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STOCHASTIC ANALYSIS OF GSB PROCESS

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Abstract. We present a modification (and partly a generalization) of STOPBREAK process, which is the stochastic model of time series with permanent, emphatic fluctuations. The threshold regime of the process is obtained by using, so called, noise indicator. Now, the model, named the General Split-BREAK (GSB) process, is investigated in terms of its basic stochastic properties. We analyze some necessary and sufficient conditions of the existence of stationary GSB process, and we describe its correlation structure. Also, we define the sequence of the increments of the GSB process, named Split-MA process. Besides the standard investigation of stochastic properties of this process, we also give the conditions of its invertibility.

1. Introduction

Starting from the fundamental results of Engle and Smith [**2**], who introduced the stochastic process of permanent fluctuations, named *Stochastic Permanent Breaking* (*STOPBREAK*) *process*, we define a new, modified version of the well-known generalization of STOPBREAK process. In our model, we set a threshold noise indicator as we already did in the time series of ARCH type, described in Stojanović and Popović [**8**]. Therefore, our model, named *the General Split-BREAK process* or, simply, *GSB process*, at the same time is a generalization of Split-BREAK model introduced by Stojanović, Popović and Popović [**9**].

In the next section we present the definition of the standard STOPBREAK model defined by Engle and Smith [**2**], as well as the definition of our GSB model and we compare their stochastic structures. In the following, Section 3, we put the special focus on the sufficient and necessary conditions of stationarity and correlation structure of the GSB process. Also, we define a specific stochastic

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process, the so called series of martingale means, which represents the stability factor of GSB model. We define this series as more general than it was done in Engle and Smith [**2**]. So, we discuss it with the special care. In Section 4, we define the sequence of increments of the GSB process, the so-called *Split-MA process*. It is similar to the standard MA processes, thus many of well-known results for MA processes can be applied. Besides the standard investigation of the stochastic properties of the Split-MA model, we give conditions for its invertibility. At last, some concluding remarks, focused on the comparison of stationary and invertible conditions of the GSB process, are given in Section 5.

2. The definition of GSB Process

Let us suppose that (y_t) is the time series with the known values at time $t \in \{0, 1, \dots T\}$ and $F = (\mathcal{F}_t)$ is a filtration defined on some probability space $(\Omega, \mathcal{F}, P)$. On the other hand, let (ε_t) be a white noise, i.e., the i.i.d. sequence of random variables adapted to the filtration F, which satisfies

$$E(\varepsilon_t | \mathcal{F}_{t-1}) = 0, \quad \mathrm{Var}(\varepsilon_t | \mathcal{F}_{t-1}) = \sigma^2$$

for each $t = 1, \dots, T$. At last, we denote as (q_t) the sequence of the $\mathcal{F}_{t-1}$ adaptive random variables which depends on the white noise (ε_t), and in addition, we suppose that $P\{0 \leqslant q_t \leqslant 1\} = 1$ for each $t = 0, 1, \dots, T$. Following Engle and Smith [**2**], firstly, we shall set their definition of the general STOPBREAK model.

Definition 2.1. The sequence (y_t) is a *general STOPBREAK process* if it satisfies the recurrent relation

$$A(L)B(L)y_t = q_{t-1}A(L)\,\varepsilon_t + (1 - q_{t-1})B(L)\,\varepsilon_t, \quad t = 1, \dots, T, \tag{2.1}$$

where $A(L) = 1 - \sum_{j=1}^{p} \alpha_j L^j$, $B(L) = 1 - \sum_{k=1}^{r} \beta_k L^k$ and L is the backshift operator.

According to the definition above, the sequence (q_t) displays the (*permanent*) *reaction* of the STOPBREAK process, because its values determine the amount of participation of previous elements of the white noise process engaged in the definition of y_t. In that way, the structure of the sequence (q_t) determines the character and the properties of the STOPBREAK process, which vary between the well-known linear stochastic models. Finally, if $A(L) = 1$ and $B(L) = 1 - L$ the mentioned model represents the basic and the simplest case of STOPBREAK process, also introduced and particularly discussed by Engle and Smith [**2**]. This model was investigated later by several authors, for instance Gonzáles [**4**], or Gonzalo and Martinez [**5**], and their works were mainly based on different variations of the reaction (q_t).

Similarly as it was done in the definition of Split-ARCH model, which was described in Stojanović and Popović [**8**], we suppose that

$$q_t = I(\varepsilon_{t-1}^2 > c) = \begin{cases} 1, & \varepsilon_{t-1}^2 > c \\ 0, & \varepsilon_{t-1}^2 \leqslant c \end{cases} \qquad t = 1, \dots, T. \tag{2.2}$$

Then, we say that permanent reaction (2.2) is *a noise indicator*, and the STOPBREAK model, obtained in this way, represents *the split-BREAK model*, introduced and discussed by Stojanović, Popović and Popović [**9**]. According to (2.2), it follows that $E(q_t\varepsilon_t|\mathcal{F}_{t-1}) = q_t E(\varepsilon_t|\mathcal{F}_{t-1}) = 0$, and it can be seen that the sequence $(q_t\varepsilon_t)$ is a martingale difference, the same as it was in the definition of the basic STOPBREAK model of Engle and Smith [**2**].

However, it seems that in the case of the general STOPBREAK process this formulation of reaction (q_t) is inadequate. The primary reason for such opinion is the fact that model (2.1) includes only "directly previous" realizations of (q_t), which are obtained at the moment $t-1$. Therefore, general STOPBREAK process (2.1) with reaction (2.2) operates in (only) two different regimes

$$\varepsilon_t = \begin{cases} A(L)y_t, & q_{t-1} = 0 \text{ (w.p. } b_c) \\ B(L)y_t, & q_{t-1} = 1 \text{ (w.p. } a_c), \end{cases} \tag{2.3}$$

where $a_c = E(q_t) = P\{\varepsilon_{t-1}^2 > c\}$, $b_c = 1 - a_c$ and "w.p." stands for "with probability". So, equality (2.3) defines the well-known Thresholds Autoregressive (TAR) model introduced by Tong [**10**] and discussed in details by Chan [**1**], Hansen [**6**] and some others. Based on these reasons, we yield a different generalization of the Split-BREAK process, in a sense, more general than (2.1).

Definition 2.2. Let L be a backshift operator, (q_t) noise indicator defined with (2.2), $A(L) = 1 - \sum_{i=1}^m \alpha_i L^i$, $B(L) = 1 - \sum_{j=1}^n \beta_j L^j$ and $C(L) = 1 - \sum_{k=1}^p \gamma_k L^k$. The sequence (y_t) represents *the General Split-BREAK* (*GSB*) *process* if it satisfies

$$A(L)y_t = B(L)q_t\varepsilon_t + C(L)(1-q_t)\varepsilon_t, \qquad t \in \mathbf{Z}. \tag{2.4}$$

The definition above represents the general stochastic model which, for its specificity, contains most of the other well-known models. In dependence of $A(L)$, $B(L)$ and $C(L)$ we have, for example, the following situations:

$$\begin{aligned} A(L) = B(L) = C(L) = 1:&\quad y_t = \varepsilon_t && \text{(White noise)} \\ A(L) = 1,\ B(L) = C(L) \neq 1:&\quad y_t = B(L)\varepsilon_t && \text{(MA model)} \\ A(L) \neq 1,\ B(L) = C(L) = 1:&\quad A(L)y_t = \varepsilon_t && \text{(AR model).} \end{aligned}$$

Finally, in the case when $A(L) = C(L) = 1 - L$ and $B(L) = 1$ we get the Split-BREAK process introduced in [**9**]. From this point of view, we shall analyze a specification of model (2.4) and suppose that $A(L) = C(L) \neq 1$ and $B(L) = 1$. Thus, the just defined model can be written in the form

$$y_t - \sum_{j=1}^{p} \alpha_j\, y_{t-j} = \varepsilon_t - \sum_{j=1}^{p} \alpha_j\, \theta_{t-j}\, \varepsilon_{t-j}, \qquad t \in \mathbf{Z}, \tag{2.5}$$

where $\alpha_j \geqslant 0,\ j = 1, \dots, p$ and $\theta_t = 1 - q_t = I(\varepsilon_{t-1}^2 \leqslant c)$. Obviously, this representation is close to linear ARMA time series, except that it includes the noise indicators (ε_t) in its own structure. They indicate the realizations of white noise which have a statistically significant value at the previous moment $t-j$. These "temporary" components imply the structure of GSB model (2.5). That makes some difficulties in usage of well-known procedures of investigating properties of the model.

3. Stationarity and correlation structure. The martingale means

We shall notice that GSB model (2.5) can be represented in the form of a linear stochastic difference equation of order one

$$\mathbf{Y}_t = \mathbf{M}\mathbf{Y}_{t-1} + \mathbf{V}_t, \qquad \mathbf{t} \in \mathbf{Z}, \tag{3.1}$$

where

$$\mathbf{Y}_t = (y_t - \theta_t \varepsilon_t \quad \cdots \quad y_{t-p+1} - \theta_{t-p+1}\varepsilon_{t-p+1})', \quad \mathbf{V}_t = (q_t \varepsilon_t \quad \cdots \quad 0)'$$

and

$$\mathbf{M} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_p \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix}$$

Without loss of generality, we suppose that for all $t \in \mathbf{Z}$ the random variable ε_t has an absolutely continuous distribution and a_c, $b_c \in (0,1)$. Then, we can specify necessary and sufficient stationarity conditions of $(\mathbf{Y_t})$.

THEOREM 3.1. *Let the sequence* $(\mathbf{Y}_t)$ *be defined by recurrent relation* (3.1). *Then, the following conditions are equivalent.*
(i) *The polynomial* $P(\lambda) = \lambda^p - \sum_{j=1}^p \alpha_j \lambda^{p-j}$ *has the roots* $\lambda_1, \ldots, \lambda_p$ *satisfying*

$$|\lambda_j| < 1, \quad \forall j = 1, \ldots, p. \tag{3.2}$$

(ii) *Equation* (3.1) *has almost sure unique, strong stationary and ergodic solution*

$$\mathbf{Y}_t = \mathbf{V}_t + \sum_{k=1}^{\infty} \mathbf{M}^k \, \mathbf{V}_{t-k}. \tag{3.3}$$

(iii) $\sum_{j=1}^p \alpha_j < 1$.

PROOF. (i) $\Rightarrow$ (ii) After some computation it can be seen that

$$\det(\mathbf{M} - \lambda \mathbf{I}) = (-1)^p P(\lambda),$$

i.e., the eigenvalues of the matrix $\mathbf{M}$ are roots of the characteristics polynomial $P(\lambda)$. Then, according to assumption (3.2) we have $\mathbf{M}^k \to \mathbf{O}_{p\times p}$, $k \to \infty$. Following Francq et al. [**3**], the existence of almost sure unique, stationary solution (3.3) of equation (3.1) is equivalent to the convergence shown above.
(ii) $\Rightarrow$ (iii) If we suppose that (ii) is true, then the sequence

$$\mathbf{U}_t = \mathbf{W}_t + \sum_{k=1}^{\infty} \mathbf{M}^k \, \mathbf{W}_{t-k}, \quad \text{where } \mathbf{W}_t = (q_t \varepsilon_t^2 \; 0 \; \cdots \; 0)', \quad \mathbf{t} \in \mathbf{Z}$$

is also strong stationary, with the mean

$$E(\mathbf{U}_t) = (\mathbf{I} - \mathbf{M})^{-1} E(\mathbf{W}_t) = a_c \, \sigma^2 \cdot \left(1 - \sum_{j=1}^p \alpha_j\right)^{-1} \cdot \mathbf{1}_{p\times 1}. \tag{3.4}$$

As the components $u_t^{(j)}$, $j = 1, \dots, p$ of the sequence $(\mathbf{U}_t)$ satisfy recurrent relations

$$u_t^{(1)} = \sum_{j=1}^{p} \alpha_j u_{t-1}^{(j)} + q_t\, \varepsilon_t^2, \qquad u_t^{(j)} = u_{t-1}^{(j-1)}, \quad j = 2, \dots, p,$$

according to the assumption of absolutely continuous distribution of ε_t, it is obvious that for any $j = 1, \dots, p$ and $t \in \mathbf{Z}$ values $u_t^{(j)}$ are strictly positive. Then, according to (3.4) and assumption $a_c > 0$, we have $E\big(u_t^{(j)}\big) > 0 \Leftrightarrow 1 - \sum_{i=1}^{p} \alpha_j > 0$ and (iii) follows.

(iii) $\Rightarrow$ (i). Let $\mathcal{S}_r(\mathbf{M}) = \max_j \{\lambda_j\}$ be the spectral radius of the matrix $\mathbf{M}$. Then $\mathcal{S}_r(\mathbf{M}) \leqslant \|\mathbf{M}\|_\infty$ it is valid (see e.g., Milovanović [**7**]) wherefrom we get $\|\mathbf{M}\|_\infty = \max\big\{ \sum_{j=1}^{p} \alpha_j,\, 1\big\} = 1$. If we suppose that $\mathcal{S}_r(\mathbf{M}) = 1$, then for some $\varphi \in [0, 2\pi)$ there exists an eigenvalue $\lambda' = e^{i\varphi}$ satisfying

$$P(\lambda') = e^{ip\varphi} - \sum_{j=1}^{p} \alpha_j\, e^{i(p-j)\varphi} = 0.$$

After that, the inequality $\big|\, e^{ip\varphi} \big| \leqslant \sum_{j=1}^{p} \alpha_j \big|\, e^{i(p-j)\varphi} \big|$ implies $\sum_{j=1}^{p} \alpha_j \geqslant 1$, which contradicts (iii). So, $\mathcal{S}_r(\mathbf{M}) < 1$ and according to the above, it is equivalent to (i). □

Corollary 3.1. *Let the GSB model be defined by* (2.2) *and* (2.5). *The sequence* (y_t) *is strong stationary iff the sequence* $(\mathbf{Y}_t)$ *satisfies the conditions of Theorem* 3.1. *In this case, it holds* $E(y_t) = 0$ *and the covariance function* $\gamma_{Y}(h) = E\big(y_{t+h}\, y_t\big)$, $h \geqslant 0$ *satisfies the recurrent relation*

$$\gamma_{Y}(h) - \sum_{j=1}^{p} \alpha_j \big[\gamma_{Y}(h-j) - s(h-j) I(h-j > 0)\big] = \begin{cases} \sigma^2, & h = 0 \\ 0, & h \neq 0 \end{cases} \tag{3.5}$$

where $s(h)$, $h \geqslant 0$ *is a solution of the difference equation*

$$s(h) - \sum_{j=1}^{p} \alpha_j\, s(h-j) = 0, \quad h \geqslant p$$

with the initial conditions $s(0) = b_c \sigma^2$, $s(k) = \sum_{j=1}^{k} \alpha_j\, s(k-j)$, $k = 1, \dots, p-1$.

Proof. Firstly, we define the sequences $u_t = y_t - \theta_t \varepsilon_t$, $v_t = q_t \varepsilon_t$, $t \in \mathbf{Z}$ so that, according to (2.5),

$$u_t = \sum_{j=1}^{p} \alpha_j u_{t-j} + v_t. \tag{3.6}$$

Therefore, (u_t) is the linear autoregressive sequence with noise (v_t) and it is obvious that it is the stationary sequence iff the sequence of vectors $(\mathbf{Y}_t)$, described in Theorem (3.1), is stationary, too. In this case, it holds $E(u_t) = 0$ and the covariance function $\gamma_{_U}(h)$ of the sequence (u_t) satisfies the recurrent relation

$$\gamma_{_U}(h) - \sum_{j=1}^{p} \alpha_j \, \gamma_{_U}(h-j) = \begin{cases} a_c\sigma^2, & h = 0 \\ 0, & h \neq 0, \end{cases} \tag{3.7}$$

where $\gamma_{_U}(-h) = \gamma_{_U}(h)$. We have that

$$E(u_t) = E(y_t) - E(\theta_t\varepsilon_t) = 0,$$
$$\gamma_{_U}(h) = E(u_t u_{t-h}) = \gamma_{_Y}(h) - s(h),$$

where $s(h) = E(\theta_{t-h}\varepsilon_{t-h}\, u_t) + b_c\sigma^2 I(h=0)$, $h \geqslant 0$. Substituting the last equality in (3.7) we obtain equation (3.5). □

The proposition above shows, at the same time, the similarity and the distinction of correlations between (3.5) and (3.7). Further, this proposition demonstrates how familiar are the stationary GSB process (y_t) and the corresponding linear model. On the other hand, similarly to the basic STOPBREAK process, equality (2.5) makes it possible to presentthe sequence (y_t) in the form of additive decomposition

$$y_t = m_t + \varepsilon_t, \tag{3.8}$$

where

$$m_t = \sum_{j=1}^{p} \alpha_j \big(y_{t-j} - \theta_{t-j}\varepsilon_{t-j}\big) = \sum_{j=1}^{p} \alpha_j \big(m_{t-j} + q_{t-j}\varepsilon_{t-j}\big) \tag{3.9}$$

is the sequence of random variables which we named *the martingale means*. In this way, the previous equality is a generalization of an analogous equality in [**2**], which can be obtained according to (3.9), for $p = \alpha_1 = 1$. Regarding this general form, the basic properties of the sequence (m_t), stationarity in particular, can be described by the following proposition which is a result of the previous analysis.

Corollary 3.2. *The sequence (m_t) is stationary iff the roots λ_j, $j = 1, \dots, p$ of the characteristic polynomial $P(\lambda)$ satisfy the condition $|\lambda_j| < 1$, or equivalently $\sum_{j=1}^{p} \alpha_j < 1$. Then, $E(m_t) = 0$ and the covariance function $\gamma_m(h) = E(m_t m_{t+h})$ satisfy the difference equation*

$$\gamma_m(h) = \sum_{j=1}^{p} \alpha_j \big[\gamma_m(h-j) + \widetilde{s}(j-h-1) I(j-h-1 \geqslant 0)\big],$$

where $\tilde{s}(h)$, $h \geqslant 0$ is a solution of the equation

$$\tilde{s}(h) - \sum_{j=1}^{p} \alpha_j \, \tilde{s}(h-j) = 0, \quad h \geqslant p$$

with the initial conditions

$$\tilde{s}(0) = \alpha_1 a_c \sigma^2, \qquad \tilde{s}(k) = \sum_{j=1}^{k} \alpha_j \, \tilde{s}(k-j) + \alpha_{k+1} a_c \sigma^2, \quad k = 1, \dots, p-1.$$

Proof. Using the same notations as in the previous proposition, we have

$$u_t = m_t + q_t\varepsilon_t, \quad v_t = q_t\varepsilon_t, \quad t \in \mathbf{Z}.$$

According to the same recurrent relation as (3.6), the whole procedure is analogue to Corollary 3.1. □

Remark 3.1. According to (3.8), regardless the stationarity conditions are fulfilled, the sequences (y_t) and (m_t) are connected as follows

$$E\big(y_t \mid \mathcal{F}_{t-1}\big) = m_t + E\big(\varepsilon_t \mid \mathcal{F}_{t-1}\big) = m_t. \tag{3.10}$$

From here, we have $E(y_t) = E(m_t) = \mu(\text{const})$, $t \in \mathbf{Z}$, i.e., the means of these two sequences are (always) equal and constant. The variance of GSB process can be determined in a similar way. As

$$\operatorname{Var}\big(y_t \mid \mathcal{F}_{t-1}\big) = E\big(y_t^2 \mid \mathcal{F}_{t-1}\big) - m_t^2 = \sigma^2, \tag{3.11}$$

we can conclude that the conditional variance of the sequence (y_t) is a constant and it is equal to the variance of the noise (ε_t). According to (3.11), the variances of the sequences (m_t) and (y_t) satisfy $\operatorname{Var}(y_t) = \operatorname{Var}(m_t) + \sigma^2$. Let us remark that (3.10) and (3.11) explain the stochastic nature of (y_t). The sequence (m_t) is predictable, and it will be a component which demonstrates the stability of the process (y_t). Opposite to this, the sequence (ε_t) is a factor which represents the deviations (or random fluctuations) from values (m_t).

4. Analysis of increments. The general Split-MA process

In this section we shall describe in details the stochastic structure of increments

$$X_t \stackrel{\text{def}}{=} A(L)y_t, \quad t \in \mathbf{Z}$$

which (according to (2.4)–(2.5)) can be written in the form of recurrent relation

$$X_t = \varepsilon_t - \sum_{j=1}^{p} \alpha_j\, \theta_{t-j}\varepsilon_{t-j}, \quad t \in \mathbf{Z}. \tag{4.1}$$

Obviously, the sequence (X_t) has the multi-regime structure, which depends on the realizations of indicators (θ_t). If all fluctuations of the white noise in time $t-j$ are large, an increment X_t will be equal to the white noise. On the other hand, the fluctuations of the white noise which do not exceed the critical value c will produce a "part of" MA(p) representation of (X_t). In this sense, the similarity of this model to the standard linear MA model is noticeable, and the sequence (X_t) we shall call *the general Split-MA model* (*of order* p), or simply *Split-MA*(p) *model*. It represents the generalization of the adequate model defined in Stojanović, Popović and Popović [**9**] and the threshold integrated moving average (TIMA) model introduced in Gonzalo and Martinez [**5**]. The main properties of this process can be expressed in the following way.

THEOREM 4.1. *The sequence* (X_t)*, defined by* (4.1)*, is stationary, meaning* $E(X_t) = 0$ *and covariance* $\gamma_X(h) = E(X_t X_{t+h})$*,* $h \geqslant 0$ *which satisfies the equality*

$$\gamma_X(h) = \begin{cases} \sigma^2\big(1 + b_c \sum_{j=1}^{p} \alpha_j^2\big), & h = 0 \\ \sigma^2\, b_c\big(\sum_{j=1}^{p-h} \alpha_j\, \alpha_{j+h} - \alpha_h\big), & 1 \leqslant h \leqslant p-1 \\ -\sigma^2\, b_c \alpha_p, & h = p \\ 0, & h > p. \end{cases}$$

PROOF. Elementary. □

Similarly to the basic STOPBREAK model, we can show (under some conditions) the invertibility of increments (X_t). This property is analyzed from different aspects by many authors who researched the STOBREAK models (see, e.g., [**2, 4**]). In order to give necessary and sufficient invertibility conditions of Split-MA process, we give the explicit stochastic representation of an invertible process.

THEOREM 4.2. *The sequence* (X_t)*, defined by* (4.1)*, is invertible iff the roots* $r_1, \dots, r_p$ *of the characteristic polynomial* $Q(\lambda) = \lambda^p - b_c \sum_{j=1}^{p} \alpha_j \lambda^{p-j}$ *satisfy* $|r_j| < 1$*,* $j = 1, \dots, p$ *or, equivalently,* $b_c \sum_{j=1}^{p} \alpha_j < 1$*. Then,*

$$\varepsilon_t = \sum_{k=0}^{\infty} \omega_k(t)\, X_{t-k}, \quad t \in \mathbf{Z}, \tag{4.2}$$

where $\omega_k(t)$ *is a solution of the stochastic difference equation*

$$\omega_k(t) = \theta_{t-k} \sum_{j=1}^{p} \alpha_j\, \omega_{k-j}(t), \quad k \geqslant p, \quad t \in \mathbf{Z}, \tag{4.3}$$

with the initial conditions $\omega_0(t) = 1$*,* $\omega_k(t) = \theta_{t-k} \sum_{j=1}^{k} \alpha_j \omega_{k-j}(t)$*,* $1 \leqslant k \leqslant p-1$*. Moreover, representation* (4.2) *is almost sure unique and the sum converges with probability one and in mean-square.*

PROOF. We shall use the similar procedure as in the analysis of the stationarity conditions of the GSB process. First of all, for any $t \in \mathbf{Z}$, we define the vectors and matrices

$$\mathbf{E}_t = \begin{pmatrix}\varepsilon_t & \varepsilon_{t-1} & \cdots & \varepsilon_{t-p+1}\end{pmatrix}', \qquad \mathbf{X}_t = \begin{pmatrix}X_t & 0 & \cdots & 0\end{pmatrix}'$$

$$\mathbf{A}_t = \begin{pmatrix} \alpha_1\, \theta_t & \alpha_2\, \theta_{t-1} & \cdots & \alpha_{p-1}\, \theta_{t-p+2} & \alpha_p\, \theta_{t-p+1} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

in order to write model (4.1) in the form of a stochastic difference equation of order one

$$\mathbf{E}_t = \mathbf{A}_{t-1}\mathbf{E}_{t-1} + \mathbf{X}_t, \qquad t \in \mathbf{Z}. \tag{4.4}$$

From here, we have

$$\mathbf{E}_t = \mathbf{X}_t + \sum_{j=1}^{k} \big(\mathbf{A}_{t-1}\cdots\mathbf{A}_{t-j}\big)\mathbf{X}_{t-j} + \Bigg(\prod_{j=1}^{k+1}\mathbf{A}_{t-j}\Bigg)\mathbf{E}_{t-k-1},$$

where $k = 1, 2, \ldots$ It can be proven (see, e.g., [**3**]) that the existence of an almost sure unique, stationary solution of equation (4.4), in the form

$$\mathbf{E}_t = \mathbf{X}_t + \sum_{k=1}^{\infty} \big(\mathbf{A}_{t-1}\cdots\mathbf{A}_{t-k}\big)\mathbf{X}_{t-k}, \qquad t \in \mathbf{Z} \tag{4.5}$$

is equivalent to the convergence $\prod_{j=1}^{k+1}\mathbf{A}_{t-j} \xrightarrow{\text{a.s.}} 0$, $k\to\infty$, i.e., to the fact that the eigenvalues r_j, $1 = 1, \ldots, p$ of the matrix

$$\mathbf{A} = E(\mathbf{A}_t) = \begin{pmatrix} \alpha_1\, b_c & \alpha_2\, b_c & \cdots & \alpha_{p-1}\, b_c & \alpha_p\, b_c \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

satisfy $|r_j| < 1,\ j = 1, \ldots, p\,$. According to the representation

$$\det\big(\mathbf{A} - \lambda\mathbf{I}\big) = (-1)^p\, Q(\lambda)$$

it is obvious that the eigenvalues r_j, $j = 1, \ldots, p$ are the roots of the characteristic polynomial $Q(\lambda)$. Then, the condition $|r_j| < 1$, $j = 1, \ldots, p$ is necessary and sufficient for the almost sure uniqueness of representation (4.5), and the almost sure convergence of the appropriate sum. In a similar manner, we can prove that the same conditions are equivalent to the mean-square convergence of the sum in (4.5). From this point on, by simple computation, we can obtain equations (4.2) and (4.3). $\square$

At the end of this section, we shall give another method for determining necessary and sufficient conditions for stationarity of the GSB process and necessary and sufficient conditions for invertibility of its increments. In order to do that, let us define the following process.

DEFINITION 4.1. Let (Z_t), $t \in \mathbf{Z}$ be some $\mathcal{F}_t$ adaptive time series. *The permanent effect of observation* of this time series is the time series $(\mathcal{P}_e(Z_t))$ such that

$$\mathcal{P}_e(Z_t) \overset{\text{def}}{=} \lim_{k\to\infty} \frac{\partial E(Z_{t+k} \mid \mathcal{F}_t)}{\partial Z_t}, \qquad t \in \mathbf{Z},$$

where the last convergence is almost sure.

Now we can state and prove the following theorem.

THEOREM 4.3. *Let (y_t) be the GSB process defined by* (2.2) *and* (2.5)*, where $a_c = E(q_t) \in (0,1)$ and (X_t) is the sequence of increments of the GSB process defined by* (4.1)*. Then:*

(i) *The sequence (y_t) is strictly stationary iff $\mathcal{P}_e(y_t) \overset{a.s.}{=} 0$;*
(ii) *The sequence (X_t) is invertible iff $\mathcal{P}_e(\varepsilon_t) \overset{a.s.}{=} 0$.*

PROOF. (i) Consider the vector time series $(\mathbf{Y}_t)$ defined by equality (3.1). For an arbitrary $t \in \mathbf{Z}$ and $k \geqslant 0$ we have $\mathbf{Y}_{t+k} = \sum_{j=0}^{k-1} \mathbf{M}^j \mathbf{V}_{t+k-j} + \mathbf{M}^k \mathbf{Y}_t$. This implies $E(\mathbf{Y}_{t+k} \mid \mathcal{F}_t) = \mathbf{M}^k \mathbf{Y}_t$, that is $E(y_{t+k} \mid \mathcal{F}_t) = c_k(t)(y_t - \theta_t \varepsilon_t)$, where

$$c_k(t) = \begin{cases} 1, & k = 0; \\ \sum_{j=1}^{k} \alpha_j \, c_{k-j}(t), & 1 \leqslant k \leqslant p-1; \\ \sum_{j=1}^{p} \alpha_j \, c_{k-j}(t), & k \geqslant p. \end{cases}$$

Then, the permanent effect of observation of time series (y_t) is

$$\mathcal{P}_e(y_t) = \left(1 - \theta_t \frac{\partial \varepsilon_t}{\partial y_t}\right) \lim_{k \to \infty} c_k(t) = q_t \lim_{k \to \infty} c_k(t).$$

The last equality is valid because of (3.8), i.e., because of the fact that $\frac{\partial \varepsilon_t}{\partial y_t} = 1$. Finally, because of $a_c = P\{q_t = 1\} > 0$, we have $\mathcal{P}_e(y_t) \overset{\text{a.s.}}{=} 0$ iff $\lim_{k\to\infty} c_k(t) = 0$. This is, according to Theorem 3.1, equivalent to the fact that (y_t) is stationary.

(ii) The proving procedure is analogous to (i), where definition (4.1) of the sequence of increments (X_t) and Theorem 4.2 should be used. □

REMARK 4.1. By using the same permanent effect of observation process, Engle and Smith [**2**] proved somewhat different (necessary and sufficient) conditions for the invertibility of the sequence of increments (X_t). Meanwhile, they considered only the simplest STOPBREAK model, the model of order $p = 1$. So, Theorem 4.3 is more general than the mentioned result.

5. Conclusion

According to Theorems 3.1 and 4.2, it is clear that the presence of the sequence (θ_t) enables that the conditions of invertibility of increments (X_t) are weaker than the appropriate conditions related to the stationarity of the series (Y_t) and (m_t). In this way, even the nonstationary time series (y_t) and (m_t) can form invertible Split-MA process which is always stationary. This situation is particularly interesting in the case of the so-called *integrated* (*standardized*) time series, where

$$\sum_{j=1}^{p} \alpha_j = 1. \tag{5.1}$$

If the value of the parameter b_c is nontrivial (i.e., $b_c \in (0,1)$), then the sequence (X_t) will be invertible, although (y_t) and (m_t) are nonstationary time series. Therefore, “normality condition" (5.1) allows us that these two series have nonzero means, which is particularly important in applications.

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