{n-}\mathrm{RV}_{\log t}(\mu_i), \quad i = 0, 1, \tag{2.13}$$

if and only if

$$\lim_{t\to\infty} \log t \int_t^\infty s\Big(q(s) - \frac{1}{4s^2}\Big)\, ds = d. \tag{2.14}$$

PROOF. Observe that equation (A) has a fundamental set of solutions $x_i(t)$, $i = 0, 1$, as described in (2.13) if and only if the functions $y_i(t) = t^{-1/2}x_i(t)$, $i = 0, 1$, are solutions of (2.4) belonging to $\mathrm{RV}_{\log t}(\mu_i)$, $i = 0, 1$, and apply Theorem 2.1 to equation (2.4) which is a special case of (2.5) with $p(t) = t(q(t) - 1/4t^2)$. □

REMARK 2.1. It should be noticed that condition (2.14) automatically implies (1.5). In fact, (2.14) guarantees the existence of regularly varying solutions of index $\frac{1}{2}$ for equation (A), which means by Lemma 2.1 that (1.5) must be satisfied.

EXAMPLE 2.1. Consider the perturbed Euler equation

$$x'' + q(t)x = 0, \quad q(t) = \frac{1}{4t^2} + \frac{d(t)}{(t\log t)^2}, \tag{2.15}$$

where $d(t)$ is a positive continuous function on $[e, \infty)$ such that $\lim_{t\to\infty} d(t) = d \in (0, \frac{1}{4})$. Since

$$\lim_{t\to\infty} \log t \int_t^\infty s\Big(q(s) - \frac{1}{4s^2}\Big)\, ds = \lim_{t\to\infty} \log t \int_t^\infty \frac{d(s)}{s(\log s)^2}\, ds = d,$$

it follows from Theorem 2.2 that equation (2.15) has two linearly independent solutions $x_i(t) = t^{1/2} y_i(t)$, $i = 0, 1$, as described in (2.13).

Example 2.2. Our next example is the following.

$$x'' + q(t)x = 0, \quad q(t) = \frac{1}{4t^2} + \frac{1}{t^2(\log t)^2 \log\log t}. \tag{2.16}$$

An easy calculation shows that (2.14) holds for $d = 0$ and so Theorem 2.2 ensures the existence of two linearly independent solutions $x_(t) = t^{1/2} y_i(t)$, $i = 0, 1$, such that $y_0(t) \in \mathrm{SV}_{\log t}$ and $y_1(t) \in \mathrm{RV}_{\log t}(1)$. Since $t^{1/2}\log\log t$ is a solution of (2.16), one can take $x_0(t) = t^{1/2}\log\log t$ and $x_1(t) = x_0(t)\int_T^t ds/x_0(s)^2$, $T > e$ being sufficiently large. As is easily seen,

$$t^{-1/2}x_1(t) = \log\log t \int_T^t \frac{ds}{s(\log\log s)^2} \sim \frac{\log t}{\log\log t} \in \mathrm{RV}_{\log t}(1).$$

Observe that Theorem 1.6 is not applicable to either of these examples.

3. Regularly varying solutions of (B) in the critical case

We now turn our attention to functional differential equations with retarded argument of the form (B)

$$x''(t) + q(t)x(g(t)) = 0,$$

where $q(t)$ and $g(t)$ are positive and continuous on $[a, \infty)$, and $g(t)$ is increasing and satisfies $g(t) < t$ and $\lim_{t\to\infty} g(t) = \infty$. We look for regularly varying solutions of (B) under the assumption that

$$\lim_{t\to\infty} t \int_t^\infty q(s)\, ds = \frac{1}{4}. \tag{3.1}$$

In view of Theorem 1.5 it is natural to expect that the nonoscillatory nature of (B) would be similar to that of the ordinary differential equation (A) provided the retarded argument $g(t)$ is a small perturbation of t, or more specifically that condition (3.1) would ensure the existence of $\mathrm{RV}(\frac{1}{2})$-solutions as described in Theorem 2.2 if $g(t)$ is "sufficiently close" to t in some sense. The objective of this section is to demonstrate the truth of the above expectation by proving the following theorem.

Theorem 3.1. *Suppose that $q(t) - 1/4t^2$ is eventually positive and that $g(t)$ has the property that*

$$\frac{g(t)}{t} = 1 + O\Big(\frac{1}{t^\alpha}\Big) \quad as \quad t \to \infty, \tag{3.2}$$

for some $\alpha > 0$. Let a constant $d \in [0, \frac{1}{4})$ be fixed and let μ_0 and $\mu_1, \mu_0 < \mu_1$, be the roots of the quadratic equation

$$\mu^2 - \mu + d = 0. \tag{3.3}$$

Then, equation (B) *possesses two* n-$\mathrm{RV}(\frac{1}{2})$*-solutions $x_i(t)$, $i = 0, 1$, such that*

$$x_i(t) = t^{1/2} y_i(t), \quad y_i(t) \in \text{n-}\mathrm{RV}_{\log t}(\mu_i), \quad i = 0, 1, \tag{3.4}$$

if and only if

$$\lim_{t\to\infty} \log t \int_t^\infty s\Big(q(s) - \frac{1}{4s^2}\Big)\, ds = d. \tag{3.5}$$

Proof. (The "only if" part) Suppose that (B) has two regularly varying solutions $x(t) = t^{1/2}y(t)$ as described in (3.4), where $y(t)$ stands for $y_i(t)$, $i = 0$ or 1. Then, $y(t)$ is an $\mathrm{RV}_{\log t}(\mu_i)$-solution, $i = 0$ or 1, of the differential equation

$$(ty'(t))' + t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2} q(t)y(g(t)) - \frac{1}{4t^2}y(t)\bigg) = 0, \tag{3.6}$$

which can be regarded as a linear ordinary differential equation

$$(ty'(t))' + q_y(t)y(t) = 0, \quad q_y(t) = t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{y(g(t))}{y(t)}q(t) - \frac{1}{4t^2}\bigg). \tag{3.7}$$

From the "only if" part of Theorem 2.2 we see that

$$\lim_{t\to\infty} \log t \int_t^\infty s\bigg(\Big(\frac{g(s)}{s}\Big)^{1/2}\frac{y(g(s))}{y(s)}q(s) - \frac{1}{4s^2}\bigg)\, ds = d. \tag{3.8}$$

We now rewrite the integrand of (3.8) as follows:

$$\begin{aligned} t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{y(g(t))}{y(t)}q(t) - \frac{1}{4t^2}\bigg) \\ = t\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{y(g(t))}{y(t)}\Big(q(t) - \frac{1}{4t^2}\Big) + \bigg(\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{y(g(t))}{y(t)} - 1\bigg)\frac{1}{4t}. \end{aligned} \tag{3.9}$$

Since $y(t) \in \text{n-}\mathrm{RV}_{\log t}(\mu_i)$, $y(t)$ is expressed in the form

$$y(t) = \exp\bigg\{\int_{t_0}^t \frac{\delta(s)}{s\log s}\, ds\bigg\}, \quad \lim_{t\to\infty}\delta(t) = \mu_i,$$

for some $t_0 > 1$, so that

$$\frac{y(g(t))}{y(t)} = \exp\bigg\{-\int_{g(t)}^t \frac{\delta(s)}{s\log s}\, ds\bigg\},$$

from which, noting that $0 \leqslant \delta(t) \leqslant 1$ for sufficiently large t, we have

$$\frac{\log g(t)}{\log t} \leqslant \frac{y(g(t))}{y(t)} \leqslant 1 \quad \text{for all large } t \tag{3.10}$$

Since (3.2) implies

$$\frac{\log g(t)}{\log t} = 1 + O\Big(\frac{1}{t^\alpha \log t}\Big) \quad \text{and} \quad \Big(\frac{g(t)}{t}\Big)^{1/2} = 1 + O\Big(\frac{1}{t^\alpha}\Big) \tag{3.11}$$

as $t \to \infty$, we find from (3.10) that

$$\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{y(g(t))}{y(t)} = 1 + O\Big(\frac{1}{t^\alpha}\Big) \quad \text{as} \quad t \to \infty,$$

which implies that

$$\lim_{t\to\infty} \log t \int_t^\infty \bigg(\Big(\frac{g(s)}{s}\Big)^{1/2}\frac{y(g(s))}{y(s)} - 1\bigg)\frac{1}{4s}\, ds = 0. \tag{3.12}$$

Then, combining (3.9) with (3.8) and (3.12), we conclude that

$$\lim_{t\to\infty} \log t \int_t^\infty s\Big(\frac{g(s)}{s}\Big)^{1/2} \frac{y(g(s))}{y(s)}\Big(q(s) - \frac{1}{4s^2}\Big)\, ds = d,$$

from which (3.5) can be derived easily.

(The “if" part) Suppose that (3.5) holds. We begin with the case where $d > 0$, so that the roots of (3.3) satisfies $0 < \mu_0 < \frac{1}{2} < \mu_1 < 1$. By means of the identity

$$\begin{aligned} t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{\log g(t)}{\log t}q(t) - \frac{1}{4t^2}\bigg) \\ = \Big(\frac{g(t)}{t}\Big)^{1/2}\frac{\log g(t)}{\log t}t\Big(q(t) - \frac{1}{4t^2}\Big) + \bigg(\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{\log g(t)}{\log t} - 1\bigg)\frac{1}{4t}, \end{aligned}$$

it can be shown that

$$\lim_{t\to\infty} \log t \int_t^\infty \bigg(\Big(\frac{g(s)}{s}\Big)^{1/2}\frac{\log g(s)}{\log s}q(s) - \frac{1}{4s^2}\bigg)\, ds = d. \tag{3.13}$$

In fact, using the relation

$$\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{\log g(t)}{\log t} = 1 + O\Big(\frac{1}{t^\alpha}\Big) \quad \text{as} \quad t\to\infty,$$

(cf. (3.11)), we see that

$$\lim_{t\to\infty} \log t \int_t^\infty \bigg(\Big(\frac{g(s)}{s}\Big)^{1/2}\frac{\log g(s)}{\log s} - 1\bigg)\frac{1}{4s}\, ds = 0$$

and

$$\lim_{t\to\infty} \log t \int_t^\infty \Big(\frac{g(s)}{s}\Big)^{1/2}\frac{\log g(s)}{\log s}s\Big(q(s) - \frac{1}{4s^2}\Big)\, ds = d,$$

from which (3.13) follows immediately.

Our first task is to prove the existence of an RV($\frac{1}{2}$)-solution $x_0(t)$ of equation (A) such that

$$x_0(t) = t^{1/2}y_0(t), \quad y_0(t) \in \text{n-RV}_{\log t}(\mu_0).$$

Let l be a positive constant such that

$$\frac{2(\mu_0+2)}{1-2\mu_0}l < 1,$$

and choose $T > a$ so large that $g(T) \geqslant a$ and

$$\bigg|\log t \int_t^\infty s\Big(q(s) - \frac{1}{4s^2}\Big)\, ds - d\bigg| \leqslant l^2, \tag{3.14}$$

and

$$\bigg|\log t \int_t^\infty s\bigg(\Big(\frac{g(s)}{s}\Big)^{1/2}\frac{\log g(s)}{\log s}q(s) - \frac{1}{4s^2}\bigg)\, ds - d\bigg| \leqslant l^2, \tag{3.15}$$

for $t \geqslant T$. Let Ξ denote the set consisting of all functions $\xi(t) \in C[g(T),\infty) \cap C^1[T,\infty)$ that are expressed in the form

$$\xi(t) = 1, \;\; g(T) \leqslant t \leqslant T, \qquad \xi(t) = \exp\bigg\{\int_T^t \frac{\delta_\xi(s)}{s\log s}\, ds\bigg\}, \;\; t \geqslant T, \tag{3.16}$$

where $\delta_\xi(t)$ satisfies $0 \leqslant \delta_\xi(t) \leqslant 1$ and $\lim_{t\to\infty} \delta_\xi(t) = \mu_0$. The set Ξ can be regarded as a closed and convex subset of the locally convex space $C^1[T,\infty)$ of all continuously differentiable functions on $[T,\infty)$ with the metric topology of uniform convergence of functions and their derivatives on compact subintervals of $[T,\infty)$. Let $\{\xi_n(t)\}$ be a sequence in Ξ consisting of

$$\xi_n(t) = \exp\left\{\int_T^t \frac{\delta_n(s)}{s\log s}\, ds\right\}, \quad t \geqslant T, \quad n = 1,2,\dots, \tag{3.17}$$

where $\delta_n(t)$ satisfies $0 \leqslant \delta_n(t) \leqslant 1, n = 1,2,..,$ and $\lim_{t\to\infty} \delta_n(t) = \mu_0$. Suppose that $\{\xi_n(t)\}$ and $\{\xi_n'(t)\}$ converge, respectively, to $\xi(t)$ and $\xi'(t)$ on compact subintervals of $[T,\infty)$. Since by (3.17)

$$t\log t\frac{\xi_n'(t)}{\xi_n(t)} = \delta_n(t), \quad t \geqslant T, \quad n = 1,2,\dots, \tag{3.18}$$

we find that

$$\delta(t) := \lim_{n\to\infty} \delta_n(t) = t\log t\frac{\xi'(t)}{\xi(t)}, \quad t \in [T,\infty).$$

It is clear that $\delta(t)$ is continuous on $[T,\infty)$, and satisfies $0 \leqslant \delta(t) \leqslant 1$ and $\lim_{t\to\infty} \delta_n(t) = \mu_0$. It follows that

$$\xi(t) = \exp\left\{\int_T^t \frac{\delta(s)}{s\log s}\, ds\right\}, \quad t \geqslant T,$$

which implies that $\xi(t) \in \Xi$, showing that Ξ is a closed subset of $C^1[T,\infty)$. To make sure the convexity of Ξ let $\xi_1(t),\dots,\xi_N(t)$ be N functions in Ξ and let $c_1,\dots,c_N$ be N positive constants such that $\sum_{k=1}^N c_k = 1$. Then, using (3.18), we see that the function $\sum_{k=1}^N c_k\xi_k(t)$ is expressed in the form

$$\sum_{k=1}^N c_k\xi_k(t) = \exp\left\{\int_T^t \frac{\delta_N(s)}{s\log s}\, ds\right\}, \quad t \geqslant T, \tag{3.19}$$

in terms of the continuous function $\delta_N(t)$ defined by

$$\delta_N(t) = \frac{\sum_{k=1}^N c_k\delta_k(t)\xi_k(t)}{\sum_{k=1}^N c_k\xi_k(t)}.$$

As easily checked, $0 \leqslant \delta_N(t) \leqslant 1$ and $\lim_{t\to\infty} \delta_N(t) = \mu_0$, and hence (3.19) ensures that $\sum_{k=1}^N c_k\xi_k(t) \in \Xi$, proving that Ξ is a convex set.

Another important property of Ξ is that

$$\frac{\log g(t)}{\log t} \leqslant \frac{\xi(g(t))}{\xi(t)} \leqslant 1, \quad t \geqslant T, \tag{3.20}$$

for all $\xi(t) \in \Xi$. This is an immediate consequence of the relation

$$\frac{\xi(g(t))}{\xi(t)} = \exp\left\{-\int_{g(t)}^t \frac{\delta_\xi(s)}{s\log s}\, ds\right\}, \quad 0 \leqslant \delta_\xi(t) \leqslant 1,$$

following from (3.16). In fact, if $t \geqslant T_1$, where $T_1 > T$ is such that $T = g(T_1)$, then we have

$$\exp\Big\{-\int_{g(t)}^{t} \frac{\delta_\xi(s)}{s \log s}\, ds\Big\} \geqslant \exp\Big\{-\int_{g(t)}^{t} \frac{1}{s \log s}\, ds\Big\} = \frac{\log g(t)}{\log t},$$

and if $T \leqslant t \leqslant T_1$, then we have

$$\frac{\xi(g(t))}{\xi(t)} \geqslant \exp\Big\{-\int_{T}^{t} \frac{1}{s \log s}\, ds\Big\} = \frac{\log T}{\log t} \geqslant \frac{\log g(t)}{\log t}.$$

Using (3.20), we find that for all $\xi(t) \in \Xi$

(3.21)
$$t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2} \frac{\log g(t)}{\log t} q(t) - \frac{1}{4t^2}\bigg) \leqslant t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2} \frac{\xi(g(t)}{\xi(t)} q(t) - \frac{1}{4t^2}\bigg) \leqslant t\Big(q(t) - \frac{1}{4t^2}\Big).$$

For simplicity we put for any $\xi(t) \in \Xi$

$$q_\xi(t) = t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2} \frac{\xi(g(t))}{\xi(t)} q(t) - \frac{1}{4t^2}\bigg), \tag{3.22}$$

and

$$Q_\xi(t) = \log t \int_t^\infty q_\xi(s)\, ds - d. \tag{3.23}$$

Because of (3.14) and (3.15) we have $|Q_\xi(t)| \leqslant l^2$, $t \geqslant T$, $\xi(t) \in \Xi$. This inequality makes it possible to apply Theorem 2.1 to each member of the family of ordinary differential equations

$$(ty')' + q_\xi(t) y = 0, \quad \xi(t) \in \Xi, \tag{3.24}$$

obtaining a solution $y_\xi(t)$ in n-$\mathrm{RV}_{\log t}(\mu_0)$ of the form

$$y_\xi(t) = \exp\Big\{\int_T^t \frac{\mu_0 + v_\xi(s) + Q_\xi(s)}{s \log s}\, ds\Big\}, \quad t \geqslant T, \tag{3.25}$$

where $v_\xi(t)$ solves the integral equation

$$v_\xi(t) = (\log t)^{1-2\mu_0} \int_t^\infty \frac{2\mu_0 Q_\xi(s) + (v_\xi(s) + Q_\xi(s))^2}{s(\log s)^{2-2\mu_0}}\, ds, \quad t \geqslant T, \tag{3.26}$$

and $|v_\xi(t)| \leqslant l$ for $t \geqslant T$ and $\lim_{t\to\infty} v_\xi(t) = 0$.

Our final procedure is to show that the set $\{y_\xi(t) : \xi(t) \in \Xi\}$ contains at least one member which provides an $\mathrm{RV}_{\log t}(\mu_0)$-solution of the retarded differential equation (3.7) with the help of the Schauder–Tychonoff fixed point theorem. To this end we define Φ to be the mapping which assigns to each $\xi(t) \in \Xi$ the function $\Phi\xi(t)$ given by

$$\Phi\xi(t) = 1 \ \text{ for } \ g(T) \leqslant t \leqslant T, \qquad \Phi\xi(t) = y_\xi(t) \ \text{ for } \ t \geqslant T. \tag{3.27}$$

(i) Φ maps Ξ into itself. This follows immediately from (3.25). Put $\delta_\xi(t) = \mu_0 + v_\xi(t) + Q_\xi(t)$. Since $|v_\xi(t) + Q_\xi(t)| \leqslant 2l$, $t \geqslant T$, $v_\xi(t) + Q_\xi(t) \to 0$ as $t \to \infty$ and $\mu_0 \in (0, \frac{1}{2})$, we have $0 \leqslant \delta_\xi(t) \leqslant 1$ for $t \geqslant T$ and $\lim_{t\to\infty} \delta_\xi(t) = \mu_0$, which ensures that $\Phi\xi(t) \in \Xi$.

(ii) $\Phi(\Xi)$ is relatively compact in $C^1[T,\infty)$. In virtue of (3.25) we obtain for any $\xi(t)\in\Xi$ and for $t\geqslant T$

$$y_\xi(t)\leqslant \exp\biggl\{\int_T^t \frac{1}{s\log s}\,ds\biggr\}=\frac{\log t}{\log T},$$

$$|y_\xi'(t)|=y_\xi(t)\frac{|\mu_0+v_\xi(t)+Q_\xi(t)|}{t\log t}\leqslant\frac{1}{T\log T},$$

and

$$|(ty_\xi'(t))'|=|q_\xi(t)|y_\xi(t)\leqslant t\Bigl(q(t)+\frac{1}{4t^2}\Bigr)\frac{\log t}{\log T}.$$

This clearly establishes, via the Arzela–Ascoli lemma, the relative compactness of the set $\Phi(\Xi)$.

(iii) Φ is a continuous mapping. Let $\{\xi_n(t)\}$ be a sequence in Ξ converging to $\xi(t)$ in $C^1[T,\infty)$. This means that $\xi_n(t)\to\xi(t)$ and $\xi_n'(t)\to\xi_n'(t)$ as $n\to\infty$ uniformly on compact subintervals of $[T,\infty)$. We have to prove that

$$\Phi\xi_n(t)\to\Phi\xi(t)\quad\text{and}\quad(\Phi\xi_n)'(t)\to(\Phi\xi)'(t)\tag{3.28}$$

uniformly on any compact subinterval of $[T,\infty)$. Using (3.25), we obtain for $t\geqslant T$

$$\begin{aligned}|\Phi\xi_n(t)-\Phi\xi(t)|&=|y_{\xi_n}(t)-y_\xi(t)|\\ &\leqslant\biggl|\exp\biggl\{\int_T^t\frac{\mu_0+v_{\xi_n}(s)+Q_{\xi_n}(s)}{s\log s}\,ds\biggr\}-\exp\biggl\{\int_T^t\frac{\mu_0+v_\xi(s)+Q_\xi(s)}{s\log s}\,ds\biggr\}\biggr|\\ &\leqslant\frac{\log t}{\log T}\int_T^t\frac{|v_{\xi_n}(s)-v_\xi(s)|+|Q_{\xi_n}(s)-Q_\xi(s)|}{s\log s}\,ds,\end{aligned}$$

and

$$\begin{aligned}|(\Phi\xi_n)'(t)-(\Phi\xi)'(t)|&=|y_{\xi_n}'(t)-y_\xi'(t)|\\ &=\biggl|y_{\xi_n}(t)\frac{\mu_0+v_{\xi_n}(t)+Q_{\xi_n}(t)}{t\log t}-y_\xi(t)\frac{\mu_0+v_\xi(t)+Q_\xi(t)}{t\log t}\biggr|\\ &\leqslant\biggl|(y_{\xi_n}(t)-y_\xi(t))\frac{\mu_0+v_{\xi_n}(t)+Q_{\xi_n}(t)}{t\log t}+y_\xi(t)\frac{(v_{\xi_n}(t)-v_\xi(t))+(Q_{\xi_n}(t)-Q_\xi(t))}{s\log s}\biggr|\\ &\leqslant\frac{|y_{\xi_n}(t)-y_\xi(t)|}{t\log t}+\frac{\log t}{\log T}\frac{|v_{\xi_n}(t)-v_\xi(t)|+|Q_{\xi_n}(t)-Q_\xi(t)|}{t\log t}.\end{aligned}$$

Consequently, in order to verify (3.28) it suffices to prove that the two sequences

$$\frac{|v_{\xi_n}(t)-v_\xi(t)|}{t\log t},\qquad\frac{|Q_{\xi_n}(t)-Q_\xi(t)|}{t\log t}\tag{3.29}$$

converge to 0 uniformly on compact subintervals of $[T,\infty)$. As a matter of fact, one can prove the uniform convergence on $[T,\infty)$ of the sequences

$$\frac{|v_{\xi_n}(t)-v_\xi(t)|}{\log t}\quad\text{and}\quad\frac{|Q_{\xi_n}(t)-Q_\xi(t)|}{\log t}.\tag{3.30}$$

As for the second sequence in (3.30) we have from (3.23)

$$\frac{|Q_{\xi_n}(t)-Q_\xi(t)|}{\log t} \leqslant \int_t^\infty |q_{\xi_n}(s)-q_\xi(s)|ds,$$

and estimate the integrand as follows:

$$
\begin{aligned}
(3.31)\quad |q_{\xi_n}(s)-q_\xi(s)| &\leqslant \Big(\frac{g(s)}{s}\Big)^{1/2}\Big|\frac{\xi_n(g(s))}{\xi_n(s)}-\frac{\xi(g(s))}{\xi(s)}\Big| s\Big(q(s)-\frac{1}{4s^2}\Big)\\
&+\Big|\Big(\frac{g(s)}{s}\Big)^{1/2}\frac{\xi_n(g(s))}{\xi_n(s)}-1\Big|\frac{1}{4s}+\Big|\Big(\frac{g(s)}{s}\Big)^{1/2}\frac{\xi(g(s))}{\xi(s)}-1\Big|\frac{1}{4s}\\
&\leqslant 2s\Big(q(s)-\frac{1}{4s^2}\Big)+2\Bigg(1-\Big(\frac{g(s)}{s}\frac{\log g(t)}{\log t}\Big)^{1/2}\Bigg)\frac{1}{4s},
\end{aligned}
$$

which shows that $|q_{\xi_n}(t)-q_\xi(t)|$ is bounded by an integrable function on $[T,\infty)$ independent of n. Since $q_{\xi_n}(t)\to q_\xi(t)$ as $n\to\infty$ for every $t\geqslant T$, from the Lebesgue convergence theorem it follows that $|Q_{\xi_n}(t)-Q_\xi(t)|/\log t$ converges to 0 uniformly on $[T,\infty)$, and hence so does the second sequence in (3.29). To examine the first sequence in (3.31) we proceed as follows. Using (3.26), we have

(3.32)

$$\frac{|v_{\xi_n}(t)-v_\xi(t)|}{(\log t)^{1-2\mu_0}} \leqslant 4l\int_t^\infty \frac{|v_{\xi_n}(s)-v_\xi(s)|}{s(\log s)^{2-2\mu_0}}\,ds+(4l+2\mu_0)\int_t^\infty \frac{|Q_{\xi_n}(s)-Q_\xi(s)|}{s(\log s)^{2-2\mu_0}}\,ds$$

for $t\geqslant T$. Putting

$$z(t)=\int_t^\infty \frac{|v_{\xi_n}(s)-v_\xi(s)|}{s(\log s)^{2-2\mu_0}}\,ds,$$

(3.32) is transformed into the differential inequality

$$(3.33)\qquad \big((\log t)^{4l}z(t)\big)' \geqslant -\frac{4l+2\mu_0}{t(\log t)^{1-4l}}\int_t^\infty \frac{|Q_{\xi_n}(s)-Q_xi(s)|}{s(\log s)^{2-2\mu_0}}\,ds,\quad t\geqslant T.$$

We now integrate (3.33) from t to ∞. Noting that $(\log t)^{4l}z(t)\to 0$ as $t\to\infty$ and that the right-hand side of (3.33) is integrable over $[T,\infty)$, we obtain

$$(3.34)\qquad z(t)\leqslant \frac{4l+2\mu_0}{4l(\log t)^{4l}}\int_t^\infty \frac{|Q_{\xi_n}(s)-Q_\xi(s)|}{s(\log s)^{2-2\mu_0-4l}}\,ds,\ \ t\geqslant T.$$

Note that $2-2\mu_0-4l>1$. Using (3.34) in (3.32), we have

$$
\begin{aligned}
&\frac{|v_{\xi_n}(t)-v_\xi(t)|}{(\log t)^{1-2\mu_0}}\\
&\quad\leqslant \frac{4l+2\mu_0}{(\log t)^{4l}}\int_t^\infty \frac{|Q_{\xi_n}(s)-Q_\xi(s)|}{s(\log s)^{2-2\mu_0-4l}}\,ds+(4l+2\mu_0)\int_t^\infty \frac{|Q_{\xi_n}(s)-Q_\xi(s)|}{s(\log s)^{2-2\mu_0}}\,ds\\
&\quad\leqslant \frac{2(4l+2\mu_0)}{(\log t)^{4l}}\int_t^\infty \frac{|Q_{\xi_n}(s)-Q_\xi(s)|}{s(\log s)^{2-2\mu_0-4l}}\,ds
\end{aligned}
$$

for $t\geqslant T$. This shows that $|v_{\xi_n}(t)-v_\xi(t)|/(\log t)^{1-2\mu_0}$ converges to 0 uniformly on $[T,\infty)$, and hence so does the sequence $|v_{\xi_n}(t)-v_\xi(t)|/\log t$. We therefore conclude that the mapping Φ defined by (3.27) is continuous in the topology of $C^1[T,\infty)$. Thus all the hypotheses of the Schauder–Tychonoff fixed point theorem are fulfilled, and so there exists $\xi_0(t) \in \Xi$ such that $\xi_0(t) = \Phi\xi_0(t)$, that is, $\xi_0(t) = y_{\xi_0}(t)$ for $t \geqslant T$. This means that $\xi_0(t)$ satisfies the linear differential equation

$$(t\xi_0(t))' + t\Big(\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{\xi_0(g(t))}{\xi_0(t)}q(t) - \frac{1}{4t^2}\Big)\xi_0(t) = 0,$$

which is rewritten as

$$(t\xi_0(t))' + t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2}q(t)\xi_0(g(t)) - \frac{1}{4t^2}\xi_0(t)\bigg), \quad t \geqslant T.$$

This implies that equation (3.6) has an $\mathrm{RV}(\mu_0)$-solution $\xi_0(t)$ existing on $[T,\infty)$. The function $x_0(t) = t^{1/2}\xi_0(t)$ then provides a solution of equation (B) enjoying the first property in (3.4) with $i = 0$.

Next, we will be concerned with the construction of a solution $x_1(t)$ of equation (B) such that $x_1(t) = t^{1/2}y_1(t)$ with $y_(t) \in \mathrm{RV}(\mu_1)$. Note that $\mu_1 \in (\frac{1}{2}, 1)$. Let m be any positive constant such that

$$\frac{2(2\mu_1+2)}{2\mu_1-1}m \leqslant 1,$$

and choose $T > a$ large enough so that $g(T) \geqslant a$,

$$\Big|\log t\int_t^\infty s\Big(q(s) - \frac{1}{4s^2}\Big)\,ds - d\Big| \leqslant m^2, \tag{3.35}$$

and

$$\bigg|\log t\int_t^\infty \bigg(\Big(\frac{g(s)}{s}\Big)^{1/2}\frac{\log g(s)}{\log s}q(s) - \frac{1}{4s^2}\bigg)\,ds - d\bigg| \leqslant m^2, \tag{3.36}$$

for $t \geqslant T$. Define H to be the set of all functions $C[g(T),\infty)\cap C^1[T,\infty)$ such that

$$\eta(t) = 1, \;\; g(T) \leqslant t \leqslant T; \qquad \eta(t) = \exp\bigg\{\int_T^t \frac{\delta_\eta(s)}{s\log s}\,ds\bigg\}, \;\; t \geqslant T, \tag{3.37}$$

where $\delta_\eta(t)$ is continuous and satisfies $0 \leqslant \delta_\eta(t) \leqslant 1$ and $\lim_{t\to\infty}\delta_\eta(t) = \mu_1$. As in the preceding case the set H can be regarded as a closed convex subset of the locally convex space $C^1[T,\infty)$. Also note that there hold the inequalities (3.20) and (3.21) with $\xi(t)$ replaced by $\eta(t)$. Put

$$q_\eta(t) = t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2}\frac{\eta(g(t))}{\eta(t)}q(t) - \frac{1}{4t^2}\bigg),$$

and

$$Q_\eta(t) = \log t\int_t^\infty q_\eta(s)\,ds - d.$$

Since $Q_\eta(t)$ satisfies $|Q_\eta(t)| \leqslant m^2$, $t \geqslant T$, for all $\eta(t) \in H$, by (3.35) and (3.36), applying Theorem 2.1 to the family of linear ordinary differential equations

$$(ty')' + q_\eta(t)y = 0, \;\; \eta(t) \in H,$$

we obtain for each $\eta \in H$ a solution $Y_\eta(t) \in \mathrm{RV}_{\log t}(\mu_1)$ having the expression

$$Y_\eta(t) = \exp\bigg\{\int_T^t \frac{\mu_1 - w_\eta(s) + Q_\eta(s)}{s\log s}\,ds\bigg\}, \;\; t \geqslant T, \tag{3.38}$$

where $w_\eta(t)$ is a solution of the integral equation

$$w_\eta(t) = (\log t)^{1-2\mu_1} \int_T^t \frac{2\mu_1 Q_\eta(s) + (w_\eta(s) - Q_\eta(s))^2}{s(\log s)^{2-2\mu_1}}\, ds, \tag{3.39}$$

satisfying $|w_\eta(t)| \leqslant m$ for $t \geqslant T$ and $\lim_{t\to\infty} w_\eta(t) = 0$.

We define by Ψ the mapping which assigns to each $\eta(t) \in H$ the function

$$\Psi\eta(t) = 1, \;\; g(T) \leqslant t \leqslant T, \qquad \Psi\eta(t) = Y_\eta(t), \;\; t \geqslant T.$$

Our final step is to verify that Ψ is continuous and sends H into a relatively compact subset of H so that the Schauder–Tychonoff fixed point theorem is applicable to Ψ. The same arguments as employed in the case of Φ defined by (3.27) are sufficient for this purpose. To avoid tedious duplications only a brief mention will be made of the continuity of Ψ. Let $\{\eta_n(t)\}$ be a sequence in H converging to $\eta(t)$ in $C^1[T,\infty)$. Our goal is attained if it is shown that

$$Y_{\eta_n}(t) \to Y_\eta(t) \quad \text{and} \quad Y'_{\eta_n}(t) \to Y'_\eta(t) \;\; \text{as} \;\; n \to \infty. \tag{3.40}$$

uniformly on any compact subinterval of $[T,\infty)$. In view of (3.38) we see that (3.40) is assured if the two sequences

$$\frac{|w_{\eta_n}(t) - w_\eta(t)|}{\log t} \;\; \text{and} \;\; \frac{|Q_{\eta_n}(t) - Q_\eta(t)|}{\log t}$$

converge to 0 uniformly on compact subintervals of $[T,\infty)$. We need only to consider the first sequence in (3.41). From (3.39) we obtain the inequality

$$\begin{aligned} &\frac{|w_{\eta_n}(t) - w_\eta(t)|}{(\log t)^{1-2\mu_1}} \\ &\leqslant 4m \int_T^t \frac{|w_{\eta_n}(s) - w_\eta(s)|}{s(\log s)^{2-2\mu_1}}\, ds + (4m+2\mu_1) \int_T^t \frac{|Q_{\eta_n}(s) - Q_\eta(s)|}{s(\log s)^{2-2\mu_1}}\, ds, \quad t \geqslant T. \end{aligned} \tag{3.41}$$

Using the function

$$z(t) = \int_T^t \frac{|w_{\eta_n}(s) - w_\eta(s)|}{s(\log s)^{2-2\mu_1}}\, ds,$$

we transform (3.41) into

$$\Big(\frac{z(t)}{(\log t)^{4m}}\Big)' \leqslant \frac{4m+2\mu_1}{t(\log t)^{4m+1}} \int_T^t \frac{|Q_{\eta_n}(s) - Q_\eta(s)|}{s(\log s)^{2-2\mu_1}}\, ds, \;\; t \geqslant T,$$

which, after integration over $[T,t]$, yields

$$z(t) \leqslant \frac{4m+2\mu_1}{4m} (\log t)^{4m} \int_T^t \frac{|Q_{\eta_n}(s) - Q_\eta(s)|}{s(\log s)^{2-2\mu_1+4m}}\, ds, \tag{3.42}$$

for $t \geqslant T$. Combining (3.41) with (3.42), we have

$$\begin{aligned} \frac{|w_{\eta_n}(t) - w_\eta(t)|}{\log t} &\leqslant \frac{4m+2\mu_1}{(\log t)^{2\mu_1-4m}} \int_T^t \frac{|Q_{\eta_n}(s) - Q_\eta(s)|}{s(\log s)^{2-2\mu_1+4m}}\, ds \\ &\quad + (4m+2\mu_1)(\log t)^{2\mu_1} \int_T^t \frac{|Q_{\eta_n}(s) - Q_\eta(s)|}{s(\log s)^{2-2\mu_1}}\, ds, \quad t \geqslant T, \end{aligned}$$

whence it follows that

$$\frac{|w_{\eta_n}(t) - w_\eta(t)|}{\log t} \leqslant 2(4m + 2\mu_1) \int_T^t \frac{|Q_{\eta_n}(s) - Q_\eta(s)|}{s(\log s)^2}\, ds, \quad t \geqslant T.$$

This ensures the desired convergence of the sequence $|w_{\eta_n}(t) - w_\eta(t)|/\log t$.

Thus the continuity of the mapping Ψ has been assured. Let $\eta_1(t) \in H$ be a fixed point of $\Psi : \eta_1(t) = \Psi\eta_1(t)$. Since $\eta_1(t) = Y_{\eta_1}(t)$ for $t \geqslant T$, $\eta_1(t)$ satisfies the differential equation

$$(t\eta_1'(t))' + t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2} \frac{\eta_1(g(t))}{\eta_1(t)} q(t) - \frac{1}{4t^2}\bigg)\eta_1(t) = 0,$$

or

$$(t\eta_1'(t))' + t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2} q(t)\eta_1(g(t)) - \frac{1}{4t^2}\eta_1(t)\bigg) = 0, \quad t \geqslant T.$$

Since $\eta_1(t) \in \mathrm{RV}_{\log t}(\mu_1)$, the function $x_1(t)$ defined by $x_1(t) = t^{1/2}\eta_1(t)$ gives a second solution of equation (B).

It remains to deal with the case where $d = 0$. In this case the roots of the quadratic equation (3.2) are $\mu_0 = 0$ and $\mu_1 = 1$, and we are required to look for the solutions $y_i(t) \in \text{n-RV}_{\log t}(i)$, $i = 0, 1$, of equation (3.6). Such solutions can be found in essentially the same manner as in the previous case $d > 0$. Only a brief explanation will be given about how to construct a solution $y_0(t) \in \text{n-RV}(0) = \text{n-RV}$. This time, instead of (3.16), we choose the set Ξ consisting of all functions $\xi(t) \in C[g(T), \infty) \cap C^1[T, \infty)$ such that

$$\xi(t) = 1, \ \ g(T) \leqslant t \leqslant T; \qquad \xi(t) = \exp\bigg\{\int_T^t \frac{\delta_\xi(s)}{s \log s}\, ds\bigg\}, \ \ t \geqslant T,$$

where $\delta_\xi(t)$ satisfies $-1 \leqslant \delta_\xi(t) \leqslant 1$ and $\lim_{t\to\infty} \delta_\xi(t) = 0$, and $T > a$ will be determined later. It is clear that

$$\frac{\log g(t)}{\log t} \leqslant \frac{\xi(g(t))}{\xi(t)} \leqslant \frac{\log t}{\log g(t)}, \quad t \geqslant T,$$

for all $\xi(t) \in \Xi$. Consequently,

$$\begin{aligned} (3.43) \quad t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2} \frac{\log g(t)}{\log t} q(t) - \frac{1}{4t^2}\bigg) &\leqslant t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2} \frac{\xi(g(t))}{\xi(t)} q(t) - \frac{1}{4t^2}\bigg) \\ &\leqslant t\bigg(\Big(\frac{g(t)}{t}\Big)^{1/2} \frac{\log t}{\log g(t)} q(t) - \frac{1}{4t^2}\bigg), \quad t \geqslant T. \end{aligned}$$

Let a constant $0 < l < \frac{1}{4}$ be fixed, and choose $T > a$ so that $g(T) \geqslant a$ and both (3.15) (with l^2 replaced by l) and the following inequality are satisfied:

$$(3.44) \qquad \bigg|\log t \int_t^\infty s\bigg(\Big(\frac{g(s)}{s}\Big)^{1/2} \frac{\log s}{\log g(s)} q(s) - \frac{1}{4s^2}\bigg) ds - d\bigg| \leqslant l, \ \ t \geqslant T.$$

Therefore, by (3.15), (3.43) and (3.44), the function $Q_\xi(t)$ defined by (3.23) and (3.22) satisfies $|Q_\xi(t)| \leqslant l$, $t \geqslant T$, for all $\xi(t) \in \Xi$, because of which Theorem 2.2

guarantees, for every $\xi(t) \in \Xi$, the existence of a solution $y_\xi(t) \in \mathrm{SV}_{\log t}$ of equation (3.24) having the representation

$$y_\xi(t) = \exp\biggl\{\int_T^t \frac{v_\xi(s) + Q_\xi(s)}{s\log s}\,ds\biggr\}, \quad t \geqslant T,$$

where $v_\xi(t)$ solves the integral equation

$$v_\xi(t) = \log t \int_t^\infty \frac{(v_\xi(s) + Q_\xi(s))^2}{s(\log s)^2}\,ds, \quad t \geqslant T.$$

Let us define the mapping Φ by (3.27) and repeat the same (somewhat simplified) argument as that applied to the case of positive d. Then, via the Schauder-Tychonoff fixed point theorem, we are led to the conclusion that Φ has a fixed point $\xi_0(t) \in \Xi$, which generates a solution $x_0(t) = t^{1/2}\xi_0(t)$ of (A) with the desired first property.

In order to obtain a solution $x_1(t)$ of (A) with the second property of (3.4) (with $\mu_1 = 1$), we need only to define H replacing (3.37) to be the set of all functions $\eta(t) \in C[g(T),\infty) \cap C^1[T,\infty)$ such that

$$\eta(t) = 1, \ \ g(T) \leqslant t \leqslant T; \qquad \eta(t) = \exp\biggl\{\int_T^t \frac{\delta_\eta(s)}{s\log s}\,ds\biggr\}, \ \ t \geqslant T,$$

where $\delta_\eta(t)$ satisfies $0 \leqslant \delta_\eta(t) \leqslant 2$ and $\lim_{t\to\infty}\delta_\eta(t) = 1$, and apply almost the same argument as in the case of $d > 0$ on the basis of the inequality

$$\Bigl(\frac{\log g(t)}{\log t}\Bigr)^2 \leqslant \frac{\eta(g(t))}{\eta(t)} \leqslant \frac{\log t}{\log g(t)}, \quad t \geqslant T,$$

holding for all $\eta(t) \in H$. No further explanation will be necessary. This completes the proof of Theorem 3.1. □

Example 3.1. Consider the retarded differential equation

$$x''(t) + q(t)x(g(t)) = 0, \quad q(t) = \frac{1}{4t^2} + \frac{d(t)}{(t\log t)^2}, \quad t \geqslant e, \tag{3.45}$$

where $g(t)$ is a retarded argument satisfying the conditions given in Theorem 3.1 and $d(t)$ is a positive continuous function such that $\lim_{t\to\infty} d(t) = d \in [0, \frac{1}{4})$. Since condition (3.5) holds for $q(t)$ (cf. Example 2.1), by Theorem 3.1 equation (3.45) possesses two solutions $x_i(t)$, $i = 0, 1$, such that $t^{-1/2}x_i(t) \in \mathrm{RV}_{\log t}(\mu_i)$, $i = 0, 1$, where μ_i, $i = 0, 1$, are the roots of (3.3). Admissible retarded arguments $g(t)$ include $t - \tau, t - t^\theta, t - \log t$, where τ and $\theta < 1$ are positive constants.

Example 3.2. Consider the retarded differential equation

$$x''(t) + q(t)x(t-\tau) = 0, \tag{3.46}$$

where

$$q(t) = \frac{1}{4t^{3/2}(t-\tau)^{1/2}}\Bigl(\frac{\log t}{\log(t-\tau)}\Bigr)^{1/8} + \frac{7}{64t^{3/2}(t-\tau)^{1/2}(\log t)^{15/8}(\log(t-\tau))^{1/8}}.$$

It ia a matter of elementary computation to see that

$$\lim_{t\to\infty} \log t \int_t^\infty s\Big(q(s) - \frac{1}{4s^2}\Big)\, ds = \frac{7}{64}.$$

This implies via Theorem 3.1 that equation (3.46) possesses two solutions $x_i(t) = t^{1/2} y_i(t)$, $i = 0, 1$, such that $y_0(t) \in \mathrm{RV}_{\log t}(\frac{1}{8})$ and $y_1(t) \in \mathrm{RV}_{\log t}(\frac{7}{8})$. One such solution is $x_0(t) = t^{1/2}(\log t)^{1/8}$.

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