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SHARP FUNCTION INEQUALITIES AND BOUNDNESS FOR TOEPLITZ TYPE OPERATOR RELATED TO GENERAL FRACTIONAL SINGULAR INTEGRAL OPERATOR

Chuangxia Huang and Lanzhe Liu

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Abstract. We establish some sharp maximal function inequalities for the Toeplitz type operator, which is related to certain fractional singular integral operator with general kernel. These results are helpful to investigate the boundedness of the operator on Lebesgue, Morrey and Triebel–Lizorkin spaces respectively.

1. Introduction

In recent decades, commutators have attracted a rapidly growing attention of the researchers in the field of harmonic analysis and have been widely studied by many authors [7, 19, 20]. In [4, 17, 18], the authors prove that the commutators generated by the singular integral operators and BMO functions are bounded on $L^p(R^n)$ for $1 < p < \infty$. Chanillo proves a similar result when singular integral operators are replaced by the fractional integral operators in [2]. The boundedness for the commutators generated by the singular integral operators and Lipschitz functions on Triebel–Lizorkin and $L^p(R^n)$ $(1 < p < \infty)$ spaces are obtained in [3, 8, 14]. Some singular integral operators with general kernel are introduced, and the boundedness for the operators and their commutators generated by BMO and Lipschitz functions are obtained [1, 11]. In [9, 10], some Toeplitz type operators related to the singular integral operators and strongly singular integral operators are introduced, and the boundedness for the operators generated by BMO and Lipschitz functions are established. The main purpose of this paper is to study the Toeplitz type operators generated by some fractional singular integral operators with general kernel and the Lipschitz and BMO functions. We will prove the

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sharp maximal inequalities for the Toeplitz type operator T_δ^b. These results are helpful to investigate the the L^p-norm inequality, the boundedness of the operator on Lebesgue, Morrey and Triebel–Lizorkin spaces respectively.

2. Preliminaries

At first, we should introduce some notations in the following. Throughout this paper, Q denotes a cube of R^n with sides parallel to the axes. For any locally integrable function f, the sharp maximal function of f is defined by

$$M^{\#}(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

where, $f_Q = |Q|^{-1} \int_Q f(x)\, dx$. It is well known that [7, 19]

$$M^{\#}(f)(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - c| dy.$$

We say that f belongs to $\mathrm{BMO}(R^n)$ if $M^{\#}(f)$ belongs to $L^\infty(R^n)$ and define $\|f\|_{\mathrm{BMO}} = \|M^{\#}(f)\|_{L^\infty}$. It is known [19] that $\|f - f_{2^kQ}\|_{\mathrm{BMO}} \leqslant Ck\|f\|_{\mathrm{BMO}}$. Let

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy.$$

For $\eta > 0$ we denote $M_\eta(f)(x) = M(|f|^\eta)^{1/\eta}(x)$. For $0 < \eta < n$ and $1 \leqslant r < \infty$ set

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left(\frac{1}{|Q|^{1 - r\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}.$$

According to [7], the A_p weight can be defined as follows, for $1 < p < \infty$,

$$A_p = \left\{ w \in L^1_{\mathrm{loc}}(R^n) : \sup_Q \left(\frac{1}{|Q|} \int_Q w(x)\, dx \right) \left(\frac{1}{|Q|} \int_Q w(x)^{-1/(p-1)} dx \right)^{p-1} < \infty \right\}$$

and $A_1 = \{w \in L^p_{\mathrm{loc}}(R^n) : M(w)(x) \leqslant Cw(x),\ \text{a.e.}\}$.

For $\beta > 0$ and $p > 1$ let $\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Triebel–Lizorkin space [14]. For $\beta > 0$ the Lipschitz space $\mathrm{Lip}_\beta(R^n)$ is the space of functions f such that

$$\|f\|_{\mathrm{Lip}_\beta} = \sup_{\substack{x,y \in R^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\beta} < \infty.$$

Definition 2.1. Let φ be a positive, increasing function on R^+ and there exists a constant $D > 0$ such that $\varphi(2t) \leqslant D\varphi(t)$ for $t \geqslant 0$. Let f be a locally integrable function on R^n. Set, for $1 \leqslant p < \infty$,

$$\|f\|_{L^{p,\varphi}} = \sup_{x \in R^n,\, d > 0} \left(\frac{1}{\varphi(d)} \int_{Q(x,d)} |f(y)|^p dy \right)^{1/p},$$

where $Q(x,d) = \{y \in R^n : |x - y| < d\}$. As usual, the generalized Morrey space can be defined by $L^{p,\varphi}(R^n) = \{f \in L^1_{\mathrm{loc}}(R^n) : \|f\|_{L^{p,\varphi}} < \infty\}$.

If $\varphi(d) = d^{\delta}$, $\delta > 0$, then $L^{p,\varphi}(R^n) = L^{p,\delta}(R^n)$ are classical Morrey spaces [15, 16]. If $\varphi(d) = 1$, then $L^{p,\varphi}(R^n) = L^p(R^n)$ are Lebesgue spaces [13].

Since Morrey spaces can be regarded as extensions of Lebesgue spaces, it is natural and important to study the boundedness of operators on the Morrey spaces [5, 6, 12, 13]. Here we study some singular integral operators, defined as follows [1].

Definition 2.2. Fix $0 < \delta < n$. Let $T_\delta : S \to S'$ be a linear operator such that T_δ is bounded on $L^2(R^n)$ and has a kernel K, that is, there exists a locally integrable function $K(x,y)$ on $R^n \times R^n \setminus \{(x,y) \in R^n \times R^n : x = y\}$ such that $T_\delta(f)(x) = \int_{R^n} K(x,y)f(y)\,dy$ for every bounded and compactly supported function f, where K satisfies: there is a sequence of positive constant numbers $\{C_j\}$ such that for any $j \geqslant 1$,

$$\int_{2|y-z|<|x-y|} (|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|)\,dx \leqslant C,$$

$$(1)\quad \left(\int_{2^k|z-y|\leqslant|x-y|<2^{k+1}|z-y|} (|K(x,y) - K(x,z)| + |K(y,x) - K(z,x)|)^q dy\right)^{1/q} \leqslant C_k(2^k|z-y|)^{-n/q'+\delta},$$

where $1 < q' < 2$ and $1/q + 1/q' = 1$. We write $T_\delta \in \mathrm{GSIO}(\delta)$.

Moreover, if b is a locally integrable function on R^n, then the Toeplitz type operator related to T_δ can be defined by $T_\delta^b = \sum_{k=1}^m T_\delta^{k,1} M_b T^{k,2}$, where $T_\delta^{k,1}$ are T_δ or $\pm I$ (the identity operator), $T^{k,2}$ are the linear operators, $k = 1,\dots,m$, $M_b(f) = bf$.

Remark 2.1. We should point out that the classical Calderón–Zygmund singular integral operator satisfies Definition 2.2 with $C_j = 2^{-j\delta}$ [7, 19].

Remark 2.2. It is obvious that the fractional integral operator with rough kernel satisfies Definition 2.2 [3], that is, for $0 < \delta < n$, let T_δ be the fractional integral operator with rough kernel defined by (see [3])

$$T_\delta f(x) = \int_{R^n} \frac{\Omega(x-y)}{|x-y|^{n-\delta}} f(y)\,dy,$$

where Ω is homogeneous of degree zero on R^n, $\int_{S^{n-1}} \Omega(x')\,d\sigma(x') = 0$ and $\Omega \in \mathrm{Lip}_\varepsilon(S^{n-1})$ for some $0 < \varepsilon \leqslant 1$, and there exists a constant $M > 0$ such that for any $x, y \in S^{n-1}$, $|\Omega(x) - \Omega(y)| \leqslant M|x-y|^\varepsilon$. When $\Omega \equiv 1$, T_δ is the Riesz potential (fractional integral operator) [2].

Remark 2.3. One can obtain that the commutator $[b, T_\delta](f) = bT_\delta(f) - T_\delta(bf)$ is a particular operator of the Toeplitz type operator T_b. The Toeplitz type operators T_δ^b are nontrivial generalizations of commutators.

3. Some Lemmas

We begin with some preliminary lemmas.

Lemma 3.1. [1] *Let T_δ be the singular integral operator as Definition* 1.2. *Then T_δ is bounded from $L^p(R^n)$ to $L^r(R^n)$ for $1 < p < n/\delta$ and $1/r = 1/p - \delta/n$.*

Lemma 3.2. [14] *For $0<\beta<1$ and $1<p<\infty$, we have*

$$\|f\|_{\dot F_p^{\beta,\infty}} \approx \left\| \sup_{Q\ni\cdot} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x)-f_Q|\,dx \right\|_{L^p} \approx \left\| \sup_{Q\ni\cdot} \inf_{c} \frac{1}{|Q|^{1+\beta/n}} \int_Q |f(x)-c|\,dx \right\|_{L^p}.$$

Lemma 3.3. [7] *Let $0<p<\infty$ and $w\in\bigcup_{1\leqslant r<\infty} A_r$. Then, for any smooth function f for which the left-hand side is finite,*

$$\int_{R^n} M(f)(x)^p w(x)\,dx \leqslant C \int_{R^n} M^{\#}(f)(x)^p w(x)\,dx.$$

Lemma 3.4. [2] *Suppose that $0<\eta<n$, $1\leqslant s<p<n/\eta$ and $1/r=1/p-\eta/n$. Then $\|M_{\eta,s}(f)\|_{L^r}\leqslant C\|f\|_{L^p}$.*

Lemma 3.5. *Let $1<p<\infty$, $0<D<2^n$. Then, for any smooth function f for which the left-hand side is finite, $\|M(f)\|_{L^{p,\varphi}}\leqslant C\|M^{\#}(f)\|_{L^{p,\varphi}}$.*

Proof. For any cube $Q=Q(x_0,d)$ in R^n, we know $M(\chi_Q)\in A_1$ for any cube $Q=Q(x,d)$ by [7]. Noticing that $M(\chi_Q)\leqslant 1$ and $M(\chi_Q)(x)\leqslant d^n/(|x-x_0|-d)^n$ if $x\in Q^c$, by Lemma 3.3, we have, for $f\in L^{p,\varphi}(R^n)$,

$$\begin{aligned}
&\int_Q M(f)(x)^p dx = \int_{R^n} M(f)(x)^p \chi_Q(x)\,dx\\
&\leqslant \int_{R^n} M(f)(x)^p M(\chi_Q)(x)\,dx \leqslant C\int_{R^n} M^{\#}(f)(x)|^p M(\chi_Q)(x)\,dx\\
&= C\bigg(\int_Q M^{\#}(f)(x)^p M(\chi_Q)(x)\,dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} M^{\#}(f)(x)^p M(\chi_Q)(x)\,dx \bigg)\\
&\leqslant C\bigg(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q\setminus 2^kQ} M^{\#}(f)(x)^p \frac{|Q|}{|2^{k+1}Q|} dx \bigg)\\
&\leqslant C\bigg(\int_Q M^{\#}(f)(x)^p dx + \sum_{k=0}^{\infty} \int_{2^{k+1}Q} M^{\#}(f)(x)^p 2^{-kn} dy \bigg)\\
&\leqslant C\|M^{\#}(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} 2^{-kn}\varphi(2^{k+1}d) \leqslant C\|M^{\#}(f)\|_{L^{p,\varphi}}^p \sum_{k=0}^{\infty} (2^{-n}D)^k \varphi(d)\\
&\leqslant C\|M^{\#}(f)\|_{L^{p,\varphi}}^p \varphi(d),
\end{aligned}$$

thus

$$\left(\frac{1}{\varphi(d)} \int_Q M(f)(x)^p dx \right)^{1/p} \leqslant C \left(\frac{1}{\varphi(d)} \int_Q M^{\#}(f)(x)^p dx \right)^{1/p}$$

and $\|M(f)\|_{L^{p,\varphi}}\leqslant C\|M^{\#}(f)\|_{L^{p,\varphi}}$. This finishes the proof. □

Lemma 3.6. *Let $0<D<2^n$, $1\leqslant s<p<n/\eta$ and $1/r=1/p-\eta/n$. Then*

$$\|M_{\eta,s}(f)\|_{L^{r,\varphi}} \leqslant C\|f\|_{L^{p,\varphi}}.$$

The proof of Lemma 3.6 is similar to that of Lemma 3.5 by Lemma 3.4, we omit the details.

4. Theorems and Their Proofs

We shall prove the following theorems.

THEOREM 4.1. *Let the sequence* $\{C_j\} \in l^1$, $0 < \beta < 1$, $q' \leqslant s < \infty$ *and* $b \in \mathrm{Lip}_\beta(R^n)$. *Suppose* T_δ *is a bounded linear operator from* $L^p(R^n)$ *to* $L^r(R^n)$ *for any* p, r *with* $1 < p < n/\delta$ *and* $1/r = 1/p - \delta/n$, *and has a kernel* K *satisfying* (1). *If* $T^1_\delta(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$, *then there exists a constant* $C > 0$ *such that, for any* $f \in C^\infty_0(R^n)$ *and* $\tilde{x} \in R^n$,

$$M^{\#}(T^b_\delta(f))(\tilde{x}) \leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}).$$

PROOF. It suffices to prove for $f \in C^\infty_0(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q \left|T^b_\delta(f)(x) - C_0\right| dx \leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T^{k,1}_\delta$ are $T_\delta (k = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. Write, for $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$,

$$T^b_\delta(f)(x) = T^{b-b_Q}_\delta(f)(x) = T^{(b-b_Q)\chi_{2Q}}_\delta(f)(x) + T^{(b-b_Q)\chi_{(2Q)^c}}_\delta(f)(x) = g(x) + h(x).$$

Then

$$\frac{1}{|Q|}\int_Q \left|T^b_\delta(f)(x) - h(x_0)\right| dx \leqslant \frac{1}{|Q|}\int_Q |g(x)|dx + \frac{1}{|Q|}\int_Q |h(x) - h(x_0)|dx = I_1 + I_2.$$

For I_1, choose $1 < r < \infty$ such that $1/r = 1/s - \delta/n$, by (L^s, L^r)-boundedness of T_δ and Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q |T^{k,1}_\delta M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|dx \\
&\leqslant \left(\frac{1}{|Q|}\int_{R^n} |T^{k,1}_\delta M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^r dx\right)^{1/r} \\
&\leqslant C|Q|^{-1/r}\left(\int_{R^n} |M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
&\leqslant C|Q|^{-1/r}\left(\int_{2Q} (|b(x) - b_Q||T^{k,2}(f)(x)|)^s dx\right)^{1/s} \\
&\leqslant C|Q|^{-1/r}\|b\|_{\mathrm{Lip}_\beta}|2Q|^{\beta/n}|2Q|^{1/s-(\beta+\delta)/n} \\
&\quad \times \left(\frac{1}{|2Q|^{1-s(\beta+\delta)/n}}\int_{2Q} |T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$
\begin{aligned}
I_1 &\leqslant \sum_{k=1}^{m} \frac{1}{|Q|} \int_Q |T_\delta^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)| dx \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

For I_2, recalling that $s > q'$, we get, for $x \in Q$,

$$
\begin{aligned}
&|T_\delta^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T_\delta^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| \\
&\quad\leqslant \int_{(2Q)^c} |b(y) - b_{2Q}\|K(x,y) - K(x_0,y)\|T^{k,2}(f)(y)| dy \\
&\quad= \sum_{j=1}^{\infty} \int_{2^j d \leqslant |y-x_0| < 2^{j+1} d} |b(y) - b_{2Q}\|K(x,y) - K(x_0,y)\|T^{k,2}(f)(y)| dy \\
&\quad\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{j=1}^{\infty} |2^{j+1}Q|^{\beta/n} \left(\int_{2^j d \leqslant |y-x_0| < 2^{j+1} d} |K(x,y) - K(x_0,y)|^q dy \right)^{1/q} \\
&\qquad\qquad \times \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy \right)^{1/q'} \\
&\quad\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{j=1}^{\infty} |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'+\delta} |2^{j+1}Q|^{1/q'-(\beta+\delta)/n} \\
&\qquad\qquad \times \left(\frac{1}{|2^{j+1}Q|^{1-s(\beta+\delta)/n}} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
&\quad\leqslant C\|b\|_{\mathrm{Lip}_\beta} M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} C_j \leqslant C\|b\|_{\mathrm{Lip}_\beta} M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}),
\end{aligned}
$$

thus

$$
\begin{aligned}
I_2 &\leqslant \frac{1}{|Q|} \int_Q \sum_{k=1}^{m} |T_\delta^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T_\delta^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| dx \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} M_{\beta+\delta,s}(T^{k,2}(f))(\tilde{x}).
\end{aligned}
$$

This completes the proof of Theorem 4.1. □

Theorem 4.2. *Let the sequence $\{2^{j\beta} C_j\} \in l^1$, $0 < \beta < 1$, $q' \leqslant s < \infty$ and $b \in \mathrm{Lip}_\beta(R^n)$. Suppose T_δ is a bounded linear operator from $L^p(R^n)$ to $L^r(R^n)$ for any p, r with $1 < p < n/\delta$ and $1/r = 1/p - \delta/n$, and has a kernel K satisfying (1). If $T_\delta^1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$, then there exists a constant $C > 0$ such that, for any $f \in C_0^\infty(R^n)$ and $\tilde{x} \in R^n$,*

$$
\sup_{Q \ni \tilde{x}} \frac{1}{|Q|^{1+\beta/n}} \int_Q \left| T_\delta^b(f)(x) - C_0 \right| dx \leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} M_{\delta,s}(T^{k,2}(f))(\tilde{x}).
$$

PROOF. It suffices to prove for $f \in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q \left|T_\delta^b(f)(x) - C_0\right| dx \leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T_\delta^{k,1}$ are $T_\delta (k = 1, \ldots, m)$. Fix a cube $Q = Q(x_0, d)$ and $\tilde{x} \in Q$. For $f_1 = f\chi_{2Q}$ and $f_2 = f\chi_{(2Q)^c}$, write

$$T_\delta^b(f)(x) = T_\delta^{b-b_Q}(f)(x) = T_\delta^{(b-b_Q)\chi_{2Q}}(f)(x) + T_\delta^{(b-b_Q)\chi_{(2Q)^c}}(f)(x) = g(x) + h(x)$$

and

$$\frac{1}{|Q|^{1+\beta/n}} \int_Q \left|T_\delta^b(f)(x) - h(x_0)\right| dx \leqslant \frac{1}{|Q|^{1+\beta/n}} \int_Q |g(x)| dx + \frac{1}{|Q|^{1+\beta/n}} \int_Q |h(x) - h(x_0)| dx = I_3 + I_4.$$

By using the same argument as in the proof of Theorem 4.1, we get, for $1 < r < \infty$ with $1/r = 1/s - \delta/n$,

$$\begin{aligned}
I_3 &\leqslant \sum_{k=1}^m \frac{C}{|Q|^{1+\beta/n}} |Q|^{1-1/r} \left(\int_{2Q} |T_\delta^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^r dx\right)^{1/r} \\
&\leqslant \sum_{k=1}^m \frac{C}{|Q|^{\beta/n}} |Q|^{-1/r} \left(\int_{2Q} |M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
&\leqslant \sum_{k=1}^m \frac{C}{|Q|^{\beta/n}} |Q|^{-1/r} \|b\|_{\mathrm{Lip}_\beta} |2Q|^{\beta/n} \left(\int_{2Q} |T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m \left(\frac{1}{|2Q|^{1-s\delta/n}} \int_{2Q} |T^{k,2}(f)(x)|^s dx\right)^{1/s} \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

$$\begin{aligned}
I_4 &\leqslant \sum_{k=1}^m \frac{1}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^\infty \int_{2^j d \leqslant |y-x_0| < 2^{j+1} d} |b(y) - b_{2Q}| \\
&\qquad \times |K(x,y) - K(x_0,y)| |T^{k,2}(f)(y)| dy dx \\
&\leqslant \sum_{k=1}^m \frac{C}{|Q|^{1+\beta/n}} \int_Q \sum_{j=1}^\infty \|b\|_{\mathrm{Lip}_\beta} |2^{j+1}Q|^{\beta/n} \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^{q'} dy\right)^{1/q'} \\
&\qquad \times \left(\int_{2^j d \leqslant |y-x_0| < 2^{j+1} d} |K(x,y) - K(x_0,y)|^q dy\right)^{1/q} dx \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^m |Q|^{-\beta/n} \sum_{j=1}^\infty |2^{j+1}Q|^{\beta/n} C_j (2^j d)^{-n/q'+\delta} |2^{j+1}Q|^{1/q'-\delta/n}
\end{aligned}$$

$$\times\left(\frac{1}{|2^{j+1}Q|^{1-s\delta/n}}\int_{2^{j+1}Q}|T^{k,2}(f)(y)|^s dy\right)^{1/s}$$
$$\leqslant C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x})\sum_{j=1}^\infty 2^{j\beta}C_j$$
$$\leqslant C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

This completes the proof of Theorem 4.2. □

Theorem 4.3. *Let the sequence $\{jC_j\}\in l^1$, $q'\leqslant s<\infty$ and $b\in \mathrm{BMO}(R^n)$. Suppose T_δ is a bounded linear operator from $L^p(R^n)$ to $L^r(R^n)$ for any p,r with $1<p<n/\delta$ and $1/r=1/p-\delta/n$, and has a kernel K satisfying* (1). *If $T_\delta^1(g)=0$ for any $g\in L^u(R^n)$ $(1<u<\infty)$, then there exists a constant $C>0$ such that, for any $f\in C_0^\infty(R^n)$ and $\tilde{x}\in R^n$,*

$$M^{\#}(T_\delta^b(f))(\tilde{x})\leqslant C\|b\|_{\mathrm{BMO}}\sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

Proof. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q\left|T_\delta^b(f)(x)-C_0\right|dx\leqslant C\|b\|_{\mathrm{BMO}}\sum_{k=1}^m M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

Without loss of generality, we may assume $T_\delta^{k,1}$ are T_δ $(k=1,\dots,m)$. Fix a cube $Q=Q(x_0,d)$ and $\tilde{x}\in Q$. For $f_1=f\chi_{2Q}$ and $f_2=f\chi_{(2Q)^c}$, similar to the proof of Theorem 4.1, we have

$$T_\delta^b(f)(x)=T_\delta^{b-b_Q}(f)(x)=T_\delta^{(b-b_Q)\chi_{2Q}}(f)(x)+T_\delta^{(b-b_Q)\chi_{(2Q)^c}}(f)(x)=g(x)+h(x),$$
$$\frac{1}{|Q|}\int_Q\left|T_\delta^b(f)(x)-h(x_0)\right|dx\leqslant\frac{1}{|Q|}\int_Q|g(x)|dx+\frac{1}{|Q|}\int_Q|h(x)-h(x_0)|dx=I_5+I_6.$$

For I_5, choose $1<t<s$, by Hölder's inequality and the boundedness of T_δ with $1<r<\infty$ and $1/r=1/t-\delta/n$, we obtain

$$\frac{1}{|Q|}\int_Q|T_\delta^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|dx$$
$$\leqslant\left(\frac{1}{|Q|}\int_{R^n}|T_\delta^{k,1}M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^r dx\right)^{1/r}$$
$$\leqslant C|Q|^{-1/r}\left(\int_{R^n}|M_{(b-b_Q)\chi_{2Q}}T^{k,2}(f)(x)|^t dx\right)^{1/t}$$
$$\leqslant C|Q|^{-1/r}\left(\int_{2Q}|T^{k,2}(f)(x)|^s dx\right)^{1/s}\left(\int_{2Q}|b(x)-b_Q|^{st/(s-t)}dx\right)^{(s-t)/st}$$
$$\leqslant C\|b\|_{\mathrm{BMO}}\left(\frac{1}{|2Q|^{1-s\delta/n}}\int_{2Q}|T^{k,2}(f)(x)|^s dx\right)^{1/s}$$

$$\leqslant C\|b\|_{\mathrm{BMO}} M_{\delta,s}(T^{k,2}(f))(\tilde{x}),$$

thus

$$I_5 \leqslant \sum_{k=1}^{l} \frac{1}{|Q|} \int_Q |T_\delta^{k,1} M_{(b-b_Q)\chi_{2Q}} T^{k,2}(f)(x)| dx \leqslant C\|b\|_{\mathrm{BMO}} \sum_{k=1}^{m} M_{\delta,s}(T^{k,2}(f))(\tilde{x}).$$

For I_6, recalling that $s > q'$, taking $1 < p < \infty$ with $1/p + 1/q + 1/s = 1$, we get, for $x \in Q$,

$$\begin{aligned}
&|T_\delta^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T_\delta^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| \\
&\quad \leqslant \sum_{j=1}^{\infty} \int_{2^j d \leqslant |y-x_0| < 2^{j+1} d} |K(x,y) - K(x_0,y)| |b(y) - b_{2Q}| |T^{k,2}(f)(y)| dy \\
&\quad \leqslant \sum_{j=1}^{\infty} \left(\int_{2^j d \leqslant |y-x_0| < 2^{j+1} d} |K(x,y) - K(x_0,y)|^q dy \right)^{1/q} \\
&\quad \times \left(\int_{2^{j+1}Q} |b(y) - b_Q|^p dy \right)^{1/p} \left(\int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
&\quad \leqslant C\|b\|_{\mathrm{BMO}} \sum_{j=1}^{\infty} C_j (2^j d)^{-n/q'+\delta} j (2^j d)^{n/p} (2^j d)^{n/s-\delta} \\
&\quad \times \left(\frac{1}{|2^{j+1}Q|^{1-s\delta/n}} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^s dy \right)^{1/s} \\
&\quad \leqslant C\|b\|_{\mathrm{BMO}} M_{\delta,s}(T^{k,2}(f))(\tilde{x}) \sum_{j=1}^{\infty} j C_j \leqslant C\|b\|_{\mathrm{BMO}} M_{\delta,s}(T^{k,2}(f))(\tilde{x}),
\end{aligned}$$

thus

$$\begin{aligned}
I_6 \leqslant \frac{1}{|Q|} \int_Q \sum_{k=1}^{l} |T_\delta^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T_\delta^{k,1} M_{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| dx \\
\leqslant C\|b\|_{\mathrm{BMO}} \sum_{k=1}^{l} M_{\delta,s}(T^{k,2}(f))(\tilde{x}).
\end{aligned}$$

This completes the proof of Theorem 4.3. □

THEOREM 4.4. *Let the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n-\delta)$, $q' < p < n/(\beta+\delta)$, $1/r = 1/p - (\beta+\delta)/n$ and $b \in \mathrm{Lip}_\beta(R^n)$. Suppose T_δ is a bounded linear operator from $L^p(R^n)$ to $L^r(R^n)$ and has a kernel K satisfying* (1). *If $T_\delta^1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$ and $T^{k,2}$ are the bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, \ldots, m$, then T_δ^b is bounded from $L^p(R^n)$ to $L^r(R^n)$.*

PROOF. Choose $q' < s < p$ in Theorem 4.1, we have, by Lemma 3.3 and 3.4,

$$\begin{aligned}
\|T_\delta^b(f)\|_{L^r} &\leqslant \|M(T_\delta^b(f))\|_{L^r} \leqslant C\|M^\#(T_\delta^b(f))\|_{L^r} \\
&\leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} \|M_{\beta+\delta,s}(T^{k,2}(f))\|_{L^r} \leqslant C\|b\|_{\mathrm{Lip}_\beta} \sum_{k=1}^{m} \|T^{k,2}(f)\|_{L^p}
\end{aligned}$$

$$\leqslant C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^p}.$$

This completes the proof. □

THEOREM 4.5. *Let the sequence $\{C_j\} \in l^1$, $0 < \beta < \min(1, n-\delta)$, $q' < p < n/(\beta+\delta)$, $1/r = 1/p - (\beta+\delta)/n$, $0 < D < 2^n$ and $b \in \mathrm{Lip}_\beta(R^n)$. Suppose T_δ is a bounded linear operator from $L^p(R^n)$ to $L^r(R^n)$ and has a kernel K satisfying* (1). *If $T_\delta^1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$ and $T^{k,2}$ are the bounded operators on $L^{p,\varphi}(R^n)$ for $1 < p < \infty$, $k = 1, \ldots, m$, then T_δ^b is bounded from $L^{p,\varphi}(R^n)$ to $L^{r,\varphi}(R^n)$.*

PROOF. Choose $q' < s < p$ in Theorem 4.1, we have, by Lemma 3.5 and 3.6,

$$\begin{aligned}\|T_\delta^b(f)\|_{L^{r,\varphi}} &\leqslant \|M(T_\delta^b(f))\|_{L^{r,\varphi}} \leqslant C\|M^{\#}(T_\delta^b(f))\|_{L^{r,\varphi}}\\ &\leqslant C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m \|M_{\delta,s}(T^{k,2}(f))\|_{L^{r,\varphi}} \leqslant C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p,\varphi}}\\ &\leqslant C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^{p,\varphi}}.\end{aligned}$$

This completes the proof. □

THEOREM 4.6. *Let the sequence $\{2^{j\beta}C_j\} \in l^1$, $0 < \beta < 1$, $q' < p < n/\delta$, $1/r = 1/p - \delta/n$ and $b \in \mathrm{Lip}_\beta(R^n)$. Suppose T_δ is a bounded linear operator from $L^p(R^n)$ to $L^r(R^n)$ and has a kernel K satisfying* (1). *If $T_\delta^1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$ and $T^{k,2}$ are the bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, \ldots, m$, then T_δ^b is bounded from $L^p(R^n)$ to $\dot F_r^{\beta,\infty}(R^n)$.*

PROOF. Choose $q' < s < p$ in Theorem 4.2, we have, by Lemma 3.2 and 3.3,

$$\begin{aligned}\|T_\delta^b(f)\|_{\dot F_r^{\beta,\infty}} &\leqslant C\left\|\sup_{Q\ni\cdot}\frac{1}{|Q|^{1+\beta/n}}\int_Q \left|T_\delta^b(f)(x) - C_0\right| dx\right\|_{L^r}\\ &\leqslant C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m \|M_{\delta,s}(T^{k,2}(f))\|_{L^r} \leqslant C\|b\|_{\mathrm{Lip}_\beta}\sum_{k=1}^m \|T^{k,2}(f)\|_{L^p}\\ &\leqslant C\|b\|_{\mathrm{Lip}_\beta}\|f\|_{L^p}.\end{aligned}$$

This completes the proof. □

THEOREM 4.7. *Let the sequence $\{jC_j\} \in l^1$, $q' < p < n/\delta$, $1/r = 1/p - \delta/n$ and $b \in \mathrm{BMO}(R^n)$. Suppose T_δ is a bounded linear operator from $L^p(R^n)$ to $L^r(R^n)$ and has a kernel K satisfying* (1). *If $T_\delta^1(g) = 0$ for any $g \in L^u(R^n)$ $(1 < u < \infty)$ and $T^{k,2}$ are the bounded operators on $L^p(R^n)$ for $1 < p < \infty$, $k = 1, \ldots, m$, then T_δ^b is bounded from $L^p(R^n)$ to $L^r(R^n)$.*

PROOF. Choose $q' < s < p$ in Theorem 4.3, we have, by Lemma 3.3 and 3.4,

$$\begin{aligned}\|T_\delta^b(f)\|_{L^r} &\leqslant \|M(T_\delta^b(f))\|_{L^p} \leqslant C\|M^{\#}(T_\delta^b(f))\|_{L^r}\\ &\leqslant C\|b\|_{\mathrm{BMO}}\sum_{k=1}^m \|M_{\delta,s}(T^{k,2}(f))\|_{L^r} \leqslant C\|b\|_{\mathrm{BMO}}\sum_{k=1}^m \|T^{k,2}(f)\|_{L^p}\\ &\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^p}.\end{aligned}$$

This completes the proof. □

THEOREM 4.8. *Let the sequence* $\{jC_j\} \in l^1$, $q' < p < n/\delta$, $1/r = 1/p - \delta/n$, $0 < D < 2^n$ *and* $b \in \mathrm{BMO}(R^n)$. *Suppose* T_δ *is a bounded linear operator from* $L^p(R^n)$ *to* $L^r(R^n)$ *and has a kernel* K *satisfying* (1). *If* $T_\delta^1(g) = 0$ *for any* $g \in L^u(R^n)$ $(1 < u < \infty)$ *and* $T^{k,2}$ *are the bounded operators on* $L^{p,\varphi}(R^n)$ *for* $1 < p < \infty$, $k = 1, \ldots, m$, *then* T_δ^b *is bounded from* $L^{p,\varphi}(R^n)$ *to* $L^{r,\varphi}(R^n)$.

PROOF. Choose $q' < s < p$ in Theorem 4.3, we have, by Lemma 3.5 and 3.6,

$$\begin{aligned}
\|T_\delta^b(f)\|_{L^{r,\varphi}} &\leqslant \|M(T_\delta^b(f))\|_{L^{r,\varphi}} \leqslant C\|M^\#(T_\delta^b(f))\|_{L^{r,\varphi}} \\
&\leqslant C\|b\|_{\mathrm{BMO}} \sum_{k=1}^m \|M_{\delta,s}(T^{k,2}(f))\|_{L^{r,\varphi}} \leqslant C\|b\|_{\mathrm{BMO}} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^{p,\varphi}} \\
&\leqslant C\|b\|_{\mathrm{BMO}}\|f\|_{L^{p,\varphi}}.
\end{aligned}$$

This completes the proof. □

COROLLARY. *Let* $T_\delta \in \mathrm{GSIO}(\delta)$, *that is* T_δ *is the singular integral operator as Definition* 1.2. *Then Theorems* 4.1–4.8 *hold for* T_δ^b.

References

1. D. C. Chang, J. F. Li and J. Xiao, *Weighted scale estimates for Calderón-Zygmund type operators*, Contemporary Math. **446** (2007), 61–70.
2. S. Chanillo, *A note on commutators*, Indiana Univ. Math. J. **31** (1982), 7–16.
3. W. G. Chen, *Besov estimates for a class of multilinear singular integrals*, Acta Math. Sinica **16** (2000), 613–626.
4. R. R. Coifman, R. Rochberg and G. Weiss, *Fractorization theorems for Hardy spaces in several variables*, Ann. of Math. **103** (1976), 611–635.
5. G. Di FaZio and M. A. Ragusa, *Commutators and Morrey spaces*, Boll. Un. Mat. Ital. **5-A(7)** (1991), 323–332.
6. G. Di Fazio and M. A. Ragusa, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Func. Anal. **112** (1993), 241–256.
7. J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*, North-Holland Math., **16**, Amsterdam, 1985.
8. S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Math. **16** (1978), 263–270.
9. S. Krantz and S. Li, *Boundedness and compactness of integral operators on spaces of homogeneous type and applications*, J. Math. Anal. Appl. **258** (2001), 629–641.
10. Y. Lin and S. Z. Lu, *Toeplitz type operators associated to strongly singular integral operator*, Sci. in China (ser. A) **36** (2006), 615–630.
11. Y. Lin, *Sharp maximal function estimates for Calderón-Zygmund type operators and commutators*, Acta Math. Scientia **31(A)** (2011), 206–215.
12. L. Z. Liu, *Interior estimates in Morrey spaces for solutions of elliptic equations and weighted boundedness for commutators of singular integral operators*, Acta Math. Scientia **25(B)(1)** (2005), 89–94.
13. T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*; in *Harmonic Analysis*, Proceedings of a conference held in Sendai, Japan, 1990, 183–189.
14. M. Paluszynski, *Characterization of the Besov spaces via the commutator operator of Coifman, Rochberg and Weiss*, Indiana Univ. Math. J. **44** (1995), 1–17.

15. J. Peetre, *On convolution operators leaving $L^{p,\lambda}$-spaces invariant*, Ann. Mat. Pura. Appl. **72** (1966), 295–304.
16. J. Peetre, *On the theory of $L^{p,\lambda}$-spaces*, J. Func. Anal. **4** (1969), 71–87.
17. C. Pérez, *Endpoint estimate for commutators of singular integral operators*, J. Func. Anal. **128** (1995), 163–185.
18. C. Pérez and R. Trujillo-Gonzalez, *Sharp weighted estimates for multilinear commutators*, J. London Math. Soc. **65** (2002), 672–692.
19. E. M. Stein, *Harmonic Analysis: Real Variable Methods, Orthogonality and Oscillatory Integrals*, Princeton Univ. Press, Princeton NJ, 1993.
20. A. Torchinsky, *Real Variable Methods in Harmonic Analysis*, Pure Applied Math. **123**, Academic Press, New York, 1986.

Department of Mathematics
Changsha University of Science and Technology
Changsha 410114
P. R. of China
cxiahuang@126.com
lanzheliu@163.com

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