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DIRECT ESTIMATIONS OF NEW GENERALIZED BASKAKOV–SZÁSZ OPERATORS

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Abstract. Several modifications of the discrete operators are available in the literature. In the recent years, certain modifications of the well-known Baskakov and Szász–Mirakyan operators have been discussed based on certain parameters. We propose mixed summation-integral type operators and estimate the quantitative asymptotic formula and a global direct result for the special case. For general case, we establish moments and some direct convergence results in ordinary approximation, which includes pointwise approximation, asymptotic formula and a direct result in terms of modulus of continuity.

1. Introduction

In order to generalize the Baskakov operators, Mihesan [**15**] proposed the following operators based on a non-negative constant a, independent of n as

$$B_n^a(f;x)=\sum_{k=0}^{\infty} b_{n,k}^a(x) f\Big(\frac{k}{n}\Big), \tag{1.1}$$

where

$$b_{n,k}^a(x)=e^{-\frac{ax}{1+x}}\frac{\sum_{i=0}^{k}\binom{k}{i}(n)_i a^{k-i}}{k!}\frac{x^k}{(1+x)^{n+k}},$$

and the rising factorial is given by $(n)_i=n(n+1)\cdots(n+i-1)$, $(n)_0=1$. Also, in order to generalize the Szász–Mirakyan operators, Jain [**13**] introduced the following operators

$$S_n^{\beta}(f,x)=\sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) f\Big(\frac{k}{n}\Big), \quad x\in[0,\infty)$$

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where $0 \leqslant \beta < 1$ and the basis function is defined as

$$L_{n,k}^{(\beta)}(x) = \frac{nx(nx+k\beta)^{k-1}}{k!}\, e^{-(nx+k\beta)}.$$

It was seen in [**15**] and [**13**] that $\sum_{k=0}^{\infty} b_{n,k}^{a}(x) = \sum_{k=0}^{\infty} L_{n,k}^{(\beta)}(x) = 1$. The integral modification of the classical Bernstein polynomial was introduced by Durrmeyer [**6**]. Gupta et al. [**10**] proposed the hybrid Durrmeyer type operators by taking the general basis function $L_{n,k}^{(\beta)}(x)$ under summation sign, the actual Durrmeyer operators for this basis function were recently considered by Gupta and Greubel [**16**]. In order to approximate Lebesgue integrable functions on the interval $[0,\infty)$, for a non-negative parameter a and $0 \leqslant \beta < 1$, we propose the mixed hybrid Durrmeyer type operators as follows:

$$D_n^{a,\beta}(f,x) = \sum_{k=0}^{\infty} \frac{\langle L_{n,k}^{(\beta)}(t), f(t)\rangle}{\langle L_{n,k}^{(\beta)}(t), 1\rangle}\, b_{n,k}^{a}(x) \tag{1.2}$$

where $\langle f,g\rangle = \int_0^\infty f(t)\, g(t)\, dt$. Some approximation properties for the particular case $\beta = 0$ were recently discussed by Agrawal et al. [**1**]. For the special case of $a = \beta = 0$, these operators reduce to the Baskakov–Szász operators introduced about twenty years ago by Gupta and Srivastava [**12**], which are defined as

$$\begin{aligned} D_n(f,x) := D_n^{0,0}(f,x) &= \sum_{k=0}^{\infty} \frac{\langle L_{n,k}^{(0)}(t), f(t)\rangle}{\langle L_{n,k}^{(0)}(t), 1\rangle}\, b_{n,k}^{0}(x) \\ &= n\sum_{k=0}^{\infty} b_{n,k}^{0}(x) \int_0^\infty L_{n,k}^{(0)}(t) f(t)dt, \end{aligned}$$

where

$$b_{n,k}^{0}(x) = \binom{n+k-1}{k}\frac{x^k}{(1+x)^{n+k}}, \quad L_{n,k}^{(0)}(t) = e^{-nt}\frac{(nt)^k}{k!}.$$

We may point out here that $\langle L_{n,k}^{(\beta)}(t), 1\rangle$ is dependent of k, while for $\beta = 0$ this term is just $1/n$. Dragomir [**3**] established interesting approximation results of continuous linear functionals in real normed spaces. Very recently, Gupta and Agarwal [**9**] presented convergence estimates on several linear positive operators. We also mention for the readers some of the related work as [**2**, **7**, **8**, **11**, **17**, **18**] etc.

Here, we first estimate the quantitative asymptotic formula for the special case and a global direct result. For general case, we establish moments and some direct results in ordinary approximation, which includes pointwise approximation, asymptotic formula, error-estimation in terms of first and second order modulus of continuity.

2. Approximation for the case $a = \beta = 0$

Very recently Luo–Milovanovic–Agarwal [**14**] established some results on the extended beta and extended hypergeometric functions. We observe that in the special case, if we take $a = \beta = 0$, then the moments can be obtained in terms of hypergeometric series as follows: Using the identities: $k! = (1)_k$ and $\Gamma(k+m+1) = \Gamma(m+1)(m+1)_k$, we have

$$\begin{aligned} D_n(t^m, x) &= n \sum_{k=0}^{\infty} b_{n,k}^0(x) \int_0^{\infty} L_{n,k}^0(x) t^m dt \\ &= n \sum_{k=0}^{\infty} \frac{(n)_k x^k}{k!(1+x)^{n+k}} \int_0^{\infty} \frac{e^{-nt}(nt)^k}{k!} t^m dt \\ &= \frac{1}{n^m} \sum_{k=0}^{\infty} \frac{(n)_k x^k}{k!(1+x)^{n+k}} \frac{\Gamma(k+m+1)}{(1)_k} \\ &= \frac{(1+x)^{-n}}{n^m} \sum_{k=0}^{\infty} \frac{(n)_k (m+1)_k \, \Gamma(m+1) x^k}{k!(1+x)^k (1)_k} \\ &= \frac{m!(1+x)^{-n}}{n^m} {}_2F_1\Big(n, m+1; 1; \frac{x}{1+x}\Big). \end{aligned}$$

Applying the well known transformation

$$ {}_2F_1(a, b; c; x) = (1-x)^{-a} {}_2F_1\Big(a, c-b; c; \frac{x}{x-1}\Big), $$

we obtain at once

$$ D_n(t^m, x) = \frac{m!}{n^m} {}_2F_1(n, -m; 1; -x). $$

Furthermore, we obtain

$$ D_n(t, x) = x + \frac{1}{n}, \quad D_n(t^2, x) = \frac{2 + 4nx + n(n+1)x^2}{n^2} $$

and

$$ D_n(t-x, x) = \frac{1}{n}, \quad D_n((t-x)^2, x) = \frac{2 + 2nx + nx^2}{n^2}. \tag{2.1} $$

In general, using the similar approach, one can show that:

$$ D_n((t-x)^r, x) = O\big(n^{-[(r+1)/2]}\big), \tag{2.2} $$

where $[\alpha]$ denotes the integral part of α.

Let $B_{x^2}[0, \infty)$ be the set of all functions f defined on $\mathbb{R}^+$ satisfying the condition $|f(x)| \leqslant M_f(1+x^2)$ with some constant M_f, depending only on f, but independent of x. $B_{x^2}[0, \infty)$ is called weighted space and it is a Banach space endowed with the norm

$$ \|f\|_{x^2} = \sup_{x \in \mathbb{R}^+} \frac{f(x)}{1+x^2}. $$

Let $C_{x^2}[0, \infty) = C[0, \infty) \cap B_{x^2}[0, \infty)$ and by $C_{x^2}^k[0, \infty)$, we denote subspace of all continuous functions $f \in B_{x^2}[0, \infty)$ for which $\lim_{x \to \infty} \frac{f(x)}{1+x^2}$ is finite.

We know that usual first modulus of continuity $\omega(\delta)$ does not tend to zero, as $\delta \to 0$ on infinite interval. Thus, we use weighted modulus of continuity $\Omega(f, \delta)$

defined on infinite interval $\mathbb{R}^+$ (see [**17**]). Let

$$\Omega(f,\delta)=\sup_{|h|<\delta,\ x\in\mathbb{R}^+}\frac{|f(x+h)-f(x)|}{(1+h^2)(1+x^2)}\quad\text{for each } f\in C_{x^2}[0,\infty).$$

Now, some elementary properties of $\Omega(f,\delta)$ are collected in the following lemma.

LEMMA 2.1. *Let $f\in C^k_{x^2}[0,\infty)$. Then,*

i) *$\Omega(f,\delta)$ is a monotonically increasing function of δ, $\delta\geqslant 0$.*
ii) *For every $f\in C^k_{x^2}[0,\infty)$, $\lim_{\delta\to 0}\Omega(f,\delta)=0$.*
iii) *For each $\lambda>0$,*

$$\Omega(f,\lambda\delta)\leqslant 2(1+\lambda)(1+\delta^2)\Omega(f,\delta). \tag{2.3}$$

From the inequality (2.3) and definition of $\Omega(f,\delta)$, we get

$$|f(t)-f(x)|\leqslant 2(1+x^2)\big(1+(t-x)^2\big)\Big(1+\frac{|t-x|}{\delta}\Big)(1+\delta^2)\Omega(f,\delta) \tag{2.4}$$

for every $f\in C_{x^2}[0,\infty)$ and $x,t\in\mathbb{R}^+$. The following estimate is quantitative Voronovskaya type asymptotic formula:

THEOREM 2.1. *Let $f''\in C^k_{x^2}[0,\infty)$, $a=\beta=0$ and $x>0$. Then we have*

$$\Big|D_n(f,x)-f(x)-\frac{1}{n}f'(x)-\frac{x(x+2)}{2n}f''(x)\Big|\leqslant\frac{f''(x)}{n^2}+8(1+x^2)O(n^{-1})\Omega\Big(f'',\frac{1}{\sqrt{n}}\Big).$$

PROOF. By Taylor's formula, there exist η lying between x and y such that

$$f(y)=f(x)+f'(x)(y-x)+\frac{f''(x)}{2}(y-x)^2+h(y,x)(y-x)^2,$$

where

$$h(y,x):=\frac{f''(\eta)-f''(x)}{2}$$

and h is a continuous function which vanishes at 0. Applying the operator D_n to above equality, we get

$$D_n(f,x)-f(x)=\frac{f'(x)}{n}+\frac{f''(x)}{2}\Big[\frac{2+2nx+nx^2}{n^2}\Big]+D_n\big(h(y,x)(y-x)^2,x\big).$$

Also, we can write that

$$\Big|D_n(f,x)-f(x)-\frac{f'(x)}{n}-\frac{f''(x)}{2}\Big(\frac{2x+x^2}{n}\Big)\Big|\leqslant\frac{f''(x)}{n^2}+D_n\big(|h(y,x)|(y-x)^2,x\big)$$

To estimate last inequality using (2.4) and the inequality $|\eta-x|\leqslant|y-x|$, we can write

$$|h(y,x)|\leqslant(1+(y-x)^2)(1+x^2)\Big(1+\frac{|y-x|}{\delta}\Big)(1+\delta^2)\Omega(f'',\delta).$$

Also,

$$|h(y,x)|\leqslant\begin{cases}2(1+x^2)(1+\delta^2)^2\Omega(f'',\delta), & |y-x|<\delta\\ (1+(y-x)^2)(1+x^2)\big(1+\frac{|y-x|}{\delta}\big)(1+\delta^2)\Omega(f'',\delta), & |y-x|\geqslant\delta\end{cases}$$

Now choosing $\delta < 1$, we have

$$\begin{aligned}|h(y,x)| &\leqslant 2(1+x^2)\Big(1+\frac{(y-x)^4}{\delta^4}\Big)(1+\delta^2)^2\Omega(f'',\delta)\\ &\leqslant 8(1+x^2)\Big(1+\frac{(y-x)^4}{\delta^4}\Big)\Omega(f'',\delta).\end{aligned}$$

Using (2.2), we deduce that

$$\begin{aligned}D_n(|h(y,x)|(y-x)^2,x) &= 8(1+x^2)\Omega(f'',\delta)\big\{D_n((t-x)^2,x)+\delta^{-4}D_n((t-x)^6,x)\big\}\\ &= 8(1+x^2)\Omega(f'',\delta)\big\{O(n^{-1})+\delta^{-4}O(n^{-3})\big\}.\end{aligned}$$

Choosing $\delta = 1/\sqrt{n}$ we have

$$D_n(|h(y,x)|(y-x)^2,x) \leqslant 8(1+x^2)\mathcal{O}(n^{-1})\Omega\big(f'',1/\sqrt{n}\big). \qquad \square$$

By $C_B[0,\infty)$, we mean the class of all real valued continuous and bounded functions f on $[0,\infty)$. The second order Ditzian–Totik modulus of smoothness is defined by

$$\omega_\varphi^2(f,\delta) = \sup_{0\leqslant h\leqslant\delta}\ \sup_{x\pm h\varphi(x)\in[0,\infty)} |f(x+h\varphi(x)) - 2f(x) + f(x-h\varphi(x))|,$$

$\varphi(x) = \sqrt{(x+1)(x+2)}$, $x \geqslant 0$. The corresponding K-functional is

$$K_{2,\varphi}(f,\delta^2) = \inf_{h\in W_\infty^2(\varphi)}\{\|f-h\| + \delta^2\|\varphi^2 h''\|\},$$

where $W_\infty^2(\varphi) = \{h \in C_B[0,\infty) : h' \in AC_{\mathrm{loc}}[0,\infty) : \varphi^2 h'' \in C_B[0,\infty)\}$. By [**5**, Thm. 2.1.1], it follows that

$$C^{-1}\omega_\varphi^2(f,\delta) \leqslant K_{2,\varphi}(f,\delta^2) \leqslant C\omega_\varphi^2(f,\delta)$$

for some absolute constant $C > 0$. Also, the Ditzian–Totik modulus of the first order is given by

$$\overrightarrow{\omega}_\varphi(f,\delta) = \sup_{0\leqslant h\leqslant\delta}\ \sup_{x\pm h\varphi(x)\in[0,\infty)} |f(x+h\varphi(x)) - f(x)|,$$

where φ is an admissible step-weight function on $[0,\infty)$.

Theorem 2.2. *If $f \in C_B[0,\infty)$ and $n \in \mathbb{N}$, then we have the inequality*

$$\|D_n(f,x) - f(x)\| \leqslant 4\omega_\varphi^2\big(f,1/\sqrt{n}\big) + \vec{\omega_\varphi}\big(f,1/n\big).$$

Proof. We set $\varphi(x) = \sqrt{(x+1)(x+2)}$,

$$W_\varphi^2[0,\infty) = \{g \in AC_{\mathrm{loc}}[0,\infty) : \varphi^2 g'' \in C_B[0,\infty)\}$$

$$\tilde{D}_n(f,x) = D_n(f,x) - f(x+1/n) + f(x).$$

Then $\frac{|t-u|}{\varphi^2(u)} \leqslant \frac{|t-x|}{\varphi^2(x)}$ for u between x and t, $\tilde{D_n}(t-x,x) = 0$ by using (2.1), and for $g \in W_\varphi^2[0,\infty)$, by Taylor's formula, we have

$$g(t) = g(x) + g'(x)(t-x) + \int_x^t g''(u)(t-u)\,du.$$

Applying the operator D_n to above equality and then taking modulus, we get

$$
\begin{aligned}
&|\tilde{D}_n(g,x)-g(x)|\\
&\qquad\leqslant D_n\Big(\Big|\int_x^t (t-u)\,g''(u)\,du\Big|,x\Big)+\Big|\int_x^{x+\frac{1}{n}}\Big(x+\frac{1}{n}-u\Big)g''(u)du\Big|\\
&\qquad\leqslant \|\varphi^2 g''\|\frac{D_n((t-x)^2,x)}{(x+1)(x+2)}+\|\varphi^2 g''\|\Big|\int_x^{x+\frac{1}{n}}\frac{|x+\frac{1}{n}-u|}{(x+1)(x+2)}du\Big|\\
&\qquad\qquad\leqslant \|\varphi^2 g''\|\Big\{\frac{1}{n}+\Big(\frac{1}{n}\Big)^2\Big\}\leqslant \frac{2}{n}\|\varphi^2 g''\|.
\end{aligned}
$$

Now for $f\in C_B[0,\infty)$, we have

$$
\begin{aligned}
&|D_n(f,x)-f(x)|\\
&\quad=|\tilde{D}_n(f-g,x)-(f-g)(x)|+|\tilde{D}_n(g,x)-g(x)|+\Big|f\Big(x+\frac{1}{n}\Big)-f(x)\Big|\\
&\quad\leqslant 4\|f-g\|+\frac{2}{n}\|\varphi^2 g''\|+\Big|f\Big(x+\sqrt{(x+1)(x+2)}\frac{1}{n\sqrt{(x+1)(x+2)}}\Big)-f(x)\Big|\\
&\quad\leqslant 4\Big\{\|f-g\|+\frac{1}{n}\|\varphi^2 g''\|\Big\}+\sup_{t\geqslant 0}\Big|f\Big(t+\varphi(t)\frac{1}{n\sqrt{(x+1)(x+2)}}\Big)-f(t)\Big|\\
&\quad\leqslant 4\Big\{\|f-g\|+\frac{1}{n}\|\varphi^2 g''\|\Big\}+\vec{\omega}_\varphi\Big(f,\frac{1}{n\sqrt{(x+1)(x+2)}}\Big)\\
&\quad\leqslant 4\Big\{\|f-g\|+\frac{1}{n}\|\varphi^2 g''\|\Big\}+\vec{\omega}_\varphi\Big(f,\frac{1}{n}\Big).
\end{aligned}
$$

Hence, by definition of $K_{2,\varphi}(f,\delta^2)$, we have the inequality

$$
\|D_n(f,x)-f(x)\leqslant 4K_{2,\varphi}\Big(f,\frac{1}{n}\Big)+\overrightarrow{\omega}_\varphi\Big(f,\frac{1}{n}\Big)\leqslant 4\omega_\varphi^2\Big(f,\frac{1}{\sqrt{n}}\Big)+\overrightarrow{\omega}_\varphi\Big(f,\frac{1}{n}\Big).\quad\square
$$

3. Moments for the general case

For the general case, we obtain moments as follows:

LEMMA 3.1. *For the operators defined by* (1.1)*, if we denote*

$$
\mu^a_{n,m}(x)=\sum_{k=0}^{\infty} b^a_{n,k}(x)\Big(\frac{k}{n}\Big)^m,
$$

then, we have the following recurrence relation:

$$
\mu^a_{n,m+1}(x)=\frac{x(1+x)}{n}[\mu^a_{n,m}(x)]'+\Big(x+\frac{ax}{n(1+x)}\Big)\mu^a_{n,m}(x).
$$

PROOF. Using the identity

$$
x(1+x)^2[b^a_{n,k}(x)]'=[(k-nx)(1+x)-ax]b^a_{n,k}(x),
$$

we have

$$
\begin{aligned}
&x(1+x)^2[\mu^a_{n,m}(x)]'\\
&\qquad=\sum_{k=0}^{\infty}[(k-nx)(1+x)-ax]b^a_{n,k}(x)\Big(\frac{k}{n}\Big)^m
\end{aligned}
$$

$$= n(1+x)\sum_{k=0}^{\infty} b_{n,k}^{a}(x)\Big(\frac{k}{n}\Big)^{m+1} - [nx(1+x)+ax]\sum_{k=0}^{\infty} b_{n,k}^{a}(x)\Big(\frac{k}{n}\Big)^{m}$$
$$= n(1+x)\mu_{n,m+1}^{a}(x) - [nx(1+x)+ax]\mu_{n,m}^{a}(x). \qquad \square$$

REMARK 3.1. Using Lemma 3.1, the first few moments are as follows:

$$\mu_{n,0}^{a}(x) = 1, \quad \mu_{n,1}^{a}(x) = x + \frac{ax}{n(1+x)},$$

$$\mu_{n,2}^{a}(x) = x^2 + \frac{1}{n}\Big[x + x^2 + \frac{2ax^2}{(1+x)}\Big] + \frac{1}{n^2}\Big[\frac{a^2x^2}{(1+x)^2} + \frac{ax}{(1+x)}\Big],$$

$$\begin{aligned}\mu_{n,3}^{a}(x) = x^3 &+ \frac{1}{n}\Big[3x^3 + 3x^2 + \frac{3ax^3}{(1+x)}\Big]\\ &+ \frac{1}{n^2}\Big[2x^3 + 3x^2 + x + \frac{3ax^3}{(1+x)} + \frac{6ax^2}{(1+x)} + \frac{3a^2x^3}{(1+x)^2}\Big]\\ &+ \frac{1}{n^3}\Big[\frac{3a^2x^2}{(1+x)^2} + \frac{ax}{(1+x)} + \frac{a^3x^3}{(1+x)^3}\Big]\end{aligned}$$

$$\begin{aligned}\mu_{n,4}^{a}(x) = x^4 &+ \frac{1}{n}\Big[6x^4 + 6x^3 + \frac{4ax^4}{(1+x)}\Big]\\ &+ \frac{1}{n^2}\Big[11x^4 + 18x^3 + 7x^2 + \frac{12ax^4}{(1+x)} + \frac{18ax^3}{(1+x)} + \frac{6a^2x^4}{(1+x)^2}\Big]\\ &+ \frac{1}{n^3}\Big[6x^4 + 12x^3 + 7x^2 + x + \frac{8ax^4}{(1+x)} + \frac{18ax^3}{(1+x)} + \frac{14ax^2}{(1+x)}\\ &\qquad + \frac{3a^2x^3(x^2+4x+3)}{(1+x)^3} + \frac{9a^2x^3}{(1+x)^2} + \frac{4a^3x^4}{(1+x)^3} + \frac{3a^2x^4}{(1+x)^2}\Big]\\ &+ \frac{1}{n^4}\Big[\frac{7a^2x^2}{(1+x)^2} + \frac{ax}{(1+x)} + \frac{6a^3x^3}{(1+x)^3} + \frac{a^4x^4}{(1+x)^4}\Big].\end{aligned}$$

LEMMA 3.2. [**16**, Lemma 2] *For $0 \leqslant \beta < 1$, we have*

$$\frac{\langle L_{n,k}^{(\beta)}(t), t^r\rangle}{\langle L_{n,k}^{(\beta)}(t), 1\rangle} = P_r(k;\beta),$$

where $\langle f, g\rangle = \int_0^\infty f(t)\,g(t)\,dt$ and $P_r(k;\beta)$ is a polynomial of order r in the variable k. In particular

$$P_0(k;\beta) = 1,$$
$$P_1(k;\beta) = \frac{1}{n}\Big[(1-\beta)k + \frac{1}{1-\beta}\Big],$$
$$P_2(k;\beta) = \frac{1}{n^2}\Big[(1-\beta)^2k^2 + 3k + \frac{2!}{1-\beta}\Big],$$
$$P_3(k;\beta) = \frac{1}{n^3}\Big[(1-\beta)^3k^3 + 6(1-\beta)k^2 + \frac{(11-8\beta)k}{1-\beta} + \frac{3!}{1-\beta}\Big],$$

$$P_4(k;\beta)=\frac{1}{n^4}\Big[(1-\beta)^4k^4+10(1-\beta)^2k^3+5(7-4\beta)k^2+\frac{10(5-3\beta)k}{1-\beta}+\frac{4!}{1-\beta}\Big],$$

$$P_5(k;\beta)=\frac{1}{n^5}\Big[(1-\beta)^5k^5+15(1-\beta)^3k^4+5(1-\beta)(17-8\beta)k^3 +\frac{15(15-20\beta+6\beta^2)k^2}{1-\beta}+\frac{(274-144\beta)k}{1-\beta}+\frac{5!}{1-\beta}\Big].$$

LEMMA 3.3. *If the r-th order moment with monomials $e_r(t)=t^r, r=0,1,\dots$ of the operators* (1.2) *be defined by*

$$T^{a,\beta}_{n,r}(x):=D^{a,\beta}_n(e_r,x)=\sum_{k=0}^{\infty}\frac{\langle L^{(\beta)}_{n,k}(t),t^r\rangle}{\langle L^{(\beta)}_{n,k}(t),1\rangle}b^a_{n,k}(x)=\sum_{k=0}^{\infty}P_r(k;\beta)b^a_{n,k}(x).$$

The first few are

$$T^{a,\beta}_{n,0}(x)=1,\quad T^{a,\beta}_{n,1}(x)=x(1-\beta)+\frac{ax(1-\beta)}{n(1+x)}+\frac{1}{n(1-\beta)},$$

$$T^{a,\beta}_{n,2}(x)=x^2(1-\beta)^2+\frac{1}{n}\Big[x^2(1-\beta)^2+x(1-\beta)^2+\frac{2ax^2(1-\beta)^2}{1+x}+3x\Big] +\frac{1}{n^2}\Big[\frac{a^2x^2(1-\beta)^2}{(1+x)^2}+\frac{ax(1-\beta)^2}{1+x}+\frac{3ax}{1+x}+\frac{2}{1-\beta}\Big],$$

$$\begin{aligned}T^{a,\beta}_{n,3}(x)&=x^3(1-\beta)^3\\&+\frac{1}{n}\Big[3x^3(1-\beta)^3+3x^2(1-\beta)^3+\frac{3ax^3(1-\beta)^3}{(1+x)}+6x^2(1-\beta)\Big]\\&+\frac{1}{n^2}\Big[2x^3(1-\beta)^3+3x^2(1-\beta)^3+x(1-\beta)^3+\frac{3ax^3(1-\beta)^3}{(1+x)}\\&\quad+\frac{6ax^2(1-\beta)^3}{(1+x)}+\frac{3a^2x^3(1-\beta)^3}{(1+x)^2}+6x^2(1-\beta)\\&\quad+6x(1-\beta)+\frac{12ax^2(1-\beta)}{(1+x)}+\frac{(11-8\beta)x}{(1-\beta)}\Big]\\&+\frac{1}{n^3}\Big[\frac{3a^2x^2(1-\beta)^3}{(1+x)^2}+\frac{ax(1-\beta)^3}{(1+x)}+\frac{a^3x^3(1-\beta)^3}{(1+x)^3}\\&\quad+\frac{6a^2x^2(1-\beta)}{(1+x)^2}+\frac{6ax(1-\beta)}{(1+x)}+\frac{(11-8\beta)ax}{(1-\beta)(1+x)}+\frac{6}{(1-\beta)}\Big]\end{aligned}$$

$$\begin{aligned}T^{a,\beta}_{n,4}(x)&=x^4(1-\beta)^4\\&+\frac{1}{n}\Big[6x^4(1-\beta)^4+6x^3(1-\beta)^4+\frac{4ax^4(1-\beta)^4}{(1+x)}+10x^3(1-\beta)^2\Big]\\&+\frac{1}{n^2}\Big[11x^4(1-\beta)^4+18x^3(1-\beta)^4+7x^2(1-\beta)^4+\frac{12ax^4(1-\beta)^4}{(1+x)}\\&\quad+\frac{18ax^3(1-\beta)^4}{(1+x)}+\frac{6a^2x^4(1-\beta)^4}{(1+x)^2}+30x^3(1-\beta)^2\end{aligned}$$

$$
\begin{aligned}
&\quad + 30x^2(1-\beta)^2 + \frac{30ax^3(1-\beta)^2}{(1+x)} + (35-20\beta)x^2\Big] \\
&+ \frac{1}{n^3}\Big[6x^4(1-\beta)^4 + 12x^3(1-\beta)^4 + 7x^2(1-\beta)^4 + x(1-\beta)^4 \\
&\quad + \frac{8ax^4(1-\beta)^4}{(1+x)} + \frac{18ax^3(1-\beta)^4}{(1+x)} + \frac{14ax^2(1-\beta)^4}{(1+x)} \\
&\quad + \frac{3a^2x^3(x^2+4x+3)(1-\beta)^4}{(1+x)^3} + \frac{9a^2x^3(1-\beta)^4}{(1+x)^2} + \frac{4a^3x^4(1-\beta)^4}{(1+x)^3} \\
&\quad + \frac{3a^2x^4(1-\beta)^4}{(1+x)^2} + 20x^3(1-\beta)^2 + 30x^2(1-\beta)^2 + 10x(1-\beta)^2 \\
&\quad + \frac{30ax^3(1-\beta)^2}{(1+x)} + \frac{60ax^2(1-\beta)^2}{(1+x)} + \frac{30a^2x^3(1-\beta)^2}{(1+x)^2} \\
&\quad + (35-20\beta)x^2 + (35-20\beta)x + \frac{(70-40\beta)ax^2}{(1+x)} + \frac{(50-30\beta)x}{(1-\beta)}\Big] \\
&+ \frac{1}{n^4}\Big[\frac{7a^2x^2(1-\beta)^4}{(1+x)^2} + \frac{ax(1-\beta)^4}{(1+x)} + \frac{6a^3x^3(1-\beta)^4}{(1+x)^3} + \frac{a^4x^4(1-\beta)^4}{(1+x)^4} \\
&\quad + \frac{30a^2x^2(1-\beta)^2}{(1+x)^2} + \frac{10ax(1-\beta)^2}{(1+x)} + \frac{10a^3x^3(1-\beta)^2}{(1+x)^3} \\
&\quad + \frac{(35-20\beta)a^2x^2}{(1+x)^2} + \frac{(35-20\beta)ax}{(1+x)} + \frac{(50-30\beta)ax}{(1-\beta)(1+x)} + \frac{24}{(1-\beta)}\Big].
\end{aligned}
$$

Proof. Obviously by (1.2), we have $T_{n,0}^{a,\beta}(x) = 1$. Next, by definition of $T_{n,r}^{a,\beta}(x)$, we have

$$
T_{n,r}^{a,\beta}(x) = \sum_{k=0}^{\infty} \frac{\langle L_{n,k}^{(\beta)}(t), t^r\rangle}{\langle L_{n,k}^{(\beta)}(t), 1\rangle} b_{n,k}^a(x) = \sum_{k=0}^{\infty} P_r(k;\beta)\, b_{n,k}^a(x).
$$

Thus, using Lemma 3.1 and Lemma 3.2, we have

$$
\begin{aligned}
T_{n,1}^{a,\beta}(x) &= \sum_{k=0}^{\infty} b_{n,k}^a(x) P_1(k;\beta) = \sum_{k=0}^{\infty} b_{n,k}^a(x)\frac{1}{n}\Big[(1-\beta)k + \frac{1}{1-\beta}\Big] \\
&= (1-\beta)\mu_{n,1}^a(x) + \frac{1}{n(1-\beta)}\mu_{n,0}^a(x) = x(1-\beta) + \frac{ax(1-\beta)}{n(1+x)} + \frac{1}{n(1-\beta)} \\
T_{n,2}^{a,\beta}(x) &= \sum_{k=0}^{\infty} b_{n,k}^a(x) P_2(k;\beta) = \sum_{k=0}^{\infty} b_{n,k}^a(x)\frac{1}{n^2}\Big[(1-\beta)^2k^2 + 3k + \frac{2}{1-\beta}\Big] \\
&= (1-\beta)^2\mu_{n,2}^a(x) + \frac{3}{n}\mu_{n,1}^a(x) + \frac{2}{n^2(1-\beta)}\mu_{n,0}^a(x) \\
&= (1-\beta)^2\Big[x^2 + \frac{x^2}{n} + \frac{x}{n} + \frac{a^2x^2}{n^2(1+x)^2} + \frac{2ax^2}{n(1+x)} + \frac{ax}{n^2(1+x)}\Big] \\
&\quad + \frac{3}{n}\Big[x + \frac{ax}{n(1+x)}\Big] + \frac{2}{n^2(1-\beta)}
\end{aligned}
$$

$$= x^2(1-\beta)^2 + \frac{1}{n}\Big[x^2(1-\beta)^2 + x(1-\beta)^2 + \frac{2ax^2(1-\beta)^2}{1+x} + 3x\Big]$$
$$+ \frac{1}{n^2}\Big[\frac{a^2x^2(1-\beta)^2}{(1+x)^2} + \frac{ax(1-\beta)^2}{1+x} + \frac{3ax}{1+x} + \frac{2}{1-\beta}\Big].$$

A continuation of this process will provide $T_{n,r}^{a,\beta}(x)$ for cases of $r \geqslant 3$. □

REMARK 3.2. If we denote the central moment as $U_{n,r}^{a,\beta}(x) = D_n^{a,\beta}((t-x)^r, x)$, then

$$U_{n,1}^{a,\beta}(x) = -x\beta + \frac{ax(1-\beta)}{n(1+x)} + \frac{1}{n(1-\beta)},$$

$$U_{n,2}^{a,\beta}(x) = x^2\beta^2 + \frac{1}{n}\Big[3x + x(1+x)(1-\beta)^2 - \frac{2ax^2\beta(1-\beta)}{1+x} - \frac{2x}{1-\beta}\Big]$$
$$+ \frac{1}{n^2}\Big[\frac{x^2a^2(1-\beta)^2}{(1+x)^2} + \frac{ax(1-\beta)^2}{1+x} + \frac{3ax}{1+x} + \frac{2}{1-\beta}\Big],$$

$$U_{n,3}^{a,\beta}(x) = x^3(1-\beta)^3 - x^3 - 3x^3(1-\beta)^2 + 3x^3(1-\beta)$$
$$+ \frac{1}{n}\Big[3x^3(1-\beta)^3 + 3x^2(1-\beta)^3 + \frac{3ax^3(1-\beta)^3}{(1+x)}$$
$$+ 6x^2(1-\beta) - 3x^3(1-\beta)^2 - \frac{6ax^3(1-\beta)^2}{(1+x)}$$
$$- 3x^2(1-\beta)^2 - 9x^2 + \frac{3ax^3(1-\beta)}{(1+x)} + \frac{3x^2}{(1-\beta)}\Big]$$
$$+ \frac{1}{n^2}\Big[2x^3(1-\beta)^3 + 3x^2(1-\beta)^3 + x(1-\beta)^3 + \frac{3ax^3(1-\beta)^3}{(1+x)}$$
$$+ \frac{6ax^2(1-\beta)^3}{(1+x)} + \frac{3a^2x^3(1-\beta)^3}{(1+x)^2} + 6x^2(1-\beta)$$
$$+ 6x(1-\beta) + \frac{12ax^2(1-\beta)}{(1+x)} + \frac{(11-8\beta)x}{(1-\beta)}$$
$$- \frac{3a^2x^3(1-\beta)^2}{(1+x)^2} - \frac{3ax^2(1-\beta)^2}{(1+x)} - \frac{9ax^2}{(1+x)} - \frac{6x}{(1-\beta)}\Big]$$
$$+ \frac{1}{n^3}\Big[\frac{3a^2x^2(1-\beta)^3}{(1+x)^2} + \frac{ax(1-\beta)^3}{(1+x)} + \frac{a^3x^3(1-\beta)^3}{(1+x)^3}$$
$$+ \frac{6a^2x^2(1-\beta)}{(1+x)^2} + \frac{6ax(1-\beta)}{(1+x)} + \frac{(11-8\beta)ax}{(1-\beta)(1+x)} + \frac{6}{(1-\beta)}\Big]$$

$$U_{n,4}^{a,\beta}(x) = x^4(1-\beta)^4 + x^4 + 6x^4(1-\beta)^2 - 4x^4(1-\beta) - 4x^4(1-\beta)^3$$
$$+ \frac{1}{n}\Big[6x^4(1-\beta)^4 + 6x^3(1-\beta)^4 + \frac{4ax^4(1-\beta)^4}{1+x}$$
$$+ 10x^3(1-\beta)^2 + 6x^4(1-\beta)^2 + \frac{12ax^4(1-\beta)^2}{1+x}$$

$$
\begin{aligned}
&+ 6x^3(1-\beta)^2 + 18x^3 - \frac{4ax^4(1-\beta)}{1+x} - \frac{4x^3}{1-\beta} \\
&- 12x^4(1-\beta)^3 - 12x^3(1-\beta)^3 - \frac{12ax^4(1-\beta)^3}{1+x} - 24x^3(1-\beta)\Big] \\
+ \frac{1}{n^2}\Big[&11x^4(1-\beta)^4 + 18x^3(1-\beta)^4 + 7x^2(1-\beta)^4 + \frac{12ax^4(1-\beta)^4}{1+x} \\
&+ \frac{18ax^3(1-\beta)^4}{1+x} + \frac{6a^2x^4(1-\beta)^4}{(1+x)^2} + 30x^3(1-\beta)^2 + 30x^2(1-\beta)^2 \\
&+ \frac{30ax^3(1-\beta)^2}{1+x} + (35-20\beta)x^2 + \frac{6a^2x^4(1-\beta)^2}{(1+x)^2} + \frac{6ax^3(1-\beta)^2}{1+x} \\
&+ \frac{18ax^3}{1+x} + \frac{12x^2}{1-\beta} - 8x^4(1-\beta)^3 - 12x^3(1-\beta)^3 - 4x^2(1-\beta)^3 \\
&- \frac{12ax^4(1-\beta)^3}{1+x} - \frac{24ax^3(1-\beta)^3}{1+x} - \frac{12a^2x^4(1-\beta)^3}{(1+x)^2} \\
&\quad - 24x^3(1-\beta) - 24x^2(1-\beta) - \frac{48ax^3(1-\beta)}{1+x} - \frac{4(11-8\beta)x^2}{1-\beta}\Big] \\
+ \frac{1}{n^3}\Big[&6x^4(1-\beta)^4 + 12x^3(1-\beta)^4 + 7x^2(1-\beta)^4 + x(1-\beta)^4 + \frac{8ax^4(1-\beta)^4}{1+x} \\
&+ \frac{18ax^3(1-\beta)^4}{1+x} + \frac{14ax^2(1-\beta)^4}{1+x} + \frac{3a^2x^3(x^2+4x+3)(1-\beta)^4}{(1+x)^3} \\
&+ \frac{9a^2x^3(1-\beta)^4}{(1+x)^2} + \frac{4a^3x^4(1-\beta)^4}{(1+x)^3} + \frac{3a^2x^4(1-\beta)^4}{(1+x)^2} + 20x^3(1-\beta)^2 \\
&+ 30x^2(1-\beta)^2 + 10x(1-\beta)^2 + \frac{30ax^3(1-\beta)^2}{1+x} + \frac{60ax^2(1-\beta)^2}{1+x} \\
&+ \frac{30a^2x^3(1-\beta)^2}{(1+x)^2} + (35-20\beta)x^2 + (35-20\beta)x + \frac{(70-40\beta)ax^2}{1+x} \\
&+ \frac{(50-30\beta)x}{1-\beta} - \frac{12a^2x^3(1-\beta)^3}{(1+x)^2} - \frac{4ax^2(1-\beta)^3}{1+x} - \frac{4a^3x^4(1-\beta)^3}{(1+x)^3} \\
&- \frac{24a^2x^3(1-\beta)}{(1+x)^2} - \frac{24ax^2(1-\beta)}{1+x} - \frac{4(11-8\beta)ax^2}{(1-\beta)(1+x)} - \frac{24x}{1-\beta}\Big] \\
+ \frac{1}{n^4}\Big[&\frac{7a^2x^2(1-\beta)^4}{(1+x)^2} + \frac{ax(1-\beta)^4}{1+x} + \frac{6a^3x^3(1-\beta)^4}{(1+x)^3} + \frac{a^4x^4(1-\beta)^4}{(1+x)^4} \\
&+ \frac{30a^2x^2(1-\beta)^2}{(1+x)^2} + \frac{10ax(1-\beta)^2}{1+x} + \frac{10a^3x^3(1-\beta)^2}{(1+x)^3} \\
&+ \frac{(35-20\beta)a^2x^2}{(1+x)^2} + \frac{(35-20\beta)ax}{1+x} + \frac{(50-30\beta)ax}{(1-\beta)(1+x)} + \frac{24}{1-\beta}\Big].
\end{aligned}
$$

4. Direct estimates for general case

In this section, we establish the following direct results:

PROPOSITION 4.1. *Let f be a continuous function on $[0,\infty)$ and $\beta=\beta_n\to 0$ as $n\to\infty$, the sequence $\{D_n^{a,\beta}(f,x)\}$ converges uniformly to $f(x)$ in $[a,b]\subset[0,\infty)$.*

PROOF. Using Lemma 3.3, we have

$$D_n^{a,\beta}(e_0,x)=1,\quad D_n^{a,\beta}(e_1,x)=x(1-\beta)+\frac{ax(1-\beta)}{n(1+x)}+\frac{1}{n(1-\beta)}$$

and

$$D_n^{a,\beta}(e_2,x)=x^2(1-\beta)^2+\frac{1}{n}\Big[x^2(1-\beta)^2+x(1-\beta)^2+\frac{2ax^2(1-\beta)^2}{1+x}+3x\Big]$$
$$+\frac{1}{n^2}\Big[\frac{a^2x^2(1-\beta)^2}{(1+x)^2}+\frac{ax(1-\beta)^2}{1+x}+\frac{3ax}{1+x}+\frac{2}{1-\beta}\Big].$$

Obviously, $D_n^{a,\beta}(e_0,x)$, $D_n^{a,\beta}(e_1,x)$ and $D_n^{a,\beta}(e_2,x)$ converges uniformly to 1, x and x^2 respectively on every compact subset of $[0,\infty)$. Thus, the required result follows from the well known Bohman–Korovkin theorem. □

THEOREM 4.1. *Let f be a bounded integrable function on $[0,\infty)$ and has the second derivative at a point $x\in[0,\infty)$, then with the condition $\beta=\beta_n\to 0$ as $n\to\infty$, it follows that*

$$\lim_{n\to\infty} n[D_n^{a,\beta_n}(f,x)-f(x)]=\Big[1+\frac{ax}{1+x}\Big]f'(x)+\Big[\frac{x^2+2x}{2}\Big]f''(x).$$

PROOF. By Taylor's expansion of f, we have

$$f(t)=f(x)+f'(x)(t-x)+\frac{1}{2}f''(x)(t-x)^2+r(t,x)(t-x)^2, \tag{4.1}$$

where $r(t,x)$ is the remainder term and $\lim_{t\to x} r(t,x)=0$. Operating $D_n^{a,\beta}$ to the equation (4.1), we obtain

$$D_n^{a,\beta}(f,x)-f(x)=D_n^{a,\beta}(t-x,x)f'(x)+D_n^{a,\beta}((t-x)^2,x)\frac{f''(x)}{2}$$
$$+D_n^{a,\beta}(r(t,x)(t-x)^2,x)$$

Using the Cauchy–Schwarz inequality, we have

$$D_n^{a,\beta}(r(t,x)(t-x)^2,x)\leqslant\sqrt{D_n^{a,\beta}(r^2(t,x),x)}\sqrt{D_n^{a,\beta}((t-x)^4,x)}. \tag{4.2}$$

As $r^2(x,x)=0$ and $r^2(t,x)\in C_2^*[0,\infty)$, we have

$$\lim_{n\to\infty} D_n^{a,\beta_n}(r^2(t,x),x)=r^2(x,x)=0 \tag{4.3}$$

uniformly with respect to $x\in[0,A]$, for some $A>0$. Now from (4.2), (4.3) and from Remark 3.2, we get $\lim_{n\to\infty} nD_n^{a,\beta_n}(r(t,x)(t-x)^2,x)=0$. Thus

$$\lim_{n\to\infty} n(D_n^{a,\beta_n}(f,x)-f(x))$$
$$=\lim_{n\to\infty} n\Big[D_n^{a,\beta_n}(t-x,x)f'(x)+\frac{1}{2}f''(x)D_n^{a,\beta_n}((t-x)^2,x)+D_n^{a,\beta_n}(r(t,x)(t-x)^2,x)\Big]$$
$$=\Big[1+\frac{ax}{1+x}\Big]f'(x)+\Big[\frac{x^2+2x}{2}\Big]f''(x).\quad\square$$

We denote the norm $\|f\| = \sup_{x\in[0,\infty)} |f(x)|$. For $f \in C_B[0,\infty)$ and $\delta > 0$, the m-th order modulus of continuity is defined as

$$\omega_m(f,\delta) = \sup_{0\leqslant h\leqslant \delta}\ \sup_{x\in[0,\infty)} |\Delta_h^m f(x)|,$$

where Δ is the forward difference. In case $m = 1$, we mean the usual modulus of continuity denoted by $\omega(f,\delta)$. Peetre's K-functional is defined as

$$K_2(f,\delta) = \inf_{g\in C_B^2[0,\infty)} \{\|f-g\| + \delta\|g''\| : g \in C_B^2[0,\infty)\},$$

where

$$C_B^2[0,\infty) = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}.$$

THEOREM 4.2. *Let $f \in C_B[0,\infty)$ and $0 < \beta < 1$, then*

$$|D_n^{a,\beta}(f,x) - f(x)| \leqslant C\omega_2\big(f, \sqrt{\delta_n}\big) + \omega\Big(f, \frac{ax(1-\beta)}{n(1+x)} + \frac{1}{n(1-\beta)} - x\beta\Big)$$

where C is a positive constant and δ_n is given by

$$\delta_n = 2x^2\beta^2 + \frac{1}{n}\Big[3x + x(x+1)(1-\beta)^2 - \frac{4ax^2\beta(1-\beta)}{1+x} - \frac{2x}{1-\beta} - \frac{2x\beta}{1-\beta}\Big] + \frac{1}{n^2}\Big[\frac{2a^2x^2(1-\beta)^2}{(1+x)^2} + \frac{5ax}{1+x} + \frac{ax(1-\beta)^2}{1+x} + \frac{2}{1-\beta} + \frac{1}{(1-\beta)^2}\Big].$$

with $\beta = \beta_n \to 0$ as $n \to \infty$.

PROOF. We introduce the auxiliary operators $\bar{D}_n^{a,\beta} : C_B[0,\infty) \to C_B[0,\infty)$ by

$$\bar{D}_n^{a,\beta}(f,x) = D_n^{a,\beta}(f,x) - f\Big(x(1-\beta) + \frac{ax(1-\beta)}{n(1+x)} + \frac{1}{n(1-\beta)}\Big) + f(x). \tag{4.4}$$

These operators are linear and preserve the linear functions in view of Lemma 3.3. Let $g \in C_B^2[0,\infty)$ and $x, t \in [0,\infty)$. By Taylor's expansion, we have

$$g(t) = g(x) + (t-x)\,g'(x) + \int_x^t (t-u)\,g''(u)\,du,$$

Hence,

$$\begin{aligned}
&|\bar{D}_n^{a,\beta}(g,x) - g(x)| \\
&\leqslant \bar{D}_n^{a,\beta}\Big(\Big|\int_x^t (t-u)g''(u)du\Big|, x\Big) \leqslant D_n^{a,\beta}\Big(\Big|\int_x^t (t-u)g''(u)du\Big|, x\Big) \\
&\quad + \Big|\int_x^{x(1-\beta)+\frac{ax(1-\beta)}{n(1+x)}+\frac{1}{n(1-\beta)}} \Big(x(1-\beta) + \frac{ax(1-\beta)}{n(1+x)} + \frac{1}{n(1-\beta)} - u\Big)g''(u)du\Big| \\
&\leqslant D_n^{a,\beta}((t-x)^2, x)\|g''\| \\
&\quad + \Big|\int_x^{x(1-\beta)+\frac{ax(1-\beta)}{n(1+x)}+\frac{1}{n(1-\beta)}} \Big(\frac{ax(1-\beta)}{n(1+x)} + \frac{1}{n(1-\beta)} - x\beta\Big)du\Big|\|g''\|
\end{aligned}$$

Next, using Remark 3.2, we have

$$
(4.5)\quad \begin{aligned} |\bar{D}_n^{a,\beta}(g,x)-g(x)| \leqslant \Big[& D_n^{a,\beta}((t-x)^2,x) \\ & + \Big(\frac{ax(1-\beta)}{n(1+x)}+\frac{1}{n(1-\beta)}-x\beta\Big)^2\Big]\|g''\| = \delta_n\|g''\|. \end{aligned}
$$

Since

$$
|D_n^{a,\beta}(f,x)| \leqslant \sum_{k=0}^{\infty}\frac{\langle L_{n,k}^{(\beta)}(t),|f(t)|\rangle}{\langle L_{n,k}^{(\beta)}(t),1\rangle}b_{n,k}^{a}(x) \leqslant \|f\|\sum_{k=0}^{\infty}\frac{\langle L_{n,k}^{(\beta)}(t),1\rangle}{\langle L_{n,k}^{(\beta)}(t),1\rangle}b_{n,k}^{a}(x) \leqslant \|f\|.
$$

Now by (4.4), we have

$$
(4.6)\qquad ||\bar{D}_n^{a,\beta}(f,x)|| \leqslant |||D_n^{a,\beta}(f,x)\| + 2\|f\| \leqslant 3\|f\|, \quad f\in C_B[0,\infty).
$$

Using (4.4), (4.5) and (4.6), we have

$$
\begin{aligned} |D_n^{a,\beta}(f,x)-f(x)| &\leqslant |\bar{D}_n^{a,\beta}(f-g,x)-(f-g)(x)| + |\bar{D}_n^{a,\beta}(g,x)-g(x)| \\ &\quad + \Big|f\Big(x(1-\beta)+\frac{ax(1-\beta)}{n(1+x)}+\frac{1}{n(1-\beta)}\Big)-f(x)\Big| \\ &\leqslant 4\|f-g\| + \delta_n\|g''\| \\ &\quad + \Big|f\Big(x(1-\beta)+\frac{ax(1-\beta)}{n(1+x)}+\frac{1}{n(1-\beta)}\Big)-f(x)\Big| \\ &\leqslant C\{\|f-g\|+\delta_n\|g''\|\} + \omega\Big(f,\frac{ax(1-\beta)}{n(1+x)}+\frac{1}{n(1-\beta)}-x\beta\Big). \end{aligned}
$$

Taking infimum over all $g\in C_B^2[0,\infty)$, and using the inequality

$$
K_2(f,\delta) \leqslant C\omega_2(f,\sqrt{\delta}), \quad \delta>0
$$

due to [**4**], we get the desired assertion. □

References

1. P. N. Agrawal, V. Gupta, A. S. Kumar, A. Kajla, *Generalized Baskakov–Szász type operators*, Appl. Math. Comput. **236** (2014), 311–324.
2. A. Aral, T. Acar, *Weighted approximation by new Bernstein–Chlodowsky–Gadjiev operators*, Filomat **27**(2) (2013), 371–380.
3. S. S. Dragomir, *Approximation of continuous linear functionals in real normed spaces*, Rend. Mat. Appl., VII. Ser. **12**(2) (1992), 357–364.
4. R. A. DeVore, G. G. Lorentz, *Constructive Approximation*, Springer, Berlin, 1993.
5. Z. Ditzian, V. Totik, *Moduli of Smoothness*, Springer-Verlag, New York, 1987.
6. J. L. Durrmeyer, *Une formule d' inversion de la Transformee de Laplace: Applications a la Theorie des Moments, These de 3e Cycle*, Faculte des Sciences de l' Universite de Paris, 1967.
7. A. Erencin, *Durrmeyer type modification of generalized Baskakov operators*, Appl. Math. Comput. **218**(3) (2011), 4384–4390.
8. N. K. Govil, V. Gupta, *Simultaneous Approximation for Stancu-Type Generalization of Certain Summation Integral-Type Operators*, in: G. V. Milovanović and Th. M. Rassias (eds.) *Analytic Number Theory, Approximation Theory, and Special Functions*, Springer, 2014, 531–548.
9. V. Gupta, R. P. Agarwal, *Convergence Estimates in Approximation Theory*, Springer, 2014.

10. V. Gupta, R. P. Agarwal, D. K. Verma, *Approximation for a new sequence of summation-integral type operators*, Adv. Math. Sci. Appl. **23**(1) (2013), 35–42.
11. V. Gupta, J. Sinha, *Simultaneous Approximation for Generalized Baskakov–Durrmeyer-Type Operators*, Mediterr. J. Math. **4**(4)(2007), 483–495.
12. V. Gupta, G. S. Srivastava, *Simultaneous approximation by Baskakov–Szász type operators*, Bull. Math. Soc. Sci. Math. Roum., Nouv. Sér. **37**(85)(3–4) (1993), 73–85.
13. G. C. Jain, *Approximation of functions by a new class of linear operators*, J. Austral. Math. Soc. **13**(3) (1972), 271–276.
14. M.-J. Luo, G. V. Milovanović, P. Agarwal, *Some results on the extended beta and extended hypergeometric functions*, Appl. Math. Comput. **248** (2014), 631–651.
15. V. Mihesan, *Uniform approximation with positive linear operators generated by generalized Baskakov method*, Automat. Comput. Appl. Math. **7**(1) (1998), 34–37.
16. V. Gupta, G. C. Greubel, *Moment Estimations of New Szász–Mirakyan–Durrmeyer Operators*, Appl. Math. Comput. **271** (2015), 540–547.
17. N. Ispir, *On modified Baskakov operators on weighted spaces*, Turk. J. Math. **26**(3) (2001), 355–365.
18. H. M. Srivastava, V. Gupta, *A certain family of summation-integral type operators*, Math. Comput. Modelling **37**(12)(2003), 1307–1315.

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