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THE SUM OF THE UNITARY DIVISOR FUNCTION

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Abstract. We establish a new upper bound on the function $\sigma^*(n)$, the sum of all coprime divisors of n. The main result is that $\sigma^*(n) \leqslant 1.3007n \log\log n$ for all $n \geqslant 570\,571$.

1. Introduction

1.1. The function $\sigma(n)$. Let $\sigma(n)$ denote the sum of the divisors of n; for example, $\sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$. In 1913 Grönwall showed that

$$\limsup \sigma(n)/(n \log\log n) = e^{\gamma} = 1.78107\ldots, \tag{1.1}$$

where γ is Euler's constant. A proof is given in [**5**, Thm. 322]. Robin showed that the manner in which (1.1) behaves is connected with the Riemann hypothesis. More precisely, he showed, in [**8**], that for $n \geqslant 5041$ the inequality

$$\sigma(n) < e^{\gamma} n \log\log n \tag{1.2}$$

is equivalent to the Riemann hypothesis. Ivić [**6**] showed that

$$\sigma(n) < 2.59\, n \log\log n, \quad (n \geqslant 7),$$

which was improved by Robin [op. cit.] to

$$\sigma(n) < \frac{\sigma(12)}{12 \log\log 12}\, n \log\log n \leqslant 2.5634\, n \log\log n, \quad (n \geqslant 7). \tag{1.3}$$

Akbary, Friggstad and Juricevic [**1**] improved this further, replacing the right-side of (1.3) with

$$\begin{aligned} \frac{\sigma(180)}{180 \log\log 180}\, n \log\log n &\leqslant 1.8414\, n \log\log n \\ &\leqslant 1.0339\, e^{\gamma} n \log\log n, \quad (n \geqslant 121). \end{aligned} \tag{1.4}$$

Given Robin's criterion for the Riemann hypothesis in (1.2) it is reasonable to suggest that (1.4) is close to the best bound that one may hope to exhibit.

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1.2. The function $\sigma^*(n)$. We say that d is a unitary divisor of n if $d \mid n$ and $(d, n/d) = 1$. Let $\sigma^*(n) = \sum_{d|n,\,(d,n/d)=1} d$ be the sum of all unitary divisors of n; for example, $\sigma^*(12) = 1 + 12 + 3 + 4 = 20$. Robin [**8**, p. 210] notes that the proof of (1.1) can be adapted to show that

$$\limsup \frac{\sigma^*(n)}{n \log\log n} = \frac{6e^\gamma}{\pi^2} = 1.08\ldots, \tag{1.5}$$

see also [**6**, p. 21]. Ivić [**6**] showed that

$$\sigma^*(n) < \frac{28}{15}\, n \log\log n, \quad (n \geqslant 31).$$

This was improved by Robin who showed that

$$\sigma^*(n) < 1.63601 n \log\log n, \quad (n \geqslant 31),$$

except for $n = 42$ when $\sigma^*(n) = 1.7366\ldots n \log\log n$. A direct comparison of these results with those in §1.1 compels us to ask the following questions.

(1) Given (1.5) can a Robin-esque criterion for the Riemann hypothesis à la (1.2) be given for $\sigma^*(n)$?
(2) Analogous to (1.4) can one obtain a relatively close approximation to (1.5) of the form

$$\sigma^*(n) < (1+\epsilon)\frac{6e^\gamma}{\pi^2}\, n \log\log n, \quad (n \geqslant n_0),$$

for reasonably small values of ϵ and n_0?

Concerning 1, Robin has conjectured [**8**, Prop. 1(i), p. 210] that there are infinitely many n for which

$$\sigma^*(n) > \frac{6e^\gamma}{\pi^2}\, n \log\log n.$$

A related conjecture is given in Proposition 1(ii) in [**8**], viz. that

$$\frac{\sigma(n)}{\sigma^*(n) \log\log n} < e^\gamma, \tag{1.6}$$

for all n sufficiently large. The interest in this conjecture stems from the limiting relation

$$\limsup \frac{\sigma(n)}{\sigma^*(n) \log\log n} = e^\gamma.$$

Derbal [**3**] proved (1.6) for all $n \geqslant 17$.

This article answers Question 2 above, at least partially, by proving

THEOREM 1.1. *For $n \geqslant 570\,571$,*

$$\sigma^*(n) \leqslant 1.3007\, n \log\log n. \tag{1.7}$$

It takes less than 40 seconds on a 1.8 GHz laptop to compute $\sigma^*(n)$ for all $1 \leqslant n \leqslant 570\,570$. One may therefore justify the number 570 571 appearing in Theorem 1.1 as being "reasonably small", as stipulated in Question 2, as least in regards to computational resources.

It would be of interest to address the following problem. Fix an $\epsilon > 0$ and determine the least value of n_0 such that $\sigma^*(n) < (1+\epsilon)\frac{6e^\gamma}{\pi^2}\, n \log\log n$ for all

$n \geqslant n_0$. The method used to prove Theorem 1.1 is incapable of reducing the right-hand side of (1.7) to anything less than $1.29887 n \log\log n$.

Theorem 1.1 is proved in §2. An application is given in §3. Two concluding questions are raised in §4.

2. Proof of Theorem 1.1

We proceed as in Robin [**8**, p. 211]. It is sufficient to verify the inequality on numbers $N_k = \prod_{i=1}^{k} p_i$, where $k \geqslant 2$, since, for $N_k \leqslant n < N_{k+1}$, we have $\sigma^*(n)/n \leqslant \sigma^*(N_k)/N_k$, whence

$$\frac{\sigma^*(n)}{n \log\log n} \leqslant \frac{\sigma^*(N_k)}{N_k \log\log N_k}. \tag{2.1}$$

Since $\sigma^*(p^\alpha) = 1 + p^\alpha$ and $\sigma^*(n)$ is a multiplicative function, the right-hand side of (2.1) is

$$\frac{\prod_{i\leqslant k}\left(1+p_i^{-1}\right)}{\log\theta(p_k)}, \tag{2.2}$$

where $\theta(x) = \sum_{p\leqslant x} \log p$. To bound the numerator in (2.2) we use

$$\sum_{p\leqslant x} \frac{1}{p} \leqslant \log\log x + B + \frac{1}{10\log^2 x} + \frac{4}{15\log^3 x}, \quad (x \geqslant 10\,372),$$

where

$$B = \gamma + \sum_{p\geqslant 2}\left\{\log\left(1-\frac{1}{p}\right) + \frac{1}{p}\right\} = 0.26149\ldots,$$

see Dusart [**4**]. To bound the denominator in (2.2) we use

$$\theta(x) \geqslant x\Big(1 - \frac{0.006788}{\log x}\Big), \quad (x \geqslant 10\,544\,111),$$

which is also found in [**4**]. Therefore, since $e^x \geqslant x+1$ we have

$$\prod_{i\leqslant k}\left(1+\frac{1}{p_i}\right) \leqslant \exp\left(\sum_{i\leqslant k}\frac{1}{p_i}\right) \leqslant A_1(p_k)\log p_k,$$

where

$$A_1(x) = \exp\Big(B + \frac{1}{10\log^2 x} + \frac{4}{15\log^2 x}\Big), \quad (x \geqslant 10\,372).$$

Also

$$\log\theta(p_k) \geqslant A_2(p_k)\log p_k,$$

where

$$A_2(x) = 1 + \frac{\log(1 - 0.006788/\log x)}{\log x}, \quad (x \geqslant 10\,544\,111).$$

It is clear that

$$A_2(x) < 1 < e^B = 1.29887\ldots < A_1(x). \tag{2.3}$$

We choose a suitably large lower bound on k in order to make $A_1(x)$ and $A_2(x)$ sufficiently close to e^B and 1 respectively. Indeed, we shall bound (2.2) for $p_k \geqslant 15\,485\,863$, which is equivalent to $k \geqslant 1\,000\,000$. Therefore

$$\frac{\prod_{i\leqslant k}\left(1+p_i^{-1}\right)}{\log\theta(p_k)} \leqslant \frac{A_1(p_k)}{A_2(p_k)} \leqslant 1.3007, \tag{2.4}$$

whence

$$\frac{\sigma^*(n)}{n\log\log n} \leqslant \frac{\sigma^*(N_k)}{N_k\log\log N_k} \leqslant 1.3007, \tag{2.5}$$

for all $k \geqslant 10^6$. One may check that (2.5) also holds for $8 \leqslant k \leqslant 10^6$, On a single core PC with 32 GB of RAM, this calculation took less than a minute using *Magma*. All that remains are the numbers $3 \leqslant n \leqslant p_1 \cdots p_8 = 9\,699\,690$. A quick computational check shows that

$$\frac{\sigma^*(570\,570)}{570\,570\log\log 570\,570} \geqslant 1.3125,$$

and that, for all $n > 570\,570$, the inequality (1.7) holds, which proves Theorem 1.1. Were this lower bound on n too large for one's tastes, one could also show

$$\sigma^*(n) \leqslant 1.3007\, n\log\log n,$$

for all $n \geqslant 53\,131$ with only two exceptions, namely

$$\sigma^*(510\,510) = (1.3245\ldots)510\,510\log\log 510\,510, \quad \text{and}$$
$$\sigma^*(570\,570) = (1.3125\ldots)570\,570\log\log 570\,570.$$

Our bounds for $\sigma^*(n)$ depend on an upper bound for $A_1(p_k)/A_2(p_k)$ in (2.4). We see at once from (2.3) that our method is incapable of reducing the bound 1.3007 in Theorem 1.1 to anything below 1.29887.

3. Application to exponential divisors

Given an $n = p_1^{a_1}\cdots p_s^{a_s}$ the integer $d = p_1^{b_1}\cdots p_s^{b_s}$ is an *exponential divisor* of n if $b_j \mid a_j$ for every $1 \leqslant j \leqslant s$. Define the functions $d^{(e)}(n)$ and $\sigma^{(e)}(n)$ to be the number of exponential divisors of n and the sum of the exponential divisors of n, respectively. Since these functions are multiplicative we have

$$d^{(e)}(n) = \prod_{j=1}^{r} d(a_j), \quad \sigma^{(e)}(n) = \prod_{j=1}^{r}\left(\sum_{b_j\mid a_j} p_j^{b_j}\right),$$

where $d(n)$ is the number of divisors of n. Minculete [**7**, Thm. 2.1 and Cor. 2.5] has given the following bounds for $\sigma^{(e)}(n)$ and $d(n)\,d^{(e)}(n)$

$$\sigma^{(e)}(n) \leqslant \frac{28}{15}\, n\log\log n, \quad (n \geqslant 6),$$
$$d^{(e)}(n)\,d(n) \leqslant \frac{28}{15}\, n\log\log n, \quad (n \geqslant 5).$$

An application of the proof of Theorem 1.1 improves these bounds.

COROLLARY 3.1. *For $n \geqslant 37$,*

$$\sigma^{(e)}(n) \leqslant 1.3007\, n \log\log n. \tag{3.1}$$

For $n \geqslant 8$,

$$d^{(e)}(n)\, d(n) \leqslant 1.3007\, n \log\log n. \tag{3.2}$$

PROOF. The displayed formula halfway down page 1529 in [**7**] gives

$$\sigma^{(e)} \leqslant n \prod_{p|n} \Big(1 + \frac{1}{p}\Big),$$

so that

$$\frac{\sigma^{(e)}(n)}{n \log\log n} \leqslant \frac{\prod_{p|n} \big(1 + \frac{1}{p}\big)}{\log\log n}. \tag{3.3}$$

As before, we need only consider (3.3) on $N_k \leqslant n < N_{k+1}$. Using (2.4) and the calculations in §2 we have

$$\frac{\sigma^{(e)}(n)}{n \log\log n} \leqslant 1.3007, \quad (n \geqslant 9\,699\,691).$$

Checking the range $37 \leqslant n \leqslant 9\,699\,691$ establishes (3.1). Minculete [**7**, Eq. (12)] showed that $d(n)\, d^{(e)}(n) \leqslant \sigma^{(e)}(n)$ for all $n \geqslant 1$. Using this, (3.1), and a simple computer check for $8 \leqslant n \leqslant 36$, establishes (3.2). □

4. Conclusion

Both of the functions $\sigma^*(n)$ and $\sigma^{(e)}(n)$ are multiplicative. We have

$$\sigma^*(p) = 1 + p > \sigma^{(e)}(p) = p,$$

and, for $a \geqslant 2$,

$$\sigma^*(p^a) = 1 + p^a < p + p^a \leqslant \sigma^{(e)}(p^a),$$

since $a = a \cdot 1$, where a and 1 are distinct. Therefore, on square-free numbers $\sigma^*(n) > \sigma^{(e)}(n)$. We conclude this section by raising two questions.

(1) What is the proportion of n for which $\sigma^*(n) > \sigma^{(e)}(n)$?
(2) Are there infinitely many values of n for which $\sigma^*(n) = \sigma^{(e)}(n)$?

The proportion in Question 1 must be at least that of the square-free numbers, viz. $6/\pi^2 \approx 0.607$. A computation shows the proportion of $1 \leqslant n \leqslant 10^9$ to be approximately 0.778307. It follows from the Erdős–Wintner theorem (see, e.g., [**9**, III.4]) that the density of n for which $\sigma^*(n) > \sigma^{(e)}(n)$ is well defined. In [**2**] the density of the set of integers n for which $\sigma(n)/n \geqslant 2$ was estimated. It seems possible that similar methods may be brought to bear on Question 1.

As for Question 2, only five values of n were found in the range $1 \leqslant n \leqslant 10^9$ for which $\sigma^*(n) = \sigma^{(e)}(n)$, namely

$$n = 20,\ 45,\ 320,\ 6615,\ 382\,200.$$

Andrew Lelechenko has also found

$$n = 680\,890\,228\,200,$$

which is the next smallest n after 382 200. He has also communicated to me that $\sigma^*(n) = \sigma^{(e)}(n)$ also for

$$n = 2\,456\,687\,209\,744\,634\,987\,008\,753\,664 = 2^{49} \times 4\,363\,953\,127\,297.$$

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References

1. A. Akbary, Z. Friggstad, R. Juricevic, *Explicit upper bounds for* $\prod_{p \leqslant p_{\omega(n)}} p/(p-1)$, Contrib. Discrete Math. **2**(2) (2007), 153–160.
2. M. Deléglise, *Bounds for the density of abundant integers*, Exp. Math. **7**(2) (1998), 137–143.
3. A. Derbal, *Grandes valeurs de la fonction* $\sigma(n)/\sigma^*(n)$, C. R. Math. Acad. Sci. Paris **346**(3–4) (2008), 125–128.
4. P. Dusart, *Inégalités explicites pour* $\psi(X)$, $\theta(X)$, $\pi(X)$ *et les nombres premiers*, C. R. Math. Acad. Sci. Soc. R. Can. **21**(2) (1999), 53–59.
5. G. H. Hardy, E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford University Press, New York, 2008.
6. A. Ivić, *Two inequalities for the sum of divisors functions*, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. **7** (1977), 17–22.
7. N. Minculete, *On certain inequalities about arithmetic functions which use the exponential divisors*, Int. J. Number Theory **8**(6) (2012), 1527–1535.
8. G. Robin, *Grandes valeurs de la fonction somme des diviseurs et hypothèse de Riemann*, J. Math. Pures Appl. (9) **63**(2) (1984), 187–213.
9. G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Camb. Stud. Adv. Math. 46, Cambridge University Press, Cambridge, 1995.

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