PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 88(102) (2010), 53–65 DOI: 10.2298/PIM1002053D

CHARACTERIZATION OF THE PSEUDO-SYMMETRIES OF IDEAL WINTGEN SUBMANIFOLDS OF DIMENSION 3

Ryszard Deszcz, Miroslava Petrović-Torgašev, Zerrin Şentürk, and Leopold Verstraelen

Communicated by Darko Milinković

Abstract. Recently, Choi and Lu proved that the Wintgen inequality $\rho \leqslant H^2 - \rho^{\perp} + k$, (where ρ is the normalized scalar curvature and H^2, respectively $\rho^{\perp}$, are the squared mean curvature and the normalized scalar normal curvature) holds on any 3-dimensional submanifold M^3 with arbitrary codimension m in any real space form $\widetilde{M}^{3+m}(k)$ of curvature k. For a given Riemannian manifold M^3, this inequality can be interpreted as follows: for all possible isometric immersions of M^3 in space forms $\widetilde{M}^{3+m}(k)$, the value of the intrinsic curvature ρ of M puts a lower bound to all possible values of the extrinsic curvature $H^2 - \rho^{\perp} + k$ that M in any case can not avoid to "undergo" as a submanifold of $\tilde{M}$. From this point of view, M is called a Wintgen ideal submanifold of $\widetilde{M}$ when this extrinsic curvature $H^2 - \rho^{\perp} + k$ actually assumes its theoretically smallest possible value, as given by its intrinsic curvature ρ, at all points of M. We show that the pseudo-symmetry or, equivalently, the property to be quasi-Einstein of such 3-dimensional Wintgen ideal submanifolds M^3 of $\widetilde{M}^{3+m}(k)$ can be characterized in terms of the intrinsic minimal values of the Ricci curvatures and of the Riemannian sectional curvatures of M and of the extrinsic notions of the umbilicity, the minimality and the pseudo-umbilicity of M in $\widetilde{M}$.

2010 *Mathematics Subject Classification*: Primary 53B25, 53B35, 53A10, 53C42.

Key words and phrases: submanifold, Wintgen inequality, pseudo-symmetric manifold, quasi-Einstein space.

The first author is supported by the Wrocław University of Environmental and Life Sciences (Poland). The first and the fourth authors are supported by the Research Foundation–Flanders project G.0432.07 (Belgium). The second author is supported by the Ministry of Science of Republic of Serbia, grant 144030.

1. Wintgen ideal submanifolds

For surfaces M^2 in the Euclidean space E^3, the *Euler inequality* $K \leqslant H^2$, where K is the (intrinsic) *Gauss curvature* of M^2 and H^2 is the (extrinsic) *squared mean curvature* of M^2 in E^3, at once follows from the fact that $K = k_1 k_2$ and $H = \frac{1}{2}(k_1 + k_2)$ where k_1 and k_2 are the *principal curvatures* of M^2 in E^3, and, obviously, $K = H^2$ everywhere on M^2 if and only if the surface M^2 is *totally umbilical* in E^3, i.e., $k_1 = k_2$ at all points of M^2, or still, by a *theorem of Meusnier*, if and only if M^2 is a part of a *plane* E^2 or of a *round sphere* S^2 in E^3.

For surfaces M^2 in the 4-dimensional Euclidean space E^4, Wintgen proved that the Gauss curvature K and the squared mean curvature H^2 and the (extrinsic) *normal curvature* $K^\perp$ always satisfy the inequality $K \leqslant H^2 - K^\perp$, and that actually the equality holds if and only if the *curvature ellipse* of M^2 in E^4 is a *circle* [36]; (cf. e.g. [6, 7] for studies also on the global differential geometry of submanifolds by a.o. Smale, Lashof, Chern, Chen and Willmore concerning the Euler characteristic of the normal bundle, the number of self-intersections and the total mean curvature). This fundamental inequality between the most important intrinsic and extrinsic scalar valued curvatures of surfaces M^2 in E^4 was later shown, by Rouxel and by Guadalupe and Rodriguez, to hold more generally for all surfaces M^2 of *arbitrary codimension* m in the *real space forms* $\tilde{M}^{2+m}(k)$ of constant sectional curvature k, inclusive the characterisation of the equality case [21, 29]. After these extensions of the above *Wintgen inequality* for submanifolds of dimension 2 and of codimension 2 to submanifolds of *dimension* 2 and *arbitrary codimension* $m \geqslant 2$, in 1999 De Smet and Dillen and Vrancken and one of the authors proved the Wintgen inequality $\rho \leqslant H^2 - \rho^\perp + k$, where ρ and $\rho^\perp$ respectively are the (intrinsic) *normalized scalar curvature* and the (extrinsic) *normalized scalar normal curvature*, for 2-*codimensional* submanifolds M^n of *arbitrary dimension* $n \geqslant 2$ in the real space forms $\tilde{M}^{n+2}(k)$, and characterized the equality situation explicitly in terms of the shape operators of M^n in $\tilde{M}^{n+2}(k)$ [12]. Moreover, in [12] it was conjectured that this Wintgen inequality holds for submanifolds M^n of *any dimension* $n \geqslant 2$ and of *any codimension* $m \geqslant 2$ in real space forms $\tilde{M}^{n+m}(k)$, (referring to the initials of the authors of [12], Suceavă recently started to call this "the DDVV conjecture" [31], and was therein followed by others, although the "*conjecture on Wintgen's inequality*" may well be a more appropriate terminology). Recently, Choi and Lu proved that this conjecture is true for all 3-dimensional submanifolds M^3 of *arbitrary codimension* $m \geqslant 2$ in $\widetilde{M}^{3+m}(k)$ and obtained characteristic expressions for the shape operators of the submanifolds M^3 in $\tilde{M}^{3+m}(k)$ which do realize the equality in this general inequality [8]. Concrete descriptions of some classes of 3-dimensional *Wintgen ideal submanifolds* were given by Bryant, Dillen, Fastenakels and Van der Veken [4, 18].

At this stage we would like further finally to mention that De Smet, Dillen, Fastenakels, Van der Veken, Vrancken and one of the authors studied the Wintgen inequality for *invariant submanifolds in Kaehler, nearly Kaehler and Sasakian spaces* [11, 17], and that Gmira, Haesen, Dillen and two of the authors studied this inequality for *submanifolds in semi-Riemannian spaces* [19, 20].

2. Pseudo-symmetric spaces

Let M^n be an n-dimensional *Riemannian manifold* with *metric* $(0,2)$ *tensor* g and *Levi-Civita connection* ∇. Let R denote the $(0,4)$ *Riemann–Christoffel curvature tensor* of M as well as the *curvature* (1,1) *operator* $R(X,Y) := \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}$, thus having

$$R(X,Y,Z,W) := g(R(X,Y)Z,W), \tag{1}$$

where X, Y, etc. denote arbitrary vector fields on M and $[.,.]$ stands for the Lie bracket. By the action of the curvature operator working as a derivation on the curvature tensor R, the following $(0,6)$ tensor $R\cdot R$ is obtained:

$$\begin{aligned}(R\cdot R)(X_1,X_2,X_3,X_4;X,Y) &:= (R(X,Y)\cdot R)(X_1,X_2,X_3,X_4)\\ &= -R(R(X,Y)X_1,X_2,X_3,X_4) - R(X_1,R(X,Y)X_2,X_3,X_4)\\ &\quad - R(X_1,X_2,R(X,Y)X_3,X_4) - R(X_1,X_2,X_3,R(X,Y)X_4).\end{aligned}$$

As was recently shown by Haesen and one of the authors [22], the tensor $R\cdot R$ can be geometrically interpreted as giving *the second-order measure of the change of the sectional curvatures $K(p,\pi)$ for tangent 2-planes π at points p after the parallel transport of π around infinitesimal co-ordinate parallelograms in M cornered at p.* Thus, the *semi-symmetric* or *Szabó symmetric spaces* [32, 33], i.e., the manifolds M for which $R\cdot R = 0$, are those Riemannian manifolds for which all sectional curvatures remain preserved after the parallel transport of their planes around all infinitesimal co-ordinate parallelograms. The *locally symmetric* or *Cartan symmetric spaces*, i.e., the manifolds M for which $\nabla R = 0$, constitute a proper subclass of the class of the Szabó symmetric spaces.

We recall that the definition (1) of the curvature tensor goes back to Schouten's geometrical interpretation of R as *the second order measure of the change of the direction of vector fields after their parallel transport around closed infinitesimal curves on M* [30]. Then the *locally flat* or *locally Euclidean* spaces, thus the manifolds M for which $R = 0$, are those Riemannian manifolds for which all directions remain preserved after parallel transport around all closed infinitesimal curves. The simplest nonflat Riemannian manifolds M are the *spaces of constant curvature* $K = k$, i.e., the spaces whose function K is *isotropic* (meaning that, at each point p, the Gauss curvature $K(p,\pi)$ at p of the local surface formed by the geodesics of M which pass through p and whose tangent vector at p lies in π, has the same value for all choices of planes π at p, thus K becoming a real function on M, which by the lemma of Schur, for $n > 2$, then necessarily has to be constant). These real *space forms* $M^n(k)$, by a theorem of Beltrami, can be obtained from the locally Euclidean spaces by *projective transformations* and their class is closed under such transformations. Further, we also recall that the knowledge of the curvature tensor R is *equivalent* to the knowledge of the sectional or Riemannian curvatures K, as was shown by Cartan. Finally, as is well known, the curvature tensor R of a space of constant curvature k is given by

$$R(X,Y,Z,W) = k\, g((X\wedge_g Y)Z,W), \tag{2}$$

where the $\wedge_g$ stands for the *metrical endomorphism* $(X \wedge_g Y)Z := g(Y,Z)X - g(X,Z)Y$. Thus for the real space forms $M^n(k)$, $n > 2$, there exists a *real valued* function K on M such that $R(X,Y,Z,W) = K\,G(X,Y,Z,W)$, where the $(0,4)$-tensor G is defined by $G(X,Y,Z,W) := g((X \wedge_g Y)Z,W)$.

A main interest of Riemann, Helmholtz, Lie, Klein,... in the spaces of constant curvature was related to the fact that these are precisely the Riemannian manifolds which satisfy the *axiom of free mobility*.

Now, similarly as proceeding from the locally Euclidean spaces to the real space forms, one can proceed from the Szabó symmetric spaces to the *pseudo-symmetric* or *Deszcz symmetric spaces* [1, 13, 22, 35]. The pseudo-symmetric spaces were defined as the manifolds M for which the $(0,6)$ tensor $R\cdot R$ and the $(0,6)$ *Tachibana tensor* $Q(g,R) := -\wedge_g \cdot R$, where the metrical endomorphism $\wedge_g$ acts on the $(0,4)$ curvature tensor R as a derivation, are proportional, say $R\cdot R = L(-\wedge_g \cdot R)$ for some real valued function L on M;

$$\begin{aligned} Q(g,R)(X_1,X_2,X_3,X_4;X,Y) &:= -((X\wedge_g Y)\cdot R)(X_1,X_2,X_3,X_4)\\ &= R((X\wedge_g Y)X_1,X_2,X_3,X_4) + R(X_1,(X\wedge_g Y)X_2,X_3,X_4)\\ &\quad + R(X_1,X_2,(X\wedge_g Y)X_3,X_4) + R(X_1,X_2,X_3,(X\wedge_g Y)X_4). \end{aligned}$$

A classical result states that the identical vanishing of this Tachibana tensor, $Q(g,R) = 0$, characterizes the real space forms. Further, results of Mikesh, Venzi, Defever and Deszcz learn that pseudo-symmetric spaces are obtained by applying *projective transformations* to the semi-symmetric spaces and that the class of the pseudo-symmetric spaces is closed under such transformations. Two 2-planes π and $\bar\pi$, spanned by vectors u,v and x,y respectively, at the same point p of M, are said to be *curvature dependent* if $Q(g,R)(u,v,v,u;x,y) \neq 0$, which is independent of the choices of bases for π and $\bar\pi$. For such planes, the *double sectional curvature* or the *sectional curvature of Deszcz* or the *Riemann curvature of Deszcz* $L(p,\pi,\bar\pi)$ is defined as the real number given by

$$L(p,\pi,\bar\pi) := \frac{(R\cdot R)(u,v,v,u;x,y)}{Q(g,R)(u,v,v,u;x,y)},$$

(which is independent of the choices of bases for π and $\bar\pi$); it is a scalar valued Riemannian invariant. The knowledge of the tensor $R\cdot R$ is *equivalent* to the knowledge of the sectional curvatures $L(p,\pi,\bar\pi)$ of Deszcz. And just like the geometrical interpretation of the sectional curvatures $K(p,\pi)$ of Riemann in terms of the *parallelogramoïds of Levi-Civita* [27], also the sectional curvatures $L(p,\pi,\bar\pi)$ of Deszcz can be interpreted in these terms (in this respect, we refer to [23] and [24] where in particular such interpretations are obtained for the sectional curvatures as well as for the *Ricci* and *conformal Weyl curvatures* of *Deszcz* in terms of the *squaroïds* of Levi-Civita). Finally the Deszcz symmetric spaces are characterized by the *isotropy* of the curvatures $L(p,\pi,\bar\pi)$, i.e., by the property that at every point p of M the scalars $L(p,\pi,\bar\pi)$ are the same for all possible pairs of curvature dependent tangent planes π and $\bar\pi$ at p. In the present situation however there is no lemma of Schur, which then would further force this real valued function

$L: M \to R$ automatically to be constant; therefore, Kowalski and Sekizawa called the pseudo-symmetric spaces for which *the double sectional curvature* L *is* indeed a *constant*, independent of the planes π and $\bar{\pi}$ as well as of the points p of M, the *pseudo-symmetric spaces of constant type* L [26]. By way of examples in this respect we would like to mention here that the 3-*dimensional Thurston geometries* [34], which in a kind of axiomatic way originated as natural *anisotropic extensions* of the spaces of constant Riemannian curvature K with their typical free mobility, all do have *constant sectional curvature* L *of Deszcz* (we set $L = 0$ for E^3 since $(K = c = 0)$, S^3 $(K = c > 0)$, H^3 $(K = c < 0)$, $S^2 \times E^1$ and $H^2 \times E^1$; $L = 1$ for $\widetilde{SL(2,R)}$ and for the 3-*dimensional Heisenberg group* H_3; and $L = -1$ for the Lie group Sol) [2].

A similar study concerning the geometrical meaning of *Ricci pseudo-symmetry in the sense of Deszcz*, i.e., of the manifolds M satisfying the curvature condition $R \cdot S = L_S Q(g,S) = L_S(-\wedge_g \cdot S)$, where S denotes the $(0,2)$ *Ricci curvature tensor* and $Q(g,S) = -\wedge_g \cdot S$ the *Ricci–Tachibana tensor* of M and L_S is a real valued function on M, was carried out by Jahanara, Haesen and two of the authors in [25], (in this respect, see also [9] and [15]). As shown in [16], a 3-dimensional Riemannian manifold M is pseudo-symmetric if and only if it is *quasi-Einstein*, i.e., if its Ricci tensor S has an eigenvalue of multiplicity $\geqslant 2$. The class of the Riemannian manifolds M with pseudo-symmetric Ricci tensor S as such is considerably larger in general than the class of the manifolds M with pseudo-symmetric Riemann–Christoffel tensor R, (which it obviously contains as a subclass). However, as shown in [10], for manifolds of dimension 3, these two pseudo-symmetry conditions are equivalent. As is well known, Schouten and Struik showed that the 3-dimensional Riemannian manifolds M are *Einstein* if and only if they have constant curvature K. In [28] two of the authors made a study of the *pseudo-symmetry in the sense of Deszcz* of the tensors R and S of the Wintgen ideal submanifolds M^n of *dimension* $n > 3$ and of *codimension* 2 in the real space forms $\widetilde{M}^{n+2}(k)$. In particular, they showed that for those Wintgen ideal submanifolds these two, a priori distinct, curvature conditions are equivalent and occur if and only if those submanifolds are either *totally umbilical* or *minimal*. In comparison, the 3-dimensional case will show to offer two additional kinds of pseudo-symmetric Wintgen ideal submanifolds.

3. On the symmetry of ideal submanifolds

Let M^n be a *submanifold* of a *real space form* $\widetilde{M}^{n+m}(k)$ of constant curvature k. Let g, ∇ and R, and, respectively, $\widetilde{g}$, $\widetilde{\nabla}$ and $\widetilde{R}$, denote the *Riemannian metric*, the *Levi-Civita connection* and the *Riemann–Cristoffel* $(0,4)$ *curvature tensor* of M and $\tilde{M}$.

The *formulae of Gauss and Weingarten* then are

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \text{and} \quad \widetilde{\nabla}_X \xi = -A_\xi X + \nabla^\perp_X \xi, \tag{3}$$

where h, A_ξ and $\nabla^\perp$ denote the *second fundamental form*, the *shape operator* or the *Weingarten map* with respect to ξ and the *normal connection* of M in $\widetilde{M}$, respectively, systematically using here and hereafter X, Y, etc. for *tangent vector* *fields* on M and ξ etc. for *normal vector fields* on M in $\tilde{M}$, (as basic references for Riemannian submanifolds, see [5] and [7]).

From (3) it follows that $\widetilde{g}(h(X,Y),\xi) = g(A_\xi(X),Y)$, such that, for any *orthonormal local normal frame* ξ_α on M in $\widetilde{M}$, (α etc. running from 1 till the codimension m),

$$h(X,Y) = \sum_\alpha g(A_\alpha(X),Y)\,\xi_\alpha, \tag{4}$$

where $A_\alpha = A_{\xi_\alpha}$. The *mean curvature vector field* $\vec{H}$ of M in $\tilde{M}$ is defined as $\vec{H} = \frac{1}{n}\operatorname{trace} h$ and its length $H = \|\vec{H}\|$ is the *mean curvature* of M in $\widetilde{M}$. By the *equation of Ricci*, the *normal curvature tensor* $R^\perp$ of M in $\widetilde{M}$ is given as follows:

$$R^\perp(X,Y;\xi,\eta) := \widetilde{g}(R^\perp(X,Y)\xi,\eta) = g([A_\xi,A_\eta](X),Y), \tag{5}$$

where

$$R^\perp(X,Y) := \nabla^\perp_X\nabla^\perp_Y - \nabla^\perp_Y\nabla^\perp_X - \nabla^\perp_{[X,Y]} \text{ and } [A_\xi,A_\eta] := A_\xi A_\eta - A_\eta A_\xi.$$

The *normalized scalar normal curvature* $\rho^\perp$ of M in $\tilde{M}$ is then defined by

$$\rho^\perp := \frac{2}{n(n-1)}\Big\{\sum_{i<j}\sum_{\alpha<\beta}\big[R^\perp(E_i,E_j;\xi_\alpha,\xi_\beta)\big]^2\Big\}^{1/2},$$

for any normal frame ξ_α and for any *orthonormal local tangent frame* E_i on M, (i etc. running from 1 till the dimension n).

We remark that $\rho^\perp = 0$ if and only if *the normal connection is flat*, which, as follows from (5) and as was already observed by Cartan, is equivalent to the *simultaneous diagonalizability of all shape operators* A_ξ. The *equation of Gauss* of M in $\widetilde{M}$ is given by

$$\begin{aligned} R(X,Y,Z,W) &= \widetilde{g}(h(Y,Z),h(X,W)) - \widetilde{g}(h(X,Z),h(Y,W)) \\ &\quad + k\ \{g(Y,Z)\ g(X,W) - g(X,Z)\ g(Y,W)\}. \end{aligned} \tag{6}$$

Let S be the $(0,2)$-*Ricci tensor* of M: $S(X,Y) = \sum_i R(E_i,X,Y,E_i)$. Then from (2), (4) and (6), we obtain that

$$\begin{aligned} S(X,Y) &= (n-1)k\,g(X,Y) + \sum_\alpha \operatorname{trace} A_\alpha\, g(A_\alpha(X),Y) \\ &\quad - \sum_\alpha\sum_i g(A_\alpha(X),E_i)\,g(A_\alpha(Y),E_i), \end{aligned} \tag{7}$$

for any choice of frames E_i and ξ_α. And the *normalized scalar curvature* ρ of M is defined by

$$\rho := \frac{2}{n(n-1)}\sum_{i<j} R(E_i,E_j,E_j,E_i).$$

Choi and Lu gave the following *affirmative solution of the conjecture concerning the inequality of Wintgen for the* 3*-dimensional submanifolds of the real space forms.*

THEOREM 1. [8] *For any M^3 in $\widetilde{M}^{3+m}(k)$, with $m \geqslant 3$:*

$$\rho \leqslant H^2 - \rho^{\perp} + k, \tag{8}$$

and the equality holds if and only if, with respect to suitably chosen local orthonormal tangent and normal frames E_1, E_2, E_3 and $\xi_1, \dots \xi_m$, the shape operators A_α of M in $\widetilde{M}$ take the forms:

$$A_1 = \begin{pmatrix} c & \mu & 0 \\ \mu & c & 0 \\ 0 & 0 & c \end{pmatrix}, \; A_2 = \begin{pmatrix} b+\mu & 0 & 0 \\ 0 & b-\mu & 0 \\ 0 & 0 & b \end{pmatrix}, \; A_3 = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \tag{9}$$

$$A_4 = \cdots = A_m = 0,$$

for some real valued functions a, b, c and μ on M.

On the other hand, from the paper of De Smet, Dillen, Vrancken and one of the authors on Wintgen's inequality, we have the following.

THEOREM 2. [12] *For any M^3 in $\widetilde{M}^5(k)$, (8) is satisfied and the equality holds if and only if, with respect to suitably chosen local orthonormal frames E_1, E_2, E_3 and ξ_1, ξ_2, the shape operators A_1 and A_2 of M in $\widetilde{M}$ are given by*

$$A_1 = \begin{pmatrix} c & \mu & 0 \\ \mu & c & 0 \\ 0 & 0 & c \end{pmatrix}, \quad A_2 = \begin{pmatrix} \mu & 0 & 0 \\ 0 & -\mu & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, for what follows and which concerns essentially dealing with the above shape operators filled into the Gauss equation of M^3 in $\widetilde{M}^{3+m}(k)$, the latter situation ($m = 2$) is in some sense algebraically included in the former one ($m > 2$) by considering $a = b = 0$. And, in the case of codimension 1 there is no question about a normal curvature, we can carry out a general study of the 3-dimensional Wintgen ideal submanifolds M^3 in arbitrary space forms $\widetilde{M}^{3+m}(k)$, $m \geqslant 2$, by dealing with the forms of the shape operators as given in (9). Frames E_1, E_2, E_3 and $\xi_1, \dots, \xi_m$ for which the corresponding shape operators A_α assume such forms which further on will be called *Choi–Lu frames* of Wintgen ideal M^3 in $\tilde{M}^{3+m}$.

From (7) and (9), the $(1,1)$ *Ricci operator* S which is metrically related to the $(0,2)$ Ricci tensor S by $g(S(X), Y) = S(X, Y)$, with respect to Choi–Lu frames E_i and ξ_α is readily found to be given by

$$S = \begin{pmatrix} S_{11} & c\mu & 0 \\ c\mu & S_{22} & 0 \\ 0 & 0 & S_{33} \end{pmatrix}, \tag{10}$$

where

$$\begin{aligned} S_{11} &= 2(a^2 + b^2 + c^2 + k) + \mu(b - 2\mu), \\ S_{22} &= 2(a^2 + b^2 + c^2 + k) - \mu(b + 2\mu), \\ S_{33} &= 2(a^2 + b^2 + c^2 + k). \end{aligned} \tag{11}$$

Hence E_3 always determines a *Ricci principal direction*, with a corresponding *Ricci curvature*

$$\rho_3 = 2\,(a^2 + b^2 + c^2 + k). \tag{12}$$

And, from (10) and (11), the other two *Ricci curvatures*, ρ_1 and ρ_2, corresponding to some orthogonal eigendirections $\widetilde{E}_1$ and $\widetilde{E}_2$ in the plane field $E_1 \wedge E_2$, are easily derived as

$$\begin{aligned} \rho_1 &= \rho_3 - 2\mu^2 + |\mu|\sqrt{b^2 + c^2}, \\ \rho_2 &= \rho_3 - 2\mu^2 - |\mu|\sqrt{b^2 + c^2}. \end{aligned} \tag{13}$$

Since M^3 is *pseudo-symmetric* or still, *symmetric in the sense of Deszcz*, if and only if its Ricci tensor has an eigenvalue of multiplicity $\geqslant 2$, from (13) we have the following.

Lemma 1. *A Wintgen ideal* 3*-dimensional submanifold in a real space form is Deszcz symmetric if and only if*

$$\text{(I)}\ \ \mu = 0 \quad \textit{or} \quad \text{(II)}\ \ \mu \neq 0,\ b = c = 0 \quad \textit{or} \quad \text{(III)}\ \ \mu \neq 0,\ b^2 + c^2 = 4\mu^2.$$

Proceeding more straightforwardly, from (6) and (9) the components, with respect to Choi–Lu frames, of the $(0,4)$ *curvature tensor* R of a Wintgen ideal M^3 in $\widetilde{M}^{3+m}(k)$ are readily found to be either zero or else to be completely determined, via the algebraic symmetries of R, by the following ones:

$$\begin{aligned} \alpha &:= K_{12} = R_{1221} = a^2 + b^2 + c^2 - 2\mu^2 + k, \\ \beta &:= K_{13} = R_{1331} = a^2 + b^2 + c^2 + b\mu + k, \\ \gamma &:= K_{23} = R_{2332} = a^2 + b^2 + c^2 - b\mu + k, \\ \delta &:= R_{1332} = c\mu, \end{aligned} \tag{14}$$

($K_{ij} = K(E_i \wedge E_j)$ are the *sectional curvatures* of M for the 2-planes $E_i \wedge E_j$ determined by a Choi–Lu frame E_1, E_2, E_3). And then also the components of the $(0,6)$ tensors $R \cdot R$ and $\wedge_g \cdot R$ are readily computable, and turn out either to be zero "together" or, when at least a priori non-zero, they appear in pairs which are, via algebraic symmetries of both these $(0,6)$ tensors (cf. [22]), completely determined by the following ones:

$$\begin{aligned} (R\cdot R)(E_1,E_3,E_1,E_3;E_1,E_2) &= -2\alpha\delta, \\ (\wedge_g\cdot R)(E_1,E_3,E_1,E_3;E_1,E_2) &= -2\delta; \\ (R\cdot R)(E_1,E_3,E_2,E_3;E_1,E_2) &= \alpha(\beta-\gamma), \\ (\wedge_g\cdot R)(E_1,E_3,E_2,E_3;E_1,E_2) &= \beta-\gamma; \\ (R\cdot R)(E_1,E_2,E_2,E_3;E_1,E_3) &= -\alpha\beta+\beta\gamma-\delta^2, \\ (\wedge_g\cdot R)(E_1,E_2,E_2,E_3;E_1,E_3) &= \gamma-\alpha. \end{aligned} \tag{15}$$

Then, by (14) and (15), the condition for the *pseudo-symmetry of* R, i.e., of the existence of a function $L\colon M \to R$ for which $R \cdot R = L\,(-\wedge_g \cdot R)$, for Wintgen ideal submanifolds M^3 in $\widetilde{M}^{3+m}(k)$, yields, of course, the previous cases (I, II, and III) of Lemma 1, but moreover in each case at the point p gives the *double sectional curvature function* L as stated in the following.

LEMMA 2. *We have*

$$\begin{aligned} &L = 0 && \textit{in case } (\mathrm{I}),\\ &L = a^2 + k && \textit{in case } (\mathrm{II}),\\ &L = a^2 + 2\mu^2 + k && \textit{in case } (\mathrm{III}). \end{aligned}$$

In case (I), M^3 is Einstein and thus a real space form and so, in particular, M^3 is then semi-symmetric and hence (also without calculations, one could know that) $L = 0$. In cases (II) and (III), according to the general theory in this respect, if M^3 is quasi-Einstein and has λ as an *eigenvalue of the Ricci tensor S of multiplicity* 1, then $L = \frac{\lambda}{2}$, which allows to obtain Lemma 2 also from the consideration of (12) and (13).

Now we aim to geometrically characterize the cases (I), (II), and (III). Clearly, from (9), we see that (I) corresponds to the *totally umbilical* Wintgen ideal submanifolds M^3 in real space forms $\widetilde{M}^{3+m}(k)$; such M^3 are intrinsically itself spaces of constant curvature $K = a^2 + b^2 + c^2 + k$. We recall that a submanifold M^n in a Riemannian manifold $\widetilde{M}^{n+m}$ is said to be *pseudo-umbilical* if its *mean curvature vector field* $\vec{H}$ determines an *umbilical* normal direction on M in $\tilde{M}$. When hereafter we call a submanifold pseudo-umbilical we mean it to be *properly* pseudo-umbilical, i.e., we exclude from it the trivial cases when it is *minimal* ($\vec{H} = \vec{0}$), or when it is *totally umbilical* (i.e., when every normal direction ξ is umbilical). The mean curvature vector field $\vec{H}$ for the submanifolds M^3 under consideration being given by $\vec{H} = c\xi_1 + b\xi_2 + a\xi_3$, it further follows from (9) that the shape operator of M in $\widetilde{M}$ with respect to $\vec{H}$ is given by

$$A_{\vec{H}} = \begin{pmatrix} a^2+b^2+c^2+b\mu & c\mu & 0\\ c\mu & a^2+b^2+c^2-b\mu & 0\\ 0 & 0 & a^2+b^2+c^2 \end{pmatrix}. \tag{16}$$

In case M is *not totally umbilical* in $\widetilde{M}$, i.e., in case $\mu \neq 0$, and in case M is *not minimal* in $\widetilde{M}$, i.e., in case not $a = b = c = 0$ (cf. (9)), then (16) shows that $\vec{H}$ determines an *umbilical normal direction* of M in $\widetilde{M}$ if and only if $b = c = 0$. In summary, from the above we know that cases (I) and (II) correspond to the Wintgen ideal submanifolds which are *totally umbilical* or *minimal* or *pseudo-umbilical*. Finally, we next aim for a geometrical characterisation of case (III): $b^2 + c^2 = 4\mu^2$ where $\mu \neq 0$. To simplify a bit the discussion, we will assume from now on that $\mu > 0$, which we can do without loss of generality, being always realizable in view of (9) by eventual changing orientations of ξ_α's and orderings of E_1 and E_2). From (12) and (13) we recall that the eigenvalues of the Ricci tensor are given by

$$\begin{aligned} \rho_1 &= 2\,(a^2+b^2+c^2+k) - 2\mu^2 + \mu\sqrt{b^2+c^2}, \\ \rho_2 &= 2\,(a^2+b^2+c^2+k) - 2\mu^2 - \mu\sqrt{b^2+c^2}, \qquad (17) \\ \rho_3 &= 2\,(a^2+b^2+c^2+k), \end{aligned}$$

where ρ_3 is the one in the E_3-direction of M and that ρ_1 and ρ_2 are the eigenvalues corresponding to certain eigendirections in the plane $\pi = E_1 \wedge E_2$ perpendicular to E_3 and of which the special character is reflected obviously, having $\mu \neq 0$, in the form of the shape operators given in (9), and which we further will call the *Choi–Lu plane* of M^3. Studying at present M^3's which are not totally umbilical, in particular, these submanifolds are not Einstein, and so, at every point, they have a *smallest* Ricci curvature, which we'll denote by inf Ric. From (17) it is clear this is ρ_2:

$$\inf \mathrm{Ric} = 2(a^2+b^2+c^2+k) - 2\mu^2 - \mu\sqrt{b^2+c^2},$$

attained by a particular direction in the Choi–Lu plane of M^3. On the other hand, we recall from (14) that the *sectional curvature* $K_{\text{Choi–Lu}}$ *of the Choi–Lu plane* is given by

$$K_{\text{Choi–Lu}} = K_{12} = a^2+b^2+c^2+k-2\mu^2.$$

Hence $\inf \mathrm{Ric} = 2\,K_{\text{Choi–Lu}}$ if and only if $-2\mu^2 - \mu\sqrt{b^2+c^2} = -4\mu^2$, i.e., if and only if (III) holds, namely when $b^2+c^2 = 4\mu^2$. From (14) we moreover see that, in this case,

$$\begin{aligned} K_{12} &= K_{\text{Choi–Lu}} = a^2 + 2\mu^2 + k, \\ K_{13} &= K_{\text{Choi–Lu}} + \mu(2\mu + b), \\ K_{23} &= K_{\text{Choi–Lu}} + \mu(2\mu - b), \end{aligned}$$

which implies, obviously having also that $b^2 \leqslant 4\mu^2$ and so, since $\mu > 0$, that

$$-2\mu \leqslant b \leqslant 2\mu,$$

and thus that as well $0 \leqslant 2\mu + b$ as $0 \leqslant 2\mu - b$, together with the equation of Gauss and (9), that in case (III) the sectional curvature K_{12} actually equals $\inf K$, the function on M giving the *minimum of the sectional curvatures* K at each point of M^3. So, in this situation we can observe on the side that the $\delta(2)$-*curvature of Chen* [7], $\delta(2) := \tau - \inf K$, where $\tau := \sum_{i<j} K_{ij}$ is the *scalar curvature* of M^3, is given by $\delta(2) = \tau - K_{12} = K_{13} + K_{23} = \rho_3 = \rho_1$ which, in this case, is $\sup \mathrm{Ric}$, the real valued function on M giving the maximum Ricci curvature at each point of M^3. Once more, in accordance with the general theory of Deszcz symmetric 3-dimensional Riemannian manifolds M^3, for the properly quasi-Einstein manifolds M^3 the sectional curvature L of Deszcz satisfies $L = \frac{\lambda}{2}$, where λ is the principal curvature with multiplicity 1, so actually $\lambda = \rho_2 = 2(a^2 + 2\mu^2 + k)$, such that

$$L = K_{\text{Choi–Lu}} = \inf K.$$

In this respect, coming back to case (II), from (14) and taking further into account (9) and the equation of Gauss, it follows that in this case, since $K_{12} = a^2 + k - 2\mu^2$,

$$L = a^2 + k = K_{23} = K_{13} = \sup K,$$

$\sup K$ denoting the *maximum of the sectional curvature function* on M, i.e., the function on M which value at each point is the maximum of all the sectional curvatures of M at this point. Taking into account at last also the case (III) of Lemma 2, in summary we can formulate the following.

THEOREM 3. *A Wintgen ideal submanifold M^3 in a real space form $\widetilde{M}^{3+m}(k)$ is Deszcz symmetric if and only if* (I) *M^3 is a totally umbilical submanifold with Deszcz sectional curvature $L = 0$ (M^3 then being a space of constant sectional curvature K), or,* (II) *M^3 is a minimal submanifold or a pseudo-umbilical submanifold, with Deszcz sectional curvature $L = \sup K$, or else,* (III) *M^3 is characterized by the curvature condition* $\inf \mathrm{Ric} = 2\,K_{\mathrm{Choi–Lu}} = 2\,\inf K$, *with Deszcz sectional curvature $L = \inf K$.*

4. Further comments and remarks

(1) We would like to refer again to the references mentioned in Section 1 for explicit descriptions of several *examples* of Wintgen ideal 3-dimensional submanifolds.

(2) Refering amongst others to Berger's discussion in his "Panorama" [3] pertaining to the *extremal values of the sectional curvature function K* of Riemannian manifolds, we observe from Theorem 3 that for the nontrivial Wintgen ideal submanifolds M^3 in $\widetilde{M}^{3+m}(k)$, i.e., the nontotally umbilical ones, *the isotropic Deszcz sectional curvatures L* are either given by the maximum or by the minimum values of K at each point. The Deszcz symmetry of those submanifolds M^3 being equivalent to being quasi-Einstein, in the above nontrivial case, $L = \sup K$ or $L = \inf K$ according to the geometrical fact that *the eigendirection of the Ricci tensor whose eigenvalue has multiplicity* 1 *is either perpendicular to the plane of Choi–Lu of these Wintgen ideal submanifolds M^3 or belongs to this plane.*

(3) Concerning the *origin of the Ricci tensor* of 3-dimensional Riemannian manifolds and some related views on the δ-curvatures of Chen, see [25].

References

1. M. Belkhelfa, R. Deszcz. M. Glogowska, M. Hotlos, D. Kowalczyk, and L. Verstraelen, *On some type of curvature conditions*, in: B. Opozda et al. (eds), *Submanifolds and Affine Differential Geometry*, Banach Center Publ. 57, Institute of Mathematics, Polish Acad. Sci., Warszawa, 2002, 179–193.
2. M. Belkhelfa, R. Deszcz and L. Verstraelen, *Symmetry Properties of* 3−*dimensional D'Atri Spaces*, Kyungpook Math. J. **46** (2006), 367–376.
3. M. Berger, *A Panoramic View of Riemannian Geometry*, Springer, Berlin, 2003.
4. R. L. Bryant, *Some remarks on the geometry of austere manifolds*, Bol. Soc. Brasil. Math. (N.S.) **21** (1991), 133–157.
5. B. Y. Chen, *Geometry of Submanifolds*, Marcel Dekker, New York, 1973.
6. B. Y. Chen, *Geometry of Submanifolds and Its Applications*, Science University of Tokyo, Department of Mathematics, Tokyo, 1981.

7. B. Y. Chen, *Riemannian Submanifolds*, Chapter 3 in: F. Dillen et al. (eds), *Handbook of Differential Geometry*, Vol. I, North-Holland, Elsevier, Amsterdam, 2000, 187–418.
8. T. Choi and Z. Lu, *On the DDVV conjecture and the comass in calibrated geometry (I)*, Math. Z. **260** (2008), 409–429.
9. F. Defever, R. Deszcz, P. Dhooghe, L. Verstraelen and Ş. Yaprak, *On Ricci pseudo-symmetric hypersurfaces in spaces of constant curvature*, Results in Math. **27** (1995), 227–236.
10. J. Deprez, R. Deszcz and L. Verstraelen, *Examples of pseudosymmetric conformally flat warped pruducts*, Chinese J. Math. **17** (1989), 51–65.
11. P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, *The normal curvature of totally real submanifolds of* $S^6(1)$, Glasgow Math. J. **40** (1998), 199–204.
12. P. J. De Smet, F. Dillen, L. Verstraelen and L. Vrancken, *A pointwise inequality in submanifold theory*, Arch. Math. **35** (1999), 115–128.
13. R. Deszcz, *On pseudosymmetric spaces*, Bull. Soc. Math. Belg., Série A, **44** (1992), 1–34.
14. R. Deszcz, M. S. Haesen and L. Verstraelen, *On Natural Symmetries*, Chapter 6 in: A. Mihai, I. Mihai and R. Miron (eds.), *Topics in Differential Geometry*, Editura Academiei Române, 2008, 249–308.
15. R. Deszcz, M. Hotloś and Z. Şentürk, *On Ricci pseudosymmetric hypersurfaces in space forms*, Demonstratio Math. **34** (2004), 203–214.
16. R. Deszcz, L. Verstraelen and Ş. Yaprak, *Warped products realizing a certain condition of pseudosymmetry type imposed on the Weyl curvature tensor*, Chinese J. Math. (Taiwan R. O. C.) **22** (1994), 139–157.
17. F. Dillen, J. Fastenakels and J. Van der Veken, *A pinching theorem for the normal scalar curvature of invariant submanifolds*, J. Geom. Phys. **57** (2007), 833–840.
18. F. Dillen, J. Fastenakels and J. Van der Veken, *Three-dimensional submanifolds of* E^5 *with extremal normal curvature*, preprint.
19. F. Dillen, S. Haesen, M. Petrović-Torgašev and L. Verstraelen, *An inequality between intrinsic and extrinsic scalar curvature invariants for codimension* 2 *embeddings*, J. Geom. Phys. **52** (2004), 101–112.
20. B. Gmira and L. Verstraelen, *A curvature inequality for Riemannian submanifolds in a semi-Riemannian space form*, in: F. Defever et al. (eds), *Geometry and Topology of Submanifolds*, Vol. IX, World Sci., Singapore, 1999, 148–159.
21. I. V. Guadalupe and L. Rodriguez, *Normal curvature of surfaces in space forms*, Pacific J. Math. **106** (1983), 95–103.
22. S. Haesen and L. Verstraelen, *Properties of a scalar curvature invariant depending on two planes*, Manuscripta Math. **122** (2007), 59–72.
23. S. Haesen and L. Verstraelen, *Natural Intrinsic Geometrical Symmetries*, SIGMA (Symmetry, Integrability and Geometry: Methods and Applications), **5**(2009), Special Issue “Elie Cartan and Differential Geometry", 15 pages.
24. B. Jahanara, S. Haesen, M. Petrović-Torgašev and L. Verstraelen, *On the Weyl conformal curvature of Deszcz*, Publ. Math. Debrecen **74** (2009), 417–431.
25. B. Jahanara, S. Haesen, Z. Şentürk and L. Verstraelen, *On the parallel transport of the Ricci curvatures*, J. Geom. Phys. **57** (2007), 1771–1777.
26. O. Kowalski, M. Sekizawa, *Pseudo-symmetric spaces of constant type in dimension three–elliptic spaces*, Rendiconti di Mathematica, Serie VII, Roma, **17** (1997), 477–512.
27. T. Levi-Civita, *Nozione di parallelismo in una varietá qualcunque e conseguente spezificazione geometrica della curvatura Riemanniana*, Rend. Circ. Matem. Palermo **42** (1917), 173–204.
28. M. Petrović-Torgašev and L. Verstraelen, *On Deszcz symmetries of Wintgen ideal submanifolds*, Arch. Math. (Brno) **44** (2008), 57–67.
29. B. Rouxel, *Sur une famille de A-surfaces d'un espace euclidien* E^4, Proc. X Österreichischer Mathematiker Kongress, Innsbruck, 1981, 185.
30. J. A. Schouten, *Die direkte Analysis zur neueren Relativitätstheorie*, Verhandelingen Kon. Akad. van Wetenschappen te Amsterdam, Sectie I, Deel **XII**, Nr. 6, (1918), 1–95.
31. B. D. Suceavă, *DDVV conjecture*, preprint.

32. Z. Szabó, *Structure theorems on Riemannian spaces satisfying* $R(X,Y)\cdot R=0$. *I. The local version*, J. Diff. Geom. **17** (1982), 531–582.
33. Z. Szabó, *Structure theorems on Riemannian spaces satisfying* $R(X,Y)\cdot R=0$. *II. The global version*, Geom. Dedicata **19** (1985), 65–108.
34. W. M. Thurston, *Three-dimensional Geometry and Topology*, Vol. I, Princeton University Press, Princeton, 1997.
35. L. Verstraelen, *Comments on the pseudo-symmetry in the sense of Deszcz*, in: F. Dillen et al. (eds), *Geometry and Topology of Submanifolds*, Vol. VI, World Sci., Singapore, 1994, 119–209.
36. P. Wintgen, *Sur l'inégalité de Chen-Willmore*, C. R. Acad. Sci. Paris **288** (1979), 993–995.

(Received 07 12 2008)

Department of Mathematics
Wrocław University of Environmental and Life Sciences
50-357 Wrocław
Poland
ryszard.deszcz@up.wroc.pl

Department of Mathematics
Faculty of Scence
34000 Kragujevac
Serbia
mirapt@kg.ac.rs

Mathematics Engineering Department
Faculty of Science and Letters
Istanbul Technical University
Istanbul
Turkey
senturk@itu.edu.tr

Departement Wiskunde
Katholieke Universiteit Leuven, Fakulteit Wetenschappen
Celestijnenlaan 200B, 3001 Heverlee
Belgium
leopold.verstraelen@wis.kuleuven.be