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IMMERSIONS AND EMBEDDINGS OF QUASITORIC MANIFOLDS OVER THE CUBE

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Abstract. A quasitoric manifold M^{2n} over the cube I^n is studied. The Stiefel–Whitney classes are calculated and used as the obstructions for immersions, embeddings and totally skew embeddings. The manifold M^{2n}, when n is a power of 2, has interesting properties: $\mathrm{imm}(M^{2n}) = 4n-2$, $\mathrm{em}(M^{2n}) = 4n-1$ and $N(M^{2n}) \geqslant 8n-3$.

1. Introduction

Immersions and embeddings of manifolds are a classical topic in algebraic topology. Almost every monograph in topology has sections devoted to manifolds and obstructions to immersions and embeddings [**2**].

A nice introduction to problems and theory of characteristic classes is given in [**11**]. The connection among Stiefel–Whitney classes, immersions and embeddings is given by the following theorem

Theorem 1.1. *If* $k := \max\{i \mid \overline{w}_i(M^n) \neq 0\}$, *then*

$$\mathrm{imm}(M^n) \geqslant n+k \;\; \textit{and} \;\; \mathrm{em}(M^n) \geqslant n+k+1, \quad \textit{where}$$
$$\mathrm{imm}(M^n) = \min\left\{d \mid M \;\textit{immerses into}\; \mathbb{R}^d\right\},$$
$$\mathrm{em}(M^n) = \min\left\{d \mid M \;\textit{embedds into}\; \mathbb{R}^d\right\}.$$

The study of skew embeddings was started by Ghomi and Tabachnikov in [**8**]. They defined a number $N(M^n) = \min\{d \mid M$ totally skew embedds into $R^d\}$, for which they obtained the bounds $2n+2 \leqslant N(M^n) \leqslant 4n+1$.

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In [**1**] the lower bound is improved for various classes of manifolds, such as projective spaces (both real and complex), products of projective spaces, Grassmannians, etc. Stiefel–Whitney classes are obstructions to totally skew embeddings as shown in [**1**, Proposition 1.] and [**1**, Corollary 4.]

THEOREM 1.2. *If* $k := \max\{i \mid \overline{w}_i(M) \neq 0\}$*, then* $N(M) \geqslant 2n + 2k + 1$.

In the same paper a conjecture [**1**, Conjecture 20] was formulated predicting that $N(M^n) \leqslant 4n - 2\alpha(n) + 1$, for compact smooth manifold M^n $(n > 1)$, where $\alpha(n)$ is the number of non-zero digits in the binary representation of n. Cohen [**5**] in 1985 resolved positively the famous *Immersion Conjecture*, by showing that each compact smooth n-manifold for $n > 1$ can be immersed in $\mathbb{R}^{2n-\alpha(n)}$.

Various types of immersions and embeddings are an interesting research topic. In [**12, 13, 14**] some more general conditions with multiple regularity are studied.

In the last decades a lot has been written about toric actions and quasitoric manifolds due to their wide applications in combinatorics, physics, topology, geometry, etc.

Quasitoric manifolds are a class of manifolds with a well understood cohomology ring which is determined by the Davis–Januszkiewicz formula [**7**, Theorem 4.14, Corollary 6.8]. Other topological invariants can be computed from the formula, and we are particulary interested in the Stiefel–Whitney classes. A nice exposition of the theory of quasitoric manifolds, including a review of their topological and combinatorial properties, can be found in the monograph [**3**] of Buchstaber and Panov.

The construction of a quasitoric manifold from the characteristic pair (P^n, l) is described in [**3**, Construction 5.12]. Recall that P^n is a simple polytope with m facets and $\Lambda = (\lambda_1, \dots, \lambda_m)$ an integer $n \times m$ matrix, where $\lambda_j \in \mathbb{Z}^n$ $j = 1, \dots, m$ corresponds to the generator of the Lie algebra isotropy subgroup of the characteristic submanifold M_j over the facet F_j. For every vertex $v = F_{i_1} \cap \dots \cap F_{i_n} \in P^n$ the matrix has the property $\det \Lambda_{I_{(v)}} = \pm 1$ where $\Lambda_{I_{(v)}}$ is a square submatrix formed by the column vectors $\lambda_{i_1}, \dots, \lambda_{i_n}$ corresponding to the facets $F_{i_1}, \dots, F_{i_n}$. The matrix Λ is called *the characteristic matrix* of M.

Let $\lambda_j = (\lambda_{1j}, \dots, \lambda_{nj})^t \in \mathbb{Z}^n$. Then we have $\theta_i := \sum_{j=1}^m \lambda_{ij} v_j$ and let $\mathcal{J}$ be the ideal in $\mathbb{Z}[v_1, \dots, v_m]$ generated by θ_i for all $i = 1, \dots, n$. Let $\mathcal{I}$ denote the Stanley-Reisner ideal of P. The ordinary cohomology of quasitoric manifolds has the following ring structure:

$$H^*(M) \simeq \mathbb{Z}[v_1, \dots, v_m]/(\mathcal{I} + \mathcal{J}).$$

The total Stiefel–Whitney class can be described by the following *Davis–Januszkiewicz formula*:

$$w(M) = \prod_{i=1}^{m}(1 + v_i) \in H^*(M; \mathbb{Z}_2),$$

where v_i is the $\mathbb{Z}_2$-reduction of the corresponding class over $\mathbb{Z}$ coefficients. The Stiefel–Whitney classes are a powerful tool for studying problems of toric topology such as cohomological rigidity [**6**].

In Section 2 one special quasitoric manifold M_I over the cube I^n is constructed by matrix Λ_{M_I}. The cohomology ring and the total Stiefel–Whitney class of this manifold are described.

Section 3 is devoted to the calculation of the total Stiefel–Whitney class of the stable normal bundle using careful manipulations of binomial coefficients in the cohomology ring (with $\mathbb{Z}_2$ coefficients). The obstruction to immersion, embedding and totally skew embedding of the manifold M_I is calculated and the main result of the paper is obtained.

2. Quasitoric manifold over the cube

2.1. Matrix Λ_{M_I} and the cube. A quasitoric manifold M is described by two key objects: its orbit polytope P and characteristic matrix Λ. Two quasitoric manifolds over the same polytope, but with distinct characteristic matrices may be different, in general, due to nonisomorphic cohomology rings. Although, the polytope P with its combinatorics yields a lot of information about the manifold itself, the characteristic matrix Λ is essential to understand important topological invariants of the quasitoric manifold.

Let I^n be a cube and M_{I^n} a quasitoric manifold over I^n. The cube has $2n$ facets $F_1, \dots, F_n, F'_1, \dots, F'_n$ such that $F_i \cap F'_i = \emptyset$ for every $i = 1, \dots, n$. Let $v_1, \dots, v_n, u_1, \dots, u_n$ be Poincaré duals to the characteristic submanifolds over the facets $F_1, \dots, F_n, F'_1, \dots, F'_n$ respectively. The Stanley–Reisner ideal is generated by $\mathcal{I} = \{v_1u_1, v_2u_2, \dots, v_nu_n\}$.

A special quasitoric manifold M_{I^n} over the cube is studied, such that the vector λ_i assigned to the facet F_i (or the generators of the Lie algebra isotropy subgroup of the characteristic submanifold M_i) is $\lambda_i = (\underbrace{0, \dots, 0}_{i-1}, 1, \underbrace{0, \dots, 0}_{n-i})^t$ for every $i = 1, \dots, n$ and vector λ_{i+n} assigned to the facet F'_i is $\lambda_{n+i} = (\underbrace{0, \dots, 0}_{i-1}, \underbrace{1, \dots, 1}_{n-i+1})^t$ for every $i = 1, \dots, n$. Then we have:

$$\Lambda_{M_{I^n}} = \begin{bmatrix} 1 & 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & 1 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 1 & 1 & \dots & 1 \end{bmatrix}.$$

The matrix $\Lambda_{M_{I^n}}$ has the property that $\det\left(\Lambda_{M_{I^n}}\right)_{(v)} = 1$ for every vertex v of I^n (all entries above the main diagonal are 0 while all entries lying on the main diagonal are 1).

The ideal $\mathcal{J}$ in $\mathbb{Z}[v_1, \dots, v_n, u_1, \dots, u_n]$ is generated by linear forms

$$\begin{aligned} &v_1 + u_1, \\ &v_2 + u_1 + u_2, \\ &\dots \\ &v_n + u_1 + u_2 + \dots + u_n. \end{aligned}$$

2.2. Cohomology ring $H^*(M_{I^n})$ and the total Stiefel–Whitney class $w(M_{I^n})$. The cohomology ring $H^*(M_{I^n})$ is determined using the Davis–Januszkiewicz theorem:

Proposition 2.1. *The cohomology ring $H^*(M_{I^n};\mathbb{Z})$ is isomorphic to*

$$H^*(M_{I^n};\mathbb{Z}) \simeq \mathbb{Z}[u_1,\dots,u_n]/\mathcal{F}_n$$

where $\mathcal{F}_n$ is an ideal in the polynomial ring $\mathbb{Z}[u_1,\dots,u_n]$ (such that $\deg(u_1) = \cdots = \deg(u_n) = 2$) generated by quadratic forms

$$u_1^2,\;\; u_2^2 + u_1u_2,\; \dots,\; u_n^2 + u_1u_n + u_2u_n + \cdots + u_{n-1}u_n.$$

Reducing modulo 2 we obtain that $H^*(M_{I^n};\mathbb{Z}_2) \simeq \mathbb{Z}_2[u_1,\dots,u_n]/\mathcal{F}_n$ where $\mathcal{F}_n$ is an ideal in the polynomial ring $\mathbb{Z}_2[u_1,\dots,u_n]$ (such that $\deg(u_1) = \cdots = \deg(u_n) = 2$) generated by quadratic forms

$$\mathcal{F}_n = \{u_1^2, u_2^2 + u_1u_2, \dots, u_n^2 + u_1u_n + u_2u_n + \cdots + u_{n-1}u_n\}.$$

It is easy to show the following relations in $H^*(M_{I^n};\mathbb{Z}_2)$:

Proposition 2.2. *For every $i = 2,\dots,n$ the following equality holds*

$$(1+u_i)(1+v_i) = 1 + u_1 + \cdots + u_{i-1} = 1 + v_{i-1}.$$

The total Stiefel–Whitney class is the characteristic class in cohomology with $\mathbb{Z}_2$ coefficients. By Davis–Januszkiewicz's formula, the total Stiefel–Whitney class of M_{I^n} is given by $w(M_{I^n}) = (1+u_1)\cdots(1+u_n)(1+v_1)\cdots(1+v_n)$, but according to Propositions 2.1 and 2.2 it easily reduces to

$$w(M_I) = (1+u_1)(1+u_1+u_2)\cdots(1+u_1+\cdots+u_{n-1}).$$

For the purposes of the main theorem, we are going to use another form of the cohomology ring $H^*(M_{I^n};\mathbb{Z}_2)$, with generators $v_1,\dots,v_n$. Recall that

$$\begin{aligned} v_1 &= u_1,\\ v_2 &= u_1 + u_2,\\ &\dots\\ v_n &= u_1 + u_2 + \cdots + u_n, \end{aligned}$$

so, we get that $H^*(M_{I^n};\mathbb{Z}_2) \simeq \mathbb{Z}_2[v_1,\dots,v_n]/\mathcal{G}_n$ where $\mathcal{G}_n$ is an ideal in the polynomial ring $\mathbb{Z}_2[v_1,\dots,v_n]$ (such that $\deg(v_1) = \cdots = \deg(v_n) = 2$) generated by quadratic forms v_1^2, $v_2^2 + v_1v_2$, $\dots$, $v_n^2 + v_{n-1}v_n$. Consequently, the total Stiefel–Whitney class is given by $w(M_I) = (1+v_1)\cdots(1+v_{n-1})$.

In the following proposition we begin the analysis of the cohomology ring $H^*(M_{I^n};\mathbb{Z}_2)$.

Proposition 2.3. *For every $i = 1,\dots,n$ the following equalities hold*

$$v_i^i = v_1v_2\cdots v_i \neq 0 \;\; \textit{and} \;\; v_i^{i+1} = 0.$$

PROOF. We easily deduce

$$v_i^{i+1} = v_i^i v_{i-1} = \cdots = v_i v_{i-1}^i = v_i v_{i-1}^{i-1} v_{i-2} = \cdots = v_i \cdots v_2 v_1^2 = 0.$$

Similarly, $v_i^i = v_1 v_2 \cdots v_i$ for all $i = 1, \ldots, n$.

To show the nontriviality of classes v_i^i, it is enough to show that $v_n^n = v_1 \cdots v_n$ is nonzero. First, we prove the following lemma.

LEMMA 2.1. *Let $i < j$ and a and b be nonnegative integers such that $a \leqslant i$ and $b \leqslant j$. Then the class $v_i^a v_j^b$ is trivial or equal to the product of some $a+b$ distinct generators v_k.*

PROOF. It is easy to get that $v_i^a v_j^b = v_{i-a+1} \cdots v_i v_{j-b+1} \cdots v_b$. If $i \leqslant j - b$ the proof is completed. Otherwise, $i = j - k$ for some positive integer $k \leqslant b - 1$ resulting in

$$\begin{aligned} v_i^a v_j^b &= v_{j-(a+k)+1} \cdot \cdots \cdot v_{j-k} \cdot v_{j-b+1} \cdot \cdots \cdot v_b \\ &= v_{j-(a+k-1)} \cdots v_{j-b} v_{j-b+1}^2 \cdots v_{j-k}^2 v_{j-k+1} \cdots v_j. \end{aligned}$$

Now we continue to remove squares and powers from the expression above using the equalities $v_m^2 = v_m v_{m-1}$. Since $v_1^2 = 0$, if $j \geqslant a + b$ then

$$v_i^a v_j^b = v_{j-(a+b)+1} \cdots v_j,$$

while in the other case $v_i^a v_j^b = 0$. □

An immediate consequence of the previous lemma is:

COROLLARY 2.1. *Every class of type $v_{i_1}^{r_1} \cdots v_{i_k}^{r_k}$ is either trivial or equal to the product of some $r_1 + \cdots + r_k$ distinct generators. Moreover, this class is non-zero if and only if for each $p = 1, 2, \ldots, n$ $r_1 + \cdots + r_p \leqslant p$.*

From the general manifold theory it is known that $H^{2n}(M_{I^n}; \mathbb{Z}_2) = \mathbb{Z}_2$. Thus, according to the previous observations, the generator of the highest cohomology group must be the class $v_1 v_2 \cdots v_n$ and the proposition is therefore proved. □

The following proposition, referred to as the 'cancellation lemma', summarizes most of the properties of the cohomology ring $H^*(M_I, \mathbb{Z}_2)$ that will be needed in Section 3. Here and later on we use the multi-index power v^α to denote the monomial $v^\alpha = v_{i_1} v_{i_2} \cdots v_{i_k}$ of degree $|\alpha| = k$ where $\alpha = \{i_1, i_2, \ldots, i_k\}$ is a subset of $[n] = \{1, 2, \ldots, n\}$. By convention $v^\alpha = 1$ if $\alpha = \emptyset$ and we always assume that $i_1 < i_2 < \cdots < i_k$.

PROPOSITION 2.4 (Cancellation Lemma). *The collection of monomials $v^\alpha = v_{i_1} v_{i_2} \cdots v_{i_k}$, where $\alpha \subset [n]$, is a graded $\mathbb{Z}_2$-vector space basis of the graded vector space $H^*(M_I, \mathbb{Z}_2)$. Moreover, if $\beta \subset [n]$ then $v^\beta v_n^p \neq 0$ if and only if $|\beta| + p \leqslant n$.*

PROOF. We already know from the proof of Proposition2.3, that the collection $\mathcal{B} = \{v^\alpha\}_{\alpha \subset [n]}$ is a spanning set for the $\mathbb{Z}_2$-vector space $H^*(M_I, \mathbb{Z}_2)$. As a consequence $\dim(H^*(M_I, \mathbb{Z}_2)) \leqslant 2^n$. By [**3**, Proposition 5.16],

$$\dim(H^*(M_I, \mathbb{Z}_2)) = h_0 + h_1 + \cdots + h_n$$

where $h = (h_0, \dots, h_n)$ is the h-vector of the associated polytope. Recall that the sum of all h_j is always equal to the number of vertices of the associated simple polytope P. In particular if $P = I^n$ is the n-dimensional cube, we obtain that $\dim(H^*(M_I, \mathbb{Z}_2)) = 2^n$ which completes the proof of the first half of the proposition. The second half is an easy consequence which can be proved by induction on p. $\square$

3. Topological obstructions to immersions and embeddings of the manifold M_I

3.1. Stiefel–Whitney class $\overline{w}(M_I)$ of the stable normal bundle. For the proof of the main theorem of this paper, we are interested in characteristic classes $\overline{w}(M_I)$ of the stable normal bundle of M_I. The Stiefel–Whitney classes $w(M_I)$ and $\overline{w}(M_I)$ are related to each other by the following equality

$$w(M_I) \cdot \overline{w}(M_I) = 1.$$

In the previous section the total Stiefel–Whitney class $w(M_{I^n})$ is determined. So, by Proposition2.3, the following holds:

Lemma 3.1. *The total Stiefel–Whitney class $\overline{w}(M_{I^n})$ of the stable normal bundle is given by*

$$\overline{w}(M_{I^n}) = (1 + v_1)(1 + v_2 + v_2^2) \cdots (1 + v_{n-1} + \cdots + v_{n-1}^{n-1}).$$

Since $\overline{w}_{2i}(M_{I^n}) = 0$ when $i \geqslant n$, it is not evident what $\overline{w}(M_{I^n})$ is in the cohomology ring $H^*(M_{I^n}; \mathbb{Z}_2)$. For small n, we could calculate $\overline{w}(M_{I^n})$ by hand:

Exercise 3.1. (1) $\overline{w}(M_{I^2}) = 1 + v_1$,
(2) $\overline{w}(M_{I^3}) = 1 + (v_1 + v_2)$,
(3) $\overline{w}(M_{I^4}) = 1 + (v_1 + v_2 + v_3) + v_1v_3 + v_1v_2v_3$,
(4) $\overline{w}(M_{I^5}) = 1 + (v_1 + v_2 + v_3 + v_4) + (v_1v_3 + v_1v_4 + v_2v_4) + (v_1v_2v_3 + v_2v_3v_4)$.

The top class $\overline{w}_{2n}$ is always zero by the theorem of Massey [**9**].

By Lemma 3.1 for the total Stiefel–Whitney classes of $\overline{w}(M_{I^n})$ and $\overline{w}(M_{I^{n+1}})$ the following recurrence relation holds (in $H^*(M_{I^{n+1}}; \mathbb{Z}_2)$):

$$\overline{w}(M_{I^{n+1}}) = \overline{w}(M_{I^n})(1 + v_n + \cdots + v_n^n),$$

or more explicitly

$$\overline{w}_{2k}(M_{I^{n+1}}) = \overline{w}_{2k}(M_{I^n}) + v_n\overline{w}_{2k-2}(M_{I^n}) + \cdots + v_n^k \text{ for all } k = 0, \dots, n-1 \tag{3.1}$$

and

$$\overline{w}_{2n}(M_{I^{n+1}}) = v_n\overline{w}_{2n-2}(M_{I^n}) + \cdots + v_n^n. \tag{3.2}$$

Here we use the fact that there is a natural homomorphism $i : H^*(M_{I^n}; \mathbb{Z}_2) \to H^*(M_{I^{n+1}}; \mathbb{Z}_2)$ which allow us to move all classes to the latter group.

By the cancellation lemma (Proposition 2.4) and working modulo 2, $\overline{w}_{2k}$ is the sum of a certain number of linearly independent square-free monomials. We consider the polynomial $\overline{W}_{2k}(v_1, \dots, v_n)$ in the ring $\mathbb{Z}_2[v_1, \dots, v_n]$ of degree $2k$, obtained after applying all possible cancellations in Lemma 3.1. Define the numbers σ_n^k for all positive integers n and $0 \leqslant k \leqslant n-1$ as follows

$$\sigma_n^k = \overline{W}_{2k}(1,\dots,1) \pmod 2$$

So by (3.1) and (3.2), we have $\sigma_{n+1}^k = \sum_{i=0}^{k} \sigma_n^i$ for every $k = 1, \dots, n-1$ and $\sigma_{n+1}^n = \sigma_{n+1}^{n-1}$. Here we tacitly used the second half of Proposition 2.4.

Let us write the first n rows of numbers σ_n^k for $k = 0, \dots, n$:

$$\begin{array}{llllllll}
1 \\
1 & 1 \\
1 & 0 & 0 \\
1 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\multicolumn{8}{l}{\dots\dots\dots\dots\dots\dots}
\end{array}$$

The previous sequence is closely related to the following sequence of binomial coefficients $\binom{n+k}{k}$:

$$\begin{array}{cccccccc}
\boxed{1} \\
\boxed{1} & \boxed{3} \\
\boxed{1} & 4 & 10 \\
\boxed{1} & \boxed{5} & \boxed{15} & \boxed{35} \\
\boxed{1} & 6 & \boxed{21} & 56 & 70 \\
\boxed{1} & \boxed{7} & \binom{8}{2} & \binom{9}{3} & \binom{10}{4} & \binom{11}{5} \\
\boxed{1} & 8 & \binom{9}{2} & \binom{10}{3} & \binom{11}{4} & \binom{12}{5} & \binom{13}{6} \\
\boxed{1} & \boxed{9} & \boxed{\binom{10}{2}} & \boxed{\binom{11}{3}} & \boxed{\binom{12}{4}} & \boxed{\binom{13}{5}} & \boxed{\binom{14}{6}} & \boxed{\binom{15}{7}} \\
\multicolumn{8}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}
\end{array}$$

An easy mathematical induction shows that:

LEMMA 3.2. *We have* $\sigma_n^k \equiv \binom{n+k}{k} \pmod 2$.

By the previous Lemma, in the case when $n = 2^r$ we have

$$\sigma_n^{n-1} \equiv \binom{2^r + (2^r - 1)}{2^r - 1} \equiv \binom{2^{r+1} - 1}{2^r - 1} \equiv 1 \pmod 2.$$

Obviously, from the definition of σ_n^k, if $\sigma_n^k = 1$, then $\overline{w}_{2k}$ is the sum of an odd number of linearly independent square-free monomials and $\overline{w}_{2k}(M_{I^n}) \neq 0$. Thus, we obtain:

THEOREM 3.1. *If* $n = 2^r$ *is a power of two then*

$$\overline{w}_{2n-2}(M_{I^n}) = v_1 v_2 \cdots v_{n-1} \neq 0.$$

Hence, Theorem 1.1 yields:

COROLLARY 3.1. *If n is a power of two then*

$$\operatorname{imm}(M_{I^n}) \geqslant 4n-2 \;\; \text{and} \;\; \operatorname{em}(M_{I^n}) \geqslant 4n-1.$$

Since $M_{I^n}^{2n}$ is orientable, it can be embedded into $\mathbb{R}^{4n-1}$. Thus,

THEOREM 3.2. *If n is a power of two, then* $\operatorname{em}(M_{I^n}) = 4n-1$.

Lemma 3.1 implies that $\overline{w}_2(M_{I^n}) = v_1+v_2+\cdots+v_{n-1}$. Due to the cancellation lemma, when n is a power of two, the characteristic class $\overline{w}_2(M_{I^n})\overline{w}_{2n-2}(M_{I^n})$ is trivial. By the result of Massey [**10**, Theorem V], it follows:

THEOREM 3.3. *If $n \geqslant 4$ is a power of two, then* $\operatorname{imm}(M_{I^n}) = 4n-2$

For totally skew embeddings, from Theorem 3.1 the lower bound obtained is:

COROLLARY 3.2. *If n is a power of two, then $N(M_{I^n}) \geqslant 8n-3$.*

3.2. Topological obstructions when n is not a power of 2. Theorem 3.1 is the sharpest possible result that one can obtain using Stiefel–Whitney classes for quasitoric manifolds. However, when n is not a power of 2 the previously constructed quasitoric manifold M_{I^n}, in general, does not achieve the maximal possible value k for which the Stiefel–Whitney class $\overline{w}_{2k}(M_{I^n}) \neq 0$.

This problem could be overcome using the results from the previous part.

Let $n = 2^{r_1}+2^{r_2}+\cdots+2^{r_t}$, $r_1 > r_2 > \cdots > r_t \geqslant 0$ be the binary representation of n and let $m_i = 2^{r_i}$ for $i = 1,\ldots,t$ and $m_0 = 0$. In the previous section we described the quasitoric manifold M_{I_j} over the cube I^{m_j}. From the result of Buchstaber and Ray [**4**, Proposition 4.7], it follows that $M_I = M_{I_1} \times \cdots \times M_{I_t}$ is a quasitoric manifold over the cube $I^n = I_1 \times \cdots \times I_t$.

The total Stiefel–Whitney class of the tangent bundle of M_I can be easily determined using the following formula (see [**11**, pp. 27, 54]):

$$w(M_I) = w(M_{I_1})\cdots w(M_{I_t}) \in H^*(M_I) \cong H^*(M_{I_1}) \otimes \cdots \otimes H^*(M_{I_t}).$$

Let $v_i^{(j)}$, for $i = 1,\ldots,m_j$, be the generators of the cohomology ring $H^*(M_{I_j},\mathbb{Z}_2)$. The total Stiefel–Whitney class is given by

$$w(M_I) = \prod_{j=1}^{t} \big(1+v_1^{(j)}\big)\cdots\big(1+v_{m_j-1}^{(j)}\big).$$

Thus, the corresponding dual Stiefel–Whitney class is given by

$$\overline{w}(M_I) = \prod_{j=1}^{t} \big(1+v_1^{(j)}\big)\Big(1+v_2^{(j)}+\big(v_2^{(j)}\big)^2\Big)\cdots\Big(1+v_{m_j-1}^{(j)}+\cdots+\big(v_{m_j-1}^{(j)}\big)^{m_j-1}\Big).$$

But, according to Theorem 3.1 we have:

$$\overline{w}(M_I) = \prod_{j=1}^{t} \Big(1+(v_1^{(j)}+\cdots+v_{m_j-1}^{(j)})+\cdots+v_1^{(j)}v_2^{(j)}\cdots v_{m_j-1}^{(j)}\Big).$$

So, the highest nontrivial dual Stiefel–Whitney class is

$$\overline{w}_{2n-2\alpha(n)}(M_I) = v_1^{(1)} \cdots v_{m_1-1}^{(1)} v_1^{(2)} \cdots v_{m_t-1}^{(t)},$$

where $\alpha(n)$ is the number of non-zero digits in the binary representation of n.

As corollary we obtain:

THEOREM 3.4 (Main theorem). *For every positive integer n there is a quasitoric manifold M_I over the cube such that*

$$\begin{aligned} \operatorname{imm}(M_I) &\geqslant 4n - 2\alpha(n), \\ \operatorname{em}(M_I) &\geqslant 4n - 2\alpha(n) + 1, \\ N(M_I) &\geqslant 8n - 4\alpha(n) + 1. \end{aligned}$$

REMARK 3.1. No similar result can be obtained in the class of toric varieties from a cube because the total Stiefel–Whitney class is trivial in that case.

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