PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 95 (109) (2014), 161–171 DOI: 10.2298/PIM1409161T

ON THE CONVERSE THEOREM OF APPROXIMATION IN VARIOUS METRICS FOR NONPERIODIC FUNCTIONS

Miloš Tomić

Communicated by Gradimir Milovanović

Abstract. The modulus of smoothness in the norm of space L_q of nonperiodic functions of several variables is estimated by best approximations by entire functions of exponential type in the metric of space L_p, $1 \leqslant p \leqslant q < \infty$.

1. Introduction and preliminaries

A converse theorem of approximation in various metrics for 2π periodic functions of several variables was proved in [**5**]. We prove the theorem of representation for the derivative of a function, and then the analogous converse theorem for nonperiodic functions defined on the space R^n. In this way we generalize and improve the results from [**4**, 6.4].

As usually we say that $f(x_1, \dots, x_n) \in L_p(R^n)$, $1 \leqslant p < \infty$ if

$$\|f\|_p = \left(\int_{R^n} |f|^p dx_1 \dots dx_n \right)^{1/p} = \left(\int_{R^n} |f|^p dx \right)^{1/p} < \infty, \quad x = (x_1, x_2, \dots, x_n)$$

The notions of the best approximation and of the modulus of smoothness are given in [**2**] and [**4**].

Let $g_\nu = g_{\nu_1 \dots \nu_n}(x_1, \dots, x_n)$, $\nu = (\nu_1, \dots, \nu_n)$, $(g_\nu \in L_p)$ be an entire function of exponential type ν_i with respect to the variable x_i $(i = 1, 2, \dots, n)$. The best approximation $E_{\nu_1,\dots,\nu_n}(f)_p$ of a function $f \in L_p(R^n)$ by entire functions of exponential type is the quantity

$$E_{\nu_1,\dots,\nu_n}(f)_p = \inf_{g_\nu} \|f - g_{\nu_1 \dots \nu_n}\|_p$$

2010 *Mathematics Subject Classification:* Primary 42B99.
Key words and phrases: modulus of smoothness, the converse theorem of approximation.

The modulus of smoothness of order k of a function f with respect to the variable x_i is

$$\omega_k(f;\delta_i)_p = \omega_k(f;0,\dots,0,\delta_i,0,\dots,0)_p = \sup_{|h_i|\leqslant\delta_i} \|\Delta^k_{h_i} f\|_p$$

where

$$\|\Delta^k_{h_i} f = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(x_1,\dots,x_{i-1},x_i+jh_i,x_{i+1},\dots,x_n).$$

The derivative of a function f is denoted by

$$f^{(\nu_1,\dots,\nu_n)} = \frac{\partial^{r_1+\dots+r_n} f}{\partial x_1^{r_1}\dots\partial x_n^{r_n}}$$

Lemma 1.1. *If $A_i \downarrow 0$ as $i\to\infty$, then for $\lambda = 1,2,\dots$ and $s \geqslant 1$ the following inequalities hold*

$$2^{(\lambda-1)s} A_{2^\lambda} \leqslant \sum_{i=2^{\lambda-1}+1}^{2^\lambda} i^{s-1} A_i \tag{1.1}$$

$$2^{(\lambda+1)s} A_{2^\lambda} \leqslant 2^{2s} \sum_{i=2^{\lambda-1}+1}^{2^\lambda} i^{s-1} A_i \tag{1.2}$$

Proof. We have

$$\sum_{i=2^{\lambda-1}+1}^{2^\lambda} i^{s-1} = (2^{\lambda-1}+1)^{s-1} + \dots + (2^\lambda)^{s-1} \geqslant (2^{\lambda-1}+1)^{s-1}\cdot 2^{\lambda-1} \geqslant (2^{\lambda-1})^s.$$

Therefore

$$2^{(\lambda-1)s} \leqslant \sum_{i=2^{\lambda-1}+1}^{2^\lambda} i^{s-1}. \tag{1.3}$$

Since the sequence A_i is monotonic, (1.1) follows from (1.3). Multiplying inequality (1.1) by 2^{2s}, we get inequality (1.2). $\square$

Lemma 1.2. *If $A_i \downarrow 0$ as $i\to\infty$, and $s\geqslant 1$, then the following inequality holds*

$$\sum_{i=2^{m-1}+1}^{2^m} i^{s-1} A_i \leqslant 2^{2s-1} \sum_{i=2^{m-2}+1}^{2^{m-1}} i^{s-1} A_i, \quad m = 2,3,\dots \tag{1.4}$$

Proof. The following inequalities hold because the sequence A_i is monotonic

$$\sum_{i=2^{m-1}+1}^{2^m} i^{s-1} A_i \leqslant A_{2^{m-1}} \sum_{i=2^{m-1}+1}^{2^m} i^{s-1}, \tag{1.5}$$

$$\sum_{i=2^{m-2}+1}^{2^{m-1}} i^{s-1} A_i \geqslant A_{2^{m-1}} \sum_{i=2^{m-2}+1}^{2^{m-1}} i^{s-1}, \tag{1.6}$$

We have

$$\sum_{i=2^{m-1}+1}^{2^m} i^{s-1} \leqslant (2^m)^{s-1} \cdot 2^{m-1},$$

$$\sum_{i=2^{m-2}+1}^{2^{m-1}} i^{s-1} \geqslant (2^{m-2})^{s-1} \cdot 2^{m-2} = 2^{1-2s} \cdot (2^m)^{s-1} \cdot 2^{m-1}.$$

From the above two inequalities it follows

$$\sum_{i=2^{m-1}+1}^{2^m} i^{s-1} \leqslant 2^{2s-1} \sum_{i=2^{m-2}+1}^{2^{m-1}} i^{s-1}. \tag{1.7}$$

Multiplying (1.7) by $A_{2^{m-1}}$ and in view of (1.5) and (1.6), we get (1.2). □

REMARK 1.1. Lemmas 1.1 and 1.2 are valid for $0 < s < 1$ also, with different constants $C = C(s)$. So inequality (1.1) becomes

$$2^{(\lambda-1)s} A_{2^\lambda} \leqslant 2^{s-1} \sum_{i=2^{\lambda-1}+1}^{2^\lambda} i^{s-1} A_i \quad (0 < s < 1).$$

2. Theorem of representation

Let $g_\nu = g_{\nu_1 \dots \nu_n}(x_1, \dots, x_n)$, $\nu = (\nu_1, \dots, \nu_n)$, be an entire L_p function of exponential type ν_i with respect to the variable x_i $(i = 1, 2, \dots, n)$, by which the best approximation $E_{\nu_1,\dots,\nu_n}(f)_p$ is achieved, i.e., let

$$E_{\nu_1,\dots,\nu_n}(f)_p = \|f - g_{\nu_1 \dots \nu_n}\|_p. \tag{2.1}$$

From these entire functions $g_{\nu_1 \dots \nu_n}(x_1, \dots, x_n)$ we create entire functions

$$\xi_\lambda = g_{2^{(\lambda+1)l_1} \dots 2^{\lambda+1} \dots 2^{(\lambda+1)l_n}} - g_{2^{\lambda l_1} \dots 2^\lambda \dots 2^{\lambda l_n}}, \quad \lambda = 0, 1, 2, \dots \tag{2.2}$$

for given natural numbers l_j $(j = 1, 2, \dots, n)$ where $l_i = 1$ for a chosen number $i \in \{1, 2, \dots, n\}$. The function ξ_λ is entire of exponential type $2^{(\lambda+1)l_j}$ with respect to x_j.

THEOREM 2.1. *Let $f \in L_p(R^n)$ and r_j be nonnegative integers, and l_j $(j = 1, \dots, n)$ be natural numbers, where $l_i = 1$ for some $i \in \{1, 2, \dots, n\}$. If the following inequality holds for the best approximation of the function*

$$\sum_{\lambda=1}^{\infty} \lambda^{q\sigma-1} E_{\lambda^{l_1} \dots \lambda \dots \lambda^{l_n}}(f)_p < \infty, \tag{2.3}$$

where

$$\sigma = \sum_{j=1}^{n} l_j \Big(r_j + \frac{1}{p} - \frac{1}{q} \Big), \quad 1 \leqslant p \leqslant q < \infty, \tag{2.4}$$

then the function f has a derivative $f^{(r_1\dots r_n)}$ belonging to L_q and in the sense of L_q the equality

$$f^{(r_1,\dots,r_n)} \overset{(q)}{=} g_{1\dots1}^{(\nu_1\dots\nu_n)} + \sum_{\lambda=0}^{\infty} \xi_\lambda^{r_1,\dots,r_n} \tag{2.5}$$

holds.

PROOF. For the sum

$$G_m = g_{1\dots1} + \sum_{\lambda=0}^{m} \xi_\lambda, \quad m = 0, 1, 2\dots \tag{2.6}$$

the equality

$$G_m = g_{2^{(m+1)l_1}\dots2^{m+1}\dots2^{(m+1)l_n}} \tag{2.7}$$

holds. In view of (2.1) and (2.7) we conclude that

$$\|f - G_m\|_p = E_{2^{(m+1)l_1}\dots2^{m+1}\dots2^{(m+1)l_n}}(f)_p$$

hence, it follows that

$$\|f - G_m\|_p \to 0 \text{ as } m \to \infty. \tag{2.8}$$

This means that the equality

$$f \overset{(p)}{=} g_{1\dots1} + \sum_{\lambda=0}^{\infty} \xi_\lambda \tag{2.9}$$

holds in L_p.

In the next step we prove (2.9) holds in L_q. For ξ_λ we have

$$\|\xi_\lambda\|_p \leqslant 2E_{2^{\lambda l_1}\dots2^{\lambda}\dots2^{\lambda l_n}}(f)_p. \tag{2.10}$$

Applying the inequality of various metrics of Nikolsky [**2**, 3.3.5] we obtain

$$\|\xi_\lambda\|_q \leqslant 2^n \bigg(\prod_{j=1}^{n} 2^{(\lambda+1)l_j}\bigg)^{1/p-1/q} \|\xi_\lambda\|_p$$

hence, in view of (2.10), it follows

$$\|\xi_\lambda\|_q \ll 2^n \bigg(\prod_{j=1}^{n} 2^{(\lambda+1)l_j}\bigg)^{1/p-1/q} E_{2^{\lambda l_1}\dots2^{\lambda}\dots2^{\lambda l_n}}(f)_p. \tag{2.11}$$

We will estimate the sum

$$G_t - G_m = \sum_{\lambda=m+1}^{t} \xi_\lambda, \quad m < t, \tag{2.12}$$

in the norm L_q. With the aim of estimating the quantity $A = \|G_t - G_m\|_q^q$ we will apply a method which has been used in several papers. For example, the method was applied in [**3**] and [**1**] (see the estimate of A in Lemma 1). The method was also applied in [**6**] to estimate quantity A from (2.6) to (2.45). Therefore, taking into account (2.11), from (2.12), we get

$$\|G_t - G_m\|_q \ll \left\{ \sum_{\lambda=m+1}^{t} \exp_2\left(\lambda q\Big(\frac{1}{p} - \frac{1}{q}\Big) \sum_{j=1}^{n} l_j\right) E^q_{2^{\lambda l_1}\dots 2^{\lambda}\dots 2^{\lambda l_n}}(f)_p \right\}^{1/q}. \tag{2.13}$$

Following the proof in [**6**] and starting from equality (2.12), we will now prove inequality (2.13). Denote

$$A = \|G_t - G_m\|_q^q = \Big\| \sum_{\lambda=m+1}^{t} \xi_\lambda \Big\|_q^q, \quad m < t. \tag{2.14}$$

For a given number q denote $[q] + 1 = k$. This means that $k \in \{2, 3, \dots\}$ and that $q/k < 1$. From (2.14) it follows that
(2.15)

$$A = \int \Big| \sum_{\lambda=m+1}^{t} \xi_\lambda \Big|^q dx = \int \Big| \sum_{\lambda=m+1}^{t} \xi_\lambda \Big|^{\frac{q}{k}k} dx \leqslant \int \Big(\sum_{\lambda=m+1}^{t} |\xi_\lambda|^{\frac{q}{k}} \Big)^k dx, \qquad \int = \int_{R^n}.$$

Denote

$$\delta_\lambda = |\xi_\lambda|^{q/k}. \tag{2.16}$$

We get

$$A \leqslant \int \Big(\sum_{\lambda=m+1}^{t} \delta_\lambda \Big)^k dx. \tag{2.17}$$

As $k = k(q)$ is an integer, then

$$\Big(\sum_{\lambda=m+1}^{t} \delta_\lambda \Big)^k \sum_{\lambda_1=m+1}^{t} \cdots \sum_{\lambda_k=m+1}^{t} \prod_{j=1}^{k} \delta_{\lambda_j}. \tag{2.18}$$

Now from (2.17), based on (2.18), we get

$$A \leqslant \sum_{\lambda_1=m+1}^{t} \cdots \sum_{\lambda_k=m+1}^{t} \int \prod_{j=1}^{k} \delta_{\lambda_j} dx. \tag{2.19}$$

Using the equality

$$\prod_{j=1}^{k} D_j = \Big(\prod_{r,s=1,\, r<s}^{k} D_r D_s \Big)^{1/(k-1)} \tag{2.20}$$

for $D_j = \delta_{\lambda_j}$ from (2.19) we obtain

$$A \leqslant \sum_{\lambda_1=m+1}^{t} \cdots \sum_{\lambda_k=m+1}^{t} \int \Big(\prod_{r,s=1,\, r<s}^{k} \delta_{\lambda_r} \delta_{\lambda_s} \Big)^{1/(k-1)} dx. \tag{2.21}$$

Applying Hölder's integral inequality to a product of $\frac{1}{2}k(k-1)$ factors, from (2.21) we get that

$$A \leqslant \sum_{\lambda_1=m+1}^{t} \cdots \sum_{\lambda_k=m+1}^{t} \prod_{r,s=1,\, r<s}^{k} \left[\int (\delta_{\lambda_r}\delta_{\lambda_s})^{k/2}dx\right]^{2/k(k-1)}. \tag{2.22}$$

Based on (2.16) we get

$$\Gamma_{rs} = \int (\delta_{\lambda_r}\delta_{\lambda_s})^{k/2}dx = \int \left(\left|\xi_{\lambda_r}\right|^{q/2}\left|\xi_{\lambda_s}\right|^{q/2}\right)dx. \tag{2.23}$$

For $\alpha = \frac{p+q}{p}$, $\alpha' = \frac{p+q}{q}$, we have $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$. Therefore by applying Hölder's inequality, we get

$$\Gamma_{rs} \leqslant \left(\left\|\xi_{\lambda_r}\right\|_{q\alpha/2}\right)^{q/2}\left(\left\|\xi_{\lambda_s}\right\|_{q\alpha'/2}\right)^{q/2}. \tag{2.24}$$

The function ξ_λ is entire of exponential type $2^{(\lambda+1)l_j}$ with respect to x_j, $j = 1, 2, \ldots, n$. Therefore applying the inequality of Nikolsky [**2**, 3.3.5] we get

$$\left(\left\|\xi_{\lambda_r}\right\|_{q\alpha/2}\right)^{q/2} \ll \left(\left\|\xi_{\lambda_r}\right\|_{p}\right)^{q/2} \exp_2\left(\left(\sum_{j=1}^{n}\lambda_r l_j\right)\left(\frac{q}{2p} - \frac{1}{\alpha}\right)\right). \tag{2.25}$$

$$\left(\left\|\xi_{\lambda_s}\right\|_{q\alpha'/2}\right)^{q/2} \ll \left(\left\|\xi_{\lambda_s}\right\|_{p}\right)^{q/2} \exp_2\left(\left(\sum_{j=1}^{n}\lambda_s l_j\right)\left(\frac{q}{2p} - \frac{1}{\alpha'}\right)\right). \tag{2.26}$$

Using the equality

$$\frac{q}{2p} - \frac{1}{\beta} = \frac{q}{2}\left(\frac{1}{p} - \frac{1}{q}\right) + \frac{1}{2} - \frac{1}{\beta}, \quad \beta \in \{\alpha, \alpha'\}, \tag{2.27}$$

from (2.24), based on (2.25), (2.26) and (2.10), we get

$$\begin{aligned} \Gamma_{rs} &\ll \exp_2\left(\left[\lambda_r\left(\frac{1}{2} - \frac{1}{\alpha}\right) + \lambda_s\left(\frac{1}{2} - \frac{1}{\alpha}'\right)\right]\sum_{j=1}^{n} l_j\right) \\ &\times\left\{\exp_2\left(\left[(\lambda_r+\lambda_s)q\left(\frac{1}{p}-\frac{1}{q}\right)\right]\sum_{j=1}^{n} l_j\right)E^q_{2^{\lambda_r l_1}\ldots 2^{\lambda_r}\ldots 2^{\lambda_r l_n}}(f)_p E^q_{2^{\lambda_s l_1}\ldots 2^{\lambda_s}\ldots 2^{\lambda_s l_n}}(f)_p\right\}^{1/2}. \end{aligned} \tag{2.28}$$

Denote

$$H_i = \exp_2\left(iq\left(\frac{1}{p} - \frac{1}{q}\right)\sum_{j=1}^{n} l_j\right)E^q_{2^{il_1}\ldots 2^{i}\ldots 2^{il_n}}(f)_p. \tag{2.29}$$

Then

$$\Gamma_{rs} \ll \exp_2\left(\left[\lambda_r\left(\frac{1}{2} - \frac{1}{\alpha}\right) + \lambda_s\left(\frac{1}{2} - \frac{1}{\alpha'}\right)\right]\sum_{j=1}^{n} l_j\right)H^{1/2}_{\lambda_r}H^{1/2}_{\lambda_s}. \tag{2.30}$$

Since $\frac{1}{\alpha'} = 1 - \frac{1}{\alpha}$, it holds that

$$\lambda_r\left(\frac{1}{2} - \frac{1}{\alpha}\right) + \lambda_s\left(\frac{1}{2} - \frac{1}{\alpha'}\right) = -(\lambda_s - \lambda_r)\left(\frac{1}{2} - \frac{1}{\alpha}\right).$$

Therefore from (2.30) it follows

$$\Gamma_{rs} \ll \exp_2\Big(-(\lambda_s-\lambda_r)\Big(\frac{1}{2}-\frac{1}{\alpha}\Big)\sum_{j=1}^{n} l_j\Big) H_{\lambda_r}^{1/2} H_{\lambda_s}^{1/2}. \tag{2.31}$$

If we apply Hölder's inequality so that α' relates to the first factor, and α to the second one, then in the same way we conclude that

$$\Gamma_{rs} \ll \exp_2\Big(-(\lambda_r-\lambda_s)\Big(\frac{1}{2}-\frac{1}{\alpha}\Big)\sum_{j=1}^{n} l_j\Big) H_{\lambda_r}^{1/2} H_{\lambda_s}^{1/2}. \tag{2.32}$$

Based on (2.31) and (2.32) we conclude that

$$\Gamma_{rs} \ll \exp_2\Big(-|\lambda_r-\lambda_s|\Big(\frac{1}{2}-\frac{1}{\alpha}\Big)\sum_{j=1}^{n} l_j\Big) H_{\lambda_r}^{1/2} H_{\lambda_s}^{1/2}. \tag{2.33}$$

Denote

$$a(\lambda_s,\lambda_r) = \exp_2\Big(-|\lambda_r-\lambda_s|\Big(\frac{1}{2}-\frac{1}{\alpha}\Big)\sum_{j=1}^{n} l_j\Big), \tag{2.34}$$

$$Q = \prod_{r,s=1,\, r<s}^{k} \big\{a(\lambda_s,\lambda_r) H_{\lambda_r}^{1/2} H_{\lambda_s}^{1/2}\big\}^{2/k(k-1)}. \tag{2.35}$$

From (2.22), based on (2.23), (2.33), (2.34) and (2.35), it follows

$$A \leqslant \sum_{\lambda_1=m+1}^{t} \cdots \sum_{\lambda_k=m+1}^{t} Q. \tag{2.36}$$

We will now estimate the product Q. Based on (2.20) it holds that

$$\prod_{r,s=1,\, r<s}^{k} \big\{H_{\lambda_r}^{1/2} H_{\lambda_s}^{1/2}\big\}^{1/(k-1)} = \prod_{j=1}^{k} H_{\lambda_j}^{1/2}$$

and then, using (2.35), we get

$$Q = \prod_{j=1}^{k} H_{\lambda_j}^{1/k} \prod_{r,s=1,\, r<s}^{k} \{a(\lambda_s,\lambda_r)\}^{2/k(k-1)}. \tag{2.37}$$

It holds $a(\lambda_s,\lambda_r) = a(\lambda_r,\lambda_s)$ and $a(\lambda_r,\lambda_r)=1$. Therefore

$$\prod_{r,s=1,\, r<s}^{k} a(\lambda_r,\lambda_s) = \prod_{r=1}^{k}\prod_{s=1}^{k} a^{1/2}(\lambda_r,\lambda_s). \tag{2.38}$$

From (2.37) based on (2.38) it follows

$$Q = \prod_{r=1}^{k} H_{\lambda_r}^{1/k} \Big\{ \prod_{s=1}^{k} [a(\lambda_s,\lambda_r)]^{1/(k-1)} \Big\}^{1/k}. \tag{2.39}$$

Now from (2.36) based on (2.39) we get

$$A \ll \sum_{\lambda_1=m+1}^{t} \cdots \sum_{\lambda_k=m+1}^{t} \prod_{r=1}^{k} H_{\lambda_r}^{1/k} \Big\{ \prod_{s=1}^{k} [a(\lambda_r, \lambda_s)]^{1/(k-1)} \Big\}^{1/k}. \tag{2.40}$$

In the inequality (2.40) the product has k factors

$$L_r = H_{\lambda_r}^{1/k} \Big\{ \prod_{s=1}^{k} [a(\lambda_r, \lambda_s)]^{1/(k-1)} \Big\}^{1/k}$$

with the exponent $1/k$. The sum of these exponents is 1. Therefore we can apply Hölder's inequality and get

$$A \ll \prod_{r=1}^{k} \Big\{ \sum_{\lambda_1=m+1}^{t} \cdots \sum_{\lambda_k=m+1}^{t} H_{\lambda_r} \prod_{s=1}^{k} \big[a(\lambda_r, \lambda_s)\big]^{1/(k-1)} \Big\}^{1/k}. \tag{2.41}$$

Denote

$$M_r = \sum_{\lambda_1=m+1}^{t} \cdots \sum_{\lambda_k=m+1}^{t} H_{\lambda_r} \prod_{s=1}^{k} \big[a(\lambda_r, \lambda_s)\big]^{1/(k-1)}, \quad r = 1, \dots, k. \tag{2.42}$$

Since $\lambda_r = m+1, \dots, t$ for every $r = 1, \dots, k$, then

$$M_1 = M_2 = \cdots = M_k = M. \tag{2.43}$$

We will estimate, for example, $M_1 = M$. Since $a(\lambda_1, \lambda_1) = 1$, then from (2.42) after some calculation we get

(2.44)

$$M = M_1 = \sum_{\lambda_1=m+1}^{t} H_{\lambda_1} \sum_{\lambda_2=m+1}^{t} [a(\lambda_1, \lambda_2)]^{1/(k-1)} \cdots \sum_{\lambda_k=m+1}^{t} [a(\lambda_1, \lambda_k)]^{1/(k-1)}.$$

Based on (2.34) we conclude that

$$\sum_{\lambda_r=m+1}^{t} [a(\lambda_1, \lambda_r)]^{1/(k-1)} \leqslant C(p, q), \quad r = 2, 3, \dots, k. \tag{2.45}$$

Now from (2.44) based on (2.45) it follows

$$M \ll \sum_{\lambda_1=m+1}^{t} H_{\lambda_1}. \tag{2.46}$$

From (2.41), using (2.42), (2.43) and (2.46), we get

$$A \ll \prod_{r=1}^{k} M^{1/k} = M \ll \sum_{i=m+1}^{t} H_i. \tag{2.47}$$

Based on (2.47) and (2.29) we conclude that

$$A \ll \sum_{i=m+1}^{t} \exp_2 \left(iq \Big(\frac{1}{p} - \frac{1}{q} \Big) \sum_{j-1}^{n} l_j \right) E^q_{2^{il_1} \dots 2^{i} \dots 2^{il_n}}(f)_p. \tag{2.48}$$

Finally, from (2.48), based on (2.14), the inequality (2.13) follows. If $r_j = 0$, then $\sigma = \left(\frac{1}{p} - \frac{1}{q}\right) \sum_{j=1}^{n} l_j$, therefore in view of (2.3) and (2.13) we deduce that the sequence $\{G_m\}$ is a Cauchy sequence in the space L_q and therefore it tends to a function f in L_q [**2**, 1.3.9]. Thus, we have

$$f \overset{(q)}{=} g_{1\dots 1} + \sum_{\lambda=0}^{\infty} \xi_\lambda. \tag{2.49}$$

In the next step we prove equality (2.5). To do it we estimate the quantity

$$B = \left\| G_t^{(r_1,\dots,r_n)} - G_m^{(r_1,\dots,r_n)} \right\|_q^q = \left\| \sum_{\lambda=m+1}^{t} \xi_\lambda^{(r_1,\dots,r_n)} \right\|_q^q. \tag{2.50}$$

Applying the inequality of the Bernstein type [**2**, 3.2.2], we get

$$\left\| \xi_\lambda^{(r_1,\dots,r_n)} \right\|_q \leqslant \left(\prod_{j=1}^{n} 2^{l_j r_j} \right) 2^{\lambda(l_1 r_1 + \dots + l_n r_n)} \|\xi_\lambda\|_q$$

hence, in view of (2.11), it follows

$$\|\xi_\lambda^{(r_1,\dots,r_n)}\| \ll 2^{\lambda\sigma} E_{2^{\lambda l_1}\dots 2^{\lambda}\dots 2^{\lambda l_n}}(f)_p. \tag{2.51}$$

Now, using for B the same procedure by which we estimated A, we get (see the estimation of B in [**6**, (2.50)–(2.65)]

$$\left\| G_t^{(r_1,\dots,r_n)} - G_m^{(r_1,\dots,r_n)} \right\|_q \ll \left\{ \sum_{\lambda=m+1}^{t} 2^{\lambda q\sigma} E^q_{2^{\lambda l_1}\dots 2^{\lambda}\dots 2^{\lambda l_n}}(f)_p \right\}^{1/q}. \tag{2.52}$$

In view of condition (2.3) and inequality (2.52) we conclude that the sequence $\{G_m^{(r_1,\dots,r_n)}\}$ is a Cauchy sequence in L_q. If we denote $G_m^{(r_1,\dots,r_n)} \to h$, $m \to \infty$, then we conclude (see [**2**, 4.4.7] or [**4**, 6.3.31]) that $h = f^{(r_1,\dots,r_n)}$. This means that equality (2.5) holds. □

3. The converse theorem of approximation

Now we are going to prove a converse theorem of approximation, analogously to the result in [**5**] and give some consequences.

THEOREM 3.1. *Let the conditions of Theorem* 2.1 *be satisfied (the condition* (2.3) *where* σ *is given by* (2.4)*), and let* k *and* m_i *be given natural numbers. Then the inequality*

$$\begin{aligned} \omega_k\Big(f^{(r_1,\dots,r_n)}; 0,\dots,0,\frac{1}{m_i},0,\dots,0\Big)_q \\ \leqslant C\Bigg\{ \frac{1}{m_i^k} \Bigg[\|f\|_p^q + \sum_{\lambda=1}^{m_i} \lambda^{q(\sigma+k)-1} E^q_{\lambda^{l_1}\dots\lambda\dots\lambda^{l_n}}(f)_p \Bigg]^{1/q} \\ + \Bigg[\sum_{\lambda=m_i+1}^{\infty} \lambda^{q\sigma-1} E^q_{\lambda^{l_1}\dots\lambda\dots\lambda^{l_n}}(f)_p \Bigg]^{1/q} \Bigg\} \end{aligned} \tag{3.1}$$

holds, where the constant C does not depend either on f or $m_i = 1, 2, \dots$.

PROOF. For the modulus of smoothness ω_k of the derivative $f^{(r_1,\dots,r_n)}$ of the function f we have

$$\omega_k(f^{(r_1,\dots,r_n)}; 1/m_i)_q \leqslant \omega_k(f^{(r_1,\dots,r_n)} - G_m^{(r_1,\dots,r_n)}; 1/m_i)_q + \omega_k(G_m^{(r_1,\dots,r_n)}; 1/m_i)_q = I_1 + I_2. \tag{3.2}$$

For I_1 we obtain

$$I_1 \ll \|f^{(r_1,\dots,r_n)} - G_m^{(r_1,\dots,r_n)}\|_q = \Big\| \sum_{\lambda=m+1}^{\infty} \xi_\lambda^{(r_1,\dots,r_n)} \Big\|_q. \tag{3.3}$$

In the same way by which inequality (2.17) was established, in view of (3.3), we conclude that

$$I_1 \ll \Big\{ \sum_{\lambda=m+1}^{\infty} 2^{\lambda q\sigma} E^q_{2^{\lambda l_1}\dots 2^{\lambda}\dots 2^{\lambda l_n}}(f)_p \Big\}^{1/q}. \tag{3.4}$$

In virtue of the properties of the modulus of smoothness [**2**, 4.4.4(2)] we have

$$I_2 = \omega_k(G_m^{(r_1,\dots,r_n)}; 1/m_i)_q \leqslant \frac{1}{m_i^k} \|G_m^{(r_1,\dots,r_i+k,\dots,r_n)}\|_q. \tag{3.5}$$

In the same way by which the inequality (2.17) was established, putting $r_i + k$ instead of r_i, and since $l_i = 1$, we get the estimate

$$\|G_m^{(r_1,\dots,r_i+k,\dots,r_n)}\|_q \ll \Big\{ \|f\|_p^q + \sum_{\lambda=0}^{\infty} 2^{\lambda q(\sigma+k)} E^q_{2^{\lambda l_1}\dots 2^{\lambda}\dots 2^{\lambda l_n}}(f)_p \Big\}^{1/q}. \tag{3.6}$$

Now, in view of (3.2), (3.4), (3.5) and (3.6), we obtain

$$\omega_k(f^{(r_1,\dots,r_n)}; 1/m_i)_q \ll \Big\{ \sum_{\lambda=m_i+1}^{\infty} 2^{\lambda q\sigma} E^q_{2^{\lambda l_1}\dots 2^{\lambda}\dots 2^{\lambda l_n}}(f)_p \Big\}^{1/q} + \frac{1}{m_i^k} \Big\{ \|f\|_p^q + \sum_{\lambda=0}^{m_i} 2^{\lambda q(\sigma+k)} E^q_{2^{\lambda l_1}\dots 2^{\lambda}\dots 2^{\lambda l_n}}(f)_p \Big\}^{1/q}. \tag{3.7}$$

Let

$$q(\sigma + k) = s, \quad E^q_{2^{\lambda l_1}\dots 2^{\lambda}\dots 2^{\lambda l_n}}(f)_p = A_{2^\lambda}. \tag{3.8}$$

Then, using inequality (1.1), (Lemma 1.1), we get

$$\sum_{\lambda=0}^{m} 2^{\lambda s} A_{2^\lambda} = A_1 + 2^s A_2 + 2^s \sum_{\lambda=2}^{m} 2^{(\lambda-1)s} A_{2^\lambda} \leqslant A_1 + 2^s A_2 + 2^s \sum_{\lambda=2}^{m} \sum_{i=2^{\lambda-1}+1}^{2^\lambda} i^{s-1} A_i$$
$$= A_1 + 2^s A_2 + 2^s \Big\{ \sum_{i=3}^{2^{m-1}} i^{s-1} A_i + \sum_{i=2^{m-1}+1}^{2^m} i^{s-1} A_i \Big\}.$$

Using Lemma 1.2, from the previous inequality, it follows

$$\sum_{\lambda=0}^{m} 2^{\lambda s} A_{2^\lambda} \ll \sum_{i=1}^{2^{m-1}} i^{s-1} A_i. \tag{3.9}$$

Choosing m so that $2^{m-1} \leqslant m_i < 2^m$, from (3.9) it follows $\sum_{\lambda=0}^{m} 2^{\lambda s} A_{2^\lambda} \ll \sum_{i=1}^{m_i} i^{s-1} A_i$, i.e.,

$$\sum_{\lambda=0}^{m} 2^{\lambda q(\sigma+k)} A_{2^\lambda} \ll \sum_{i=1}^{m_i} i^{q(\sigma+k)-1} A_i. \tag{3.10}$$

To estimate the first sum in (3.7) we use (1.2), (Lemma 1.1.), and get

$$\begin{aligned}\sum_{\lambda=m+1}^{\infty} 2^{\lambda q\sigma} A_{2^\lambda} &= 2^{-q\sigma} \sum_{\lambda=m+1}^{\infty} 2^{(\lambda+1)q\sigma} A_{2^\lambda} \leqslant 2^{-q\sigma} 2^{2q\sigma} \sum_{\lambda=m+1}^{\infty} \sum_{i=2^{\lambda-1}+1}^{2^\lambda} i^{q\sigma-1} A_i \\ &= 2^{q\sigma}\{(2^m+1)^{q\sigma-1} A_{2^m+1} + \cdots + (2^{m+1})^{q\sigma-1} A_{2^{m+1}} + \cdots\},\end{aligned}$$

hence, using that $m_i < 2^m$, it follows

$$\sum_{\lambda=m+1}^{\infty} 2^{\lambda q\sigma} A_{2^\lambda} \leqslant 2^{q\sigma} \sum_{\lambda=m_i+1}^{\infty} i^{q\sigma-1} A_i. \tag{3.11}$$

Putting $A_i = E_i^q$ (equality (3.8)), from (3.7) and (3.11), it follows (3.1). □

Corollary 3.1. *For $n = 1$ it holds that $l_j = 1$, $r_j = r$, $\sigma = r + \frac{1}{p} - \frac{1}{q}$ and we get the corresponding theorems and inequalities for a function of one variable.*

Corollary 3.2. *If $l_j = 1$, $j = 1, 2, \ldots, n$ and $r_j = 0$, $j \neq i$, $r_i = r$, then $\sigma = n(\frac{1}{p} - \frac{1}{q}) + r$. Therefore, the condition*

$$\sum_{\lambda=1}^{\infty} \lambda^{q[r+n(1/p-1/q)]-1} E^q_{\lambda\ldots\lambda\ldots\lambda}(f)_p < \infty$$

implies that the function f has a derivative $\partial^r f/\partial x^r$ with respect to any variable x_i belonging to L_q. For the modulus of smoothness the corresponding inequality holds.

Corollary 3.3. *Applying the inequality $(\sum a_k)^s \leqslant \sum (a_k)^s$, $a_k \geqslant 0$, $0 < s \leqslant 1$, for $s = 1/q$, from (3.7) it follows*

$$\begin{aligned}\omega_k(f^{(r_1,\ldots,r_n)}; 1/m_i)_q \ll &\sum_{\lambda=m_i+1}^{\infty} 2^{\lambda\sigma} E_{2^{\lambda l_1}\ldots 2^{\lambda}\ldots 2^{\lambda l_n}}(f)_p \\ &+ \frac{1}{m_i^k}\Big\{\|f\|_p^q + \sum_{\lambda=0}^{m_i} 2^{\lambda(\sigma+k)} E_{2^{\lambda l_1}\ldots 2^{\lambda}\ldots 2^{\lambda l_n}}(f)_p\Big\}.\end{aligned}$$

wherefrom

$$\omega_k(f^{(r_1,\dots,r_n)};1/m_i)_q \ll \sum_{\lambda=m_i+1}^{\infty} \lambda^{\sigma-1} E_{\lambda^{l_1}\dots\lambda\dots\lambda^{l_n}}(f)_p + \frac{1}{m_i^k}\Big\{\|f\|_p + \sum_{\lambda=1}^{m_i} \lambda^{\sigma+k-1} E_{\lambda^{l_1}\dots\lambda\dots\lambda^{l_n}}(f)_p\Big\}. \tag{3.12}$$

For $n = 1$ *inequality* (3.12) *implies inequality* 6.4.1(3) *in* [**4**]. *For* $r_j = 0$, $j \neq i$, $r_i = r$ $(j = 1,\dots,n)$ *it holds that* $\sigma = r + (\frac{1}{p} - \frac{1}{q})\sum_{j=1}^{n} l_j$, *and from* (3.12) *it follows inequality* 6.4.3(8) *in* [**4**].

Corollary 3.4. *For* $p = q$ *it holds that* $\sigma = \sum_{j=1}^{n} l_j r_j$, *and from* (3.12) *we get the corresponding result in* L_p.

Remark 3.1. Some results of this paper were presented at the First Mathematical Conference of the Republic of Srpska (Pale, May 21-22, 2011).

References

1. N. A. Il'yasov, *An inverse theorem of approximation theory in different metrics*, Mat. Zametki 50:6 (1991), 57–65, 158 (in Russian); translated in Math. Notes 50:5-6 (1991), 1253–1260
2. S. M. Nikol'skiĭ, *Approximation of Functions of Several Variables and Imbedding Theorems*, Second edition, revised and supplemented, Nauka, Moscow, 1977 (in Russian).
3. M. K. Potapov, *Imbedding theorems in the mixed metric*, Trudy Mat. inst. AN SSSR 156 (1980), 143–156 (in Russian)
4. A. F. Timan, *Theory of Approximation of Functions of Real Variable*, Gos. Izdat. FM, Moscow, 1960 (in Russian).
5. M. Tomić, *On converse theorem of approximation in various metrics for periodic functions of several variables*, Facta Univ., Ser. Math. Inform. 15 (2000), 49–56.
6. ______ *On representation of derivatives of functions in* L_p, Mat. Vesn. 62(3) (2010), 235–250.

University of East Sarajevo
East Sarajevo
Republic of Srpska

(Received 12 08 2010)
(Revised 30 09 2013)