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GENERALIZATION OF THE GRACE–HEAWOOD THEOREM

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Abstract. Using the theorem of Walsh we give a generalization of the theorems of Grace.

1. Introduction

All the polynomials in this paper are complex. The following theorem is well known and is due to Walsh [**1**].

Theorem 1.1. *If all the zeros of a polynomial* $f(z) = a_0 + a_1 z + \cdots + z^n$ *lie in the circle* $|z| \leqslant r$*, then all the zeros of the polynomial* $F(z) = f(z) + c$*, lie in the circle* $|z| \leqslant r + \sqrt[n]{|c|}$*.*

Due to translation, previous theorem can be slightly generalized:

Theorem 1.2. *If all the zeros of a polynomial* $f(z) = a_0 + a_1 z + \cdots + z^n$ *lie in the circle* $|z - d| \leqslant r$*, then all the zeros of the polynomial* $F(z) = f(z) + c$*, lie in the circle* $|z - d| \leqslant r + \sqrt[n]{|c|}$*.*

For polynomials $f(z) = a_0 + a_1 z + \cdots + z^n$ and $g(z) = b_0 + b_1 z + \cdots + z^n$ let us define

$$A(f,g) = a_0 - \frac{a_1 b_{n-1}}{\binom{n}{1}} + \frac{a_2 b_{n-2}}{\binom{n}{2}} - \cdots + (-1)^{n-1} \frac{a_{n-1} b_1}{\binom{n}{n-1}} + (-1)^n b_0$$

The polynomials $f(z)$ and $g(z)$ are called apolar if $A(f,g) = 0$. The following theorem, due to Grace, is the basic one for us.

Theorem 1.3. *If* $f(z)$ *and* $g(z)$ *are apolar and if one of them has all its zeros in a circular region* C*, then at least one zero of the other one is in* C*.*

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2. Main result

A question arises what happens when $f(z)$ and $g(z)$ are not apolar. We prove the following simple result.

THEOREM 2.1. *For polynomials $f(z) = a_0 + a_1 z + \cdots + z^n$ and $g(z) = b_0 + b_1 z + \cdots + z^n$ the following holds: if all the zeros of $f(z)$ are contained in some circular region of radius r, then a zero of $g(z)$ is contained in concentric circular region of radius $r + \sqrt[n]{|A(f,g)|}$.*

PROOF. Polynomials $f(z) - A(f,g)$ and $g(z)$ are apolar. If C is a circular region of a radius r containing all zeros of $f(z)$, then the concentric circular region of a radius $r + \sqrt[n]{|A(f,g)|}$ contains all the zeros of the polynomial $f(z) - A(f,g)$ (Theorem 1.2). Then, applying Theorem 1.3, we obtain the desired result. □

Now we want to prove a theorem that is, in a sense, dual to Theorem 1.1.

LEMMA 2.1. *Let $f(z) = a_0 + a_1 z + \cdots + z^n$, be a polynomial having a zero in a circular region $|z| \leqslant r$. Then $f(z) + c$ has a zero in the circular region*

$$|z| \leqslant r + \sqrt[n]{|c|}.$$

PROOF. Let b be a zero of $f(z)$, $|b| \leqslant r$. It is easy to see that $0 = A(f(z), (z-b)^n) = A(f(z)+c, (z-b)^n - c)$. Hence, polynomials $f(z)+c$ and $(z-b)^n - c$ are apolar. Since all zeros of $(z-b)^n$ are in the circle $|z| \leqslant r$, by Theorem 1.1. it follows that all the zeros of $(z-b)^n - c$ are in the circle $|z| \leqslant r + \sqrt[n]{|c|}$. Finally, because of apolarity of $f(z)+c$ and $(z-b)^n - c$, we conclude that $f(z)+c$, has a zero in the circle $|z| \leqslant r + \sqrt[n]{|c|}$. □

Similarly as before, due to translation, we can slightly reformulate the statement of the previous lemma.

LEMMA 2.2. *Let $f(z) = a_0 + a_1 z + \cdots + z^n$, be a polynomial having a zero in a circular region $|z-d| \leqslant r$. Then $f(z) + c$ has a zero in the circular region*

$$|z-d| \leqslant r + \sqrt[n]{|c|}.$$

Our main result is a generalization of the following theorem due to Grace and Heawood [**2**].

THEOREM 2.2. *If z_1 and z_2 are zeros of the polynomial $f(z)$, $\deg(f) = n > 1$, then its derivative has a zero in the circular region*

$$\left|z - \frac{1}{2}(z_1+z_2)\right| \leqslant \frac{1}{2}|z_1 - z_2| \cot\left(\frac{\pi}{n}\right).$$

Now we are going to prove the following theorem.

THEOREM 2.3. *Let $f(z) = a_0 + a_1 z + \cdots + z^n$ be a polynomial. Then for any distinct z_1 and z_2, the circular region*

$$\left|z - \frac{1}{2}(z_1+z_2)\right| \leqslant \frac{1}{2}|z_1 - z_2| \cot\left(\frac{\pi}{n}\right) + \sqrt[n-1]{\frac{|a|}{n-1}}, \text{ where } a = -\frac{f(z_2) - f(z_1)}{z_2 - z_1}$$

contains a zero of the derivative $f'(z)$.

PROOF. Choose some distinct z_1 and z_2, and consider the polynomial

$$F(z) = f(z) + az + b, \text{ where } a = -\frac{f(z_2) - f(z_1)}{z_2 - z_1}, \quad b = \frac{z_1 f(z_2) - z_2 f(z_1)}{z_2 - z_1}.$$

Then we have $F(z_1) = F(z_2) = 0$, and $F'(z) = f'(z) + a$. By Theorem 2.2, the circular region

$$\left|z - \frac{1}{2}(z_1 + z_2)\right| \leqslant \frac{1}{2}|z_1 - z_2| \cot\left(\frac{\pi}{n}\right)$$

contains a zero of $F'(z)$. Then, by Lemma 2.2 applied on $F'(z) - a = f'(z)$ and $r = \frac{1}{2}|z_1 - z_2| \cot\left(\frac{\pi}{n}\right)$, we conclude that a zero of $f'(z)$ is contained in the circular region

$$\left|z - \frac{1}{2}(z_1 + z_2)\right| \leqslant \frac{1}{2}|z_1 - z_2| \cot\left(\frac{\pi}{n}\right) + \sqrt[n-1]{\frac{|a|}{n-1}},$$

and the theorem is proved. □

References

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