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LOGARITHMIC BLOCH SPACE AND ITS PREDUAL

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ABSTRACT. We consider the space $\mathfrak{B}^1_{\log^\alpha}$, of analytic functions on the unit disk $\mathbb{D}$, defined by the requirement $\int_{\mathbb{D}} |f'(z)|\phi(|z|)\,dA(z) < \infty$, where $\phi(r) = \log^\alpha(1/(1-r))$ and show that it is a predual of the "$\log^\alpha$-Bloch" space and the dual of the corresponding little Bloch space. We prove that a function $f(z) = \sum_{n=0}^\infty a_n z^n$ with $a_n \downarrow 0$ is in $\mathfrak{B}^1_{\log^\alpha}$ iff $\sum_{n=0}^\infty \log^\alpha(n+2)/(n+1) < \infty$ and apply this to obtain a criterion for membership of the Libera transform of a function with positive coefficients in $\mathfrak{B}^1_{\log^\alpha}$. Some properties of the Cesàro and the Libera operator are considered as well.

1. Introduction and some results

Let $H(\mathbb{D})$ denote the space of all functions analytic in the unit disk $\mathbb{D}$ of the complex plane. Endowed with the topology of uniform convergence on compact subsets of $\mathbb{D}$, the class $H(\mathbb{D})$ becomes a complete locally convex space. In this paper we are concerned with the predual of the space $\mathfrak{B}_{\log^\alpha}$, $\alpha \in \mathbb{R}$,

$$\mathfrak{B}_{\log^\alpha} = \Big\{ f \in H(\mathbb{D}) : |f'(z)| = \mathcal{O}\Big((1-|z|)^{-1} \log^\alpha \frac{2}{1-|z|}\Big)\Big\}.$$

The norm in $\mathfrak{B}_{\log^\alpha}$ is defined by

$$\|f\|_{\mathfrak{B}_{\log^\alpha}} = |f(0)| + \sup_{z\in\mathbb{D}} |f'(z)|(1-|z|)\log^{-\alpha}\frac{1}{1-|z|}.$$

The subspace, $\mathfrak{b}_{\log^\alpha}$, of $\mathfrak{B}_{\log^\alpha}$ is defined by replacing "$\mathcal{O}$" with "o". It will be proved:
(A) The dual of $\mathfrak{b}_{\log^\alpha}$ is isomorphic to $\mathfrak{B}^1_{\log^\alpha}$,

$$\mathfrak{B}^1_{\log^\alpha} = \Big\{ f : \|f\|_{\mathfrak{B}^1_{\log^\alpha}} = |f(0)| + \int_{\mathbb{D}} |f'(z)| \log^\alpha \frac{2}{1-|z|}\, dA(z) < \infty \Big\}, \tag{1.1}$$

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and the dual of $\mathfrak{B}^1_{\log^\alpha}$ is isomorphic to $\mathfrak{B}_{\log^\alpha}$, in both cases with respect to the bilinear form

$$\langle f, g\rangle = \lim_{r\uparrow 1}\sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n)r^{2n}. \tag{1.2}$$

(In (1.1) dA stands for the normalized Lebesgue measure on $\mathbb{D}$.) This extends the well-known result on the Bloch space and the little Bloch space $\mathfrak{b} := \mathfrak{b}_{\log^0}$.

These spaces are Banach spaces, and the space $\mathfrak{b}_{\log^\alpha}$ coincides with the closure in $\mathfrak{B}_{\log^\alpha}$ of the set of all polynomials. The space $\mathfrak{B}_{\log} := \mathfrak{B}_{\log^1}$ occurs naturally in the study of pointwise multipliers on the usual Bloch space $\mathfrak{B} := \mathfrak{B}_{\log^0}$ (see [**3**]).

One of interesting properties of $\mathfrak{B}^1_{\log^\alpha}$ is described in the following theorem:

THEOREM 1.1. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, where $\{a_n\}$ is a nonincreasing sequence, of real numbers, tending to zero. Let $\alpha \geqslant -1$. Then f belongs to $\mathfrak{B}^1_{\log^\alpha}$ if and only if*

$$S_\alpha(f) := \sum_{n=0}^{\infty} \frac{a_n \log^\alpha(n+2)}{n+1} < \infty. \tag{1.3}$$

Moreover, there is a constant C independent of $\{a_n\}$ such that $S_\alpha(f)/C \leqslant \|f\|_{\mathfrak{B}^1_{\log^\alpha}} \leqslant CS_\alpha(f)$.

PROOF. See Section 4. □

In the case $\alpha = 0$, this assertion is proved in [**17**]. We can take a_n to be the coefficients of the Libera transform of a function with positive coefficients. Namely, if $g(z) = \sum_{n=0}^{\infty} \hat{g}(n) z^n$ and

$$\sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} < \infty, \tag{1.4}$$

then the Libera transform $\mathcal{L}g$ of g is well defined as

$$\mathcal{L}g(z) = \frac{1}{1-z}\int_z^1 f(\zeta)\,d\zeta = \sum_{n=0}^{\infty} z^n \sum_{k=n}^{\infty} \frac{\hat{g}(k)}{k+1} \tag{1.5}$$

(see, e.g., [**12**]). If $\hat{g} \geqslant 0$, then condition (1.4) is also necessary for the existence of the integral in (1.8): take $z = 0$ to conclude that (1.8) implies the convergence of the integral

$$\int_0^1 g(t)\,dt = \sum_{n=0}^{\infty} \frac{\hat{g}(n)}{n+1}.$$

Then, as an application of Theorem 1.1 we get

THEOREM 1.2. *Let $\alpha > -1$, let $g \in H(\mathbb{D})$, and $\hat{g} \geqslant 0$. Then $\mathcal{L}g$ is in $\mathfrak{B}^1_{\log^\alpha}$ if and only if*

$$K_\alpha(g) := \sum_{n=0}^{\infty} \frac{\hat{g}(n)\log^{\alpha+1}(n+2)}{n+1} < \infty.$$

We have $K_\alpha(g)/C \leqslant \|\mathcal{L}g\|_{\mathfrak{B}^1_{\log^\alpha}} \leqslant CK_\alpha(g)$.

PROOF. See Section 4. □

In the general case, the integral in (1.5) need not exist, but it certainly exists if $g \in H(\overline{\mathbb{D}})$, which means that g is analytic in a neighborhood of the closed disk. By using Theorem 1.1 we shall prove that $\overline{\mathcal{L}}$ cannot be extended to a bounded operator from $\mathfrak{B}^1_{\log^\alpha}$ to $H(\mathbb{D})$, if $\alpha < 0$. In the case $\alpha \geqslant 0$, every function $g \in \mathfrak{B}^1_{\log^\alpha}$ satisfies (1.4), whence $\mathcal{L}$ is well defined, and we will show that $\mathcal{L}$ maps this space into $\mathfrak{B}^1_{\log^{\alpha-1}}$, when $\alpha > 0$. If $\alpha = 0$ we need a sort of "iterated" logarithmic space.

Cesàro operator. The dual of $H(\mathbb{D})$ is equal to $H(\overline{\mathbb{D}})$, where "$g \in H(\overline{\mathbb{D}})$" means that g is holomorphic in a neighborhood of $\overline{\mathbb{D}}$ (depending on g). The duality pairing is given

$$\langle f, g\rangle = \sum_{n=0}^{\infty} \hat{f}(n)\hat{g}(n), \tag{1.6}$$

where $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n \in H(\mathbb{D})$ and $g(z) = \sum_{n=0}^{\infty} \hat{g}(n)z^n \in H(\overline{\mathbb{D}})$, and the series is absolutely convergent (see, e.g., [**8**]). The Cesàro operator is defined on $H(\mathbb{D})$ as

$$\mathcal{C}f(z) = \sum_{n=0}^{\infty} z^n \frac{1}{n+1} \sum_{k=0}^{n} a_k, \quad f \in H(\mathbb{D}). \tag{1.7}$$

It is easy to verify that the adjoint of $\mathcal{C} : H(\mathbb{D}) \mapsto H(\mathbb{D})$ is equal to $\overline{\mathcal{L}} : H(\overline{\mathbb{D}}) \mapsto H(\overline{\mathbb{D}})$, under the pairing (1.6), and vice versa (see, e.g., [**12**]).

The operators $\mathcal{C}$ and $\overline{\mathcal{L}}$ acting on H^p spaces were first studied by Siskakis in 1987. In [**21**] he proved that $\mathcal{C}$ is bounded on H^p for $1 < p < \infty$, and that $\overline{\mathcal{L}}$ can be extended to a bounded operator on H^p, $1 < p < \infty$, and obtained some results on their spectra and norms. A few years later he proved the boundedness of the Cesàro operator on H^1 ([**22**]), while Miao proved its boundedness on H^p for $0 < p < 1$ ([**10**]). A short proof of the boundedness of $\mathcal{C}$ on H^p, $0 < p < \infty$, as well as a stronger result, can be also found in Nowak [**11**]. However, H^∞ is not mapped into itself by $\mathcal{C}$ (see [**4**]). If we write (1.7) as

$$z\mathcal{C}f(z) = \int_0^z \frac{f(\zeta)}{1-\zeta}\, d\zeta,$$

and hence

$$(z\mathcal{C}f(z))' = \frac{f(z)}{1-z},$$

we conclude that $\mathcal{C}$ maps H^∞ into the Bloch space (see [**4**]).

On the other hand, by using the inequality

$$|f(z)| = \mathcal{O}\Big(\log \frac{2}{1-|z|}\Big), \quad f \in \mathfrak{B},$$

and the analogous inequality for $f \in \mathfrak{b}$ (replace "$\mathcal{O}$" with "o"), we get:

(B) The operator $\mathcal{C}$ maps the space $\mathfrak{B}$ into $\mathfrak{B}_{\log}$, and $\mathfrak{b}$ into $\mathfrak{b}_{\log}$.

One of our aims is to generalize this assertion to some other values of α and then use assertion **(A)** together with the duality between $\mathcal{C}$ and $\overline{\mathcal{L}}$ to obtain an alternative proof of some results on the action of $\mathcal{L}$ from $\mathfrak{B}^1_{\alpha+1}$ to $\mathfrak{B}^1_{\alpha}$, where

$$\mathcal{L}f(z) = \int_0^1 f(t + (1-t)z)\, dt. \tag{1.8}$$

In particular we have:

(C) The operator $\mathcal{L}$ is well defined on $\mathfrak{B}^1_{\log}$ and maps it into $\mathfrak{B}^1$.

It should be noted that: (a) $\mathfrak{B}^1 \subsetneq H^1$; (b) $\mathcal{L}$ does not map $\mathfrak{B}^1$ into H^1 (see [**17**]); and (c) $\mathcal{L}$ maps $\mathfrak{B}$ into BMOA [**12**], which improves an earlier result, namely that $\mathcal{L}$ maps $\mathfrak{B}$ into $\mathfrak{B}$ ([**5, 24**]).

The formula (1.8) is obtained from (1.5) by integrating over the straight line joining z and 1. A sufficient (not necessary [**18**]) condition for the possibility of such integration is (1.4) ($g = f$).

In proving some of our results, in particular assertions (A) and (B), we use a sequence of polynomials constructed in [**6**] (see also [**7**] and [**16**]) to decompose the space into a sum which resembles a sum of finite-dimensional spaces (see Section 3).

2. Some more results

Some elementary facts concerning the cases when $\mathcal{L}f$ is well defined are collected in the following theorem, where

$$\ell^1_{-1} = \Big\{ g \in H(\mathbb{D}) : \|g\|_{\ell^1_{-1}} = \sum_{n=0}^{\infty} \frac{|\hat{g}(n)|}{n+1} < \infty \Big\}.$$

Theorem 2.1. *Let $\alpha \in \mathbb{R}$. Then:*

(a) *$\mathfrak{B}_{\log^\alpha} \subset \ell^1_{-1}$ for all α;*
(b) *$\mathfrak{B}^1_{\log^\alpha} \subset \ell^1_{-1}$ if and only if $\alpha \geqslant 0$;*
(c) *if $\alpha < 0$, then $\overline{\mathcal{L}}$ cannot be extended to a continuous operator from $\mathfrak{B}^1_{\log^\alpha}$ to $H(\mathbb{D})$.*

Proof. See Section 4. □

Remark 2.1. The inclusions in (a) and (b) are continuous. Assertion (c) says much more than simply that $\mathfrak{B}^1_{\log^\alpha} \not\subset \ell^1_{-1}$.

In the context of the action of $\mathcal{C}$ and $\mathcal{L}$ some new spaces occur: the space $\mathfrak{B}_{\mathrm{logg}}$ is defined by the requirement

$$|f'(z)| = \mathcal{O}\Big(\log\log\frac{4}{1-|z|}\Big),$$

the space $\mathfrak{b}_{\mathrm{logg}}$ defined by replacing "$\mathcal{O}$" with "o", and the space $\mathfrak{B}^1_{\mathrm{logg}}$ defined by

$$\int_{\mathbb{D}} |f'(z)| \log\log\frac{4}{1-|z|}\, dA(z) < \infty.$$

Our next result is

THEOREM 2.2. (a) *If* $\alpha > -1$*, then* $\mathcal{C}$ *maps the space* $\mathfrak{B}_{\log^\alpha}$*, resp.* $\mathfrak{b}_{\log^\alpha}$*, into* $\mathfrak{B}_{\log^{\alpha+1}}$*, resp.* $\mathfrak{b}_{\log^{\alpha+1}}$*.*

(b) $\mathcal{C}$ *maps the space* $\mathfrak{B}_{\log^{-1}}$*, resp.* $\mathfrak{b}_{\log^{-1}}$*, into* $\mathfrak{B}_{\mathrm{logg}}$*, resp.* $\mathfrak{b}_{\mathrm{logg}}$*.*

PROOF. See Section 5. □

REMARK 2.2. If $f \in \mathfrak{B}_{\log^\alpha}$ and $\alpha < -1$, then, as it can easily be shown, $f \in A(\mathbb{D})$, where $A(\mathbb{D})$ is the disk-algebra, i.e., the subset of H^∞ consisting of those f which have a continuous extension to the closed disk. Moreover, the modulus of continuity of the boundary function $f_*(\zeta)$, $\zeta \in \partial\mathbb{D}$, satisfies the condition

$$\omega(f_*, t) = \mathcal{O}\Big(t\Big(\log\frac{2}{t}\Big)^{\alpha+1}\Big), \quad t \downarrow 0.$$

This follows from the inequality

$$\omega(f_*, t) \leqslant C\int_{1-t}^{1} M_\infty(r, f')\,dr,$$

see [**15**, Theorem 2.2]. It should be noted that the modulus of continuity of f_* is "proportional" to that of $f(z)$, $z \in \mathbb{D}$, see [**23, 19**].

Concerning the Libera operator we shall prove, besides Theorem 2.1(c), the following facts.

THEOREM 2.3. (a) *If* $\alpha > 0$*, then* $\mathcal{L}$ *is well defined on* $\mathfrak{B}^1_{\log^\alpha}$ *and maps this space to* $\mathfrak{B}^1_{\log^{\alpha-1}}$*.*

(b) $\mathcal{L}$ *is well defined on* $\mathfrak{B}^1_{\mathrm{logg}}$ *and maps this space into* $\mathfrak{B}^1_{\log^{-1}}$*.*

(c) $\mathcal{L}$ *is well defined on* $\mathfrak{B}^1$ *and maps it into* $\mathfrak{B}^1_\alpha$ *for all* $\alpha < -1$*.*

PROOF. See Section 5. □

THEOREM 2.4. *Let* $\alpha \in \mathbb{R}$*. Then the dual of* $\mathfrak{b}_{\log^\alpha}$*, resp.* $\mathfrak{B}^1_{\log^\alpha}$*, is isomorphic to* $\mathfrak{B}^1_{\log^\alpha}$*, resp.* $\mathfrak{B}_{\log^\alpha}$ *under the pairing* (1.2)*. Similarly, the dual of* $\mathfrak{b}_{\mathrm{logg}}$*, resp.* $\mathfrak{B}^1_{\mathrm{logg}}$*, is isomorphic to* $\mathfrak{B}^1_{\mathrm{logg}}$*, resp.* $\mathfrak{B}_{\mathrm{logg}}$*, under the same pairing.*

PROOF. See Section 6. □

REMARK 2.3. The phrase "the dual of X is isomorphic to Y under the pairing (1.2)" means that if $f \in X$ and $g \in Y$, then the limit in (1.2) exists and the functional $\Phi(f) = \langle f, g\rangle$ is bounded on X; and on the other hand, if $\Phi \in X^*$, then there exists $g \in Y$ such that $\Phi(f) = \langle f, g\rangle$, and moreover, there exists a constant C independent of g such that $\|g\|_Y/C \leqslant \|\Phi\| \leqslant C\|g\|_Y$.

As an application of Theorems 2.2, 2.3, and 2.4, one can prove the following fact.

THEOREM 2.5. *Let* $\alpha > 0$*. Then the adjoint (with respect to* (1.2)*) of the operator* $\mathcal{L} : \mathfrak{B}^1_{\log^\alpha} \mapsto \mathfrak{B}^1_{\log^{\alpha-1}}$ *is equal to* $\mathcal{C} : \mathfrak{B}_{\log^{\alpha-1}} \mapsto \mathfrak{B}_{\log^\alpha}$*. The adjoint of the operator* $\mathcal{C} : \mathfrak{b}_{\log^{\alpha-1}} \mapsto \mathfrak{b}_{\log^\alpha}$ *is equal to* $\mathcal{L} : \mathfrak{B}^1_{\log^\alpha} \mapsto \mathfrak{B}^1_{\log^{\alpha-1}}$*. The analogous assertions hold in the case when* $\alpha = 0$*.*

3. Decompositions

In [**6**], a sequence $\{V_n\}_0^\infty$ was constructed in the following way.
Let ω be a C^∞-function on $\mathbb{R}$ such that

(1) $\omega(t) = 1$ for $t \leqslant 1$,
(2) $\omega(t) = 0$ for $t \geqslant 2$,
(3) ω is decreasing and positive on the interval $(1, 2)$.

Let $\varphi(t) = \omega(t/2) - \omega(t)$, and let $V_0(z) = 1 + z$, and, for $n \geqslant 1$,

$$V_n(z) = \sum_{k=0}^{\infty} \varphi(k/2^{n-1}) z^k = \sum_{k=2^{n-1}}^{2^{n+1}-1} \varphi(k/2^{n-1}) z^k.$$

The polynomials V_n have the following properties:

$$g(z) = \sum_{n=0}^{\infty} V_n * g(z), \text{ for } g \in H(\mathbb{D}); \tag{3.1}$$

$$\|V_n * g\|_p \leqslant C\|g\|_p, \quad \text{for } g \in H^p,\ p > 0; \tag{3.2}$$

$$\|V_n\|_p \asymp 2^{n(1-1/p)}, \text{ for all } p > 0, \tag{3.3}$$

where $*$ denotes the Hadamard product. Here $\|h\|_p$ denotes the norm in the p-Hardy space H^p,

$$\|h\|_p = \sup_{0<r<1} \Big(\frac{1}{2\pi}\int_0^{2\pi} |h(re^{i\theta})|\, d\theta\Big)^{1/p} = \sup_{0<r<1} M_p(r, g).$$

We need additional properties.

Lemma 3.1. *Let $P(z) = \sum_{k=m}^{j} a_k z^k$, $m < j$. Then*

$$r^j \|P\|_p \leqslant M_p(r, P) \leqslant r^m \|P\|_p, \quad 0 < r < 1.$$

When applied to the polynomial $P = V_n * g'$, this gives:

$$r^{2^{n+1}-1}\|V_n * g'\|_p \leqslant M_p(r, V_n * g') \leqslant r^{2^{n-1}-1}\|V_n * g'\|_p \text{ for } \ n \geqslant 1. \tag{3.4}$$

Another inequality will be used (see [**16**, Exercise 7.3.5]):

$$2^{n-1}\|V_n * g\|_p / C \leqslant \|V_n * g'\|_p \leqslant C 2^{n+1}\|V_n * g\|_p \text{ for } n \geqslant 1, \tag{3.5}$$

where C is a constant independent of n and g.

Theorem 3.1. *Let $\alpha \in \mathbb{R}$, and $f \in H(\mathbb{D})$. Then:*

(i) *$f \in \mathfrak{B}_{\log^\alpha}$ if and only if $\sup_{n\geqslant 0}(n+1)^{-\alpha}\|V_n * f\|_\infty < \infty$.*
(ii) *$f \in \mathfrak{b}_{\log^\alpha}$ if and only if $\lim_{n\to\infty}(n+1)^{-\alpha}\|V_n * f\|_\infty = 0$.*
(iii) *$f \in \mathfrak{B}^1_{\log^\alpha}$ if and only if $\sum_{n=0}^{\infty}(n+1)^{\alpha}\|V_n * f\|_1 < \infty$.*

Moreover, the inequality

$$C^{-1}\|f\|_{\mathfrak{B}_{\log^\alpha}} \leqslant \sup_{n\geqslant 0}(n+1)^{-\alpha}\|V_n * g\|_\infty \leqslant C^{-1}\|f\|_{\mathfrak{B}_{\log^\alpha}}$$

holds, where C is independent of f. The analogous inequality holds in the case of (iii) *as well.*

For the proof we need the following reformulation of [**9**, Proposition 4.1].

LEMMA 3.2. *Let φ be a continuous function on the interval $(0,1]$ such that $\varphi(x)/x^\gamma$ $(0<x<1)$ is nonincreasing, and $\varphi(x)/x^\beta$ $(0<x<1)$ is nondecreasing, where β and γ are positive constants independent of x.* [1] *Let*

$$F_1(r)=(1-r)^{-1/q}\varphi(1-r)\sup_{n\geqslant 1}\lambda_n r^{2^{n+1}-1},$$
$$F_2(r)=(1-r)^{1/q}\varphi(1-r)\sum_{n=0}^{\infty}\lambda_n r^{2^{n-1}-1},$$

where $\lambda_n\geqslant 0$, $0<q\leqslant\infty$. If $F=F_1$ or $F=F_2$, then

$$C^{-1}\|F\|_{L^q(0,1)}\leqslant\|\{\varphi(2^{-n})\lambda_n\}\|_{\ell^q}\leqslant C\|F\|_{L^q(0,1)}.$$

PROOF OF THEOREM 3.1. CASE (i). Let $\varphi(x)=x\log^{-\alpha}(2/x)$, and $q=\infty$. That φ is normal follows from the relation

$$\lim_{x\downarrow 0}\frac{x\varphi_\alpha'(x)}{\varphi_\alpha(x)}=1.$$

Let $\lambda_n=2^n\|V_n*f\|_\infty$. By (3.1), (3.2), (3.4), and (3.5), we have

$$C^{-1}|\hat f(1)|+C^{-1}\sup_{n\geqslant 1}\lambda_n r^{2^{n+1}-1}\leqslant M_\infty(r,f')\leqslant C|\hat f(1)|+C\sum_{n=1}^{\infty}\lambda_n r^{2^{n-1}-1}.$$

Hence, by Lemma 3.2, we obtain the desired result.

CASE (ii). In this case we can proceed in two ways:

1° Modify the proof of Lemma 3.2 to get the inequalities

$$C^{-1}\|F\|_{C_0[0,1]}\leqslant\|\{\varphi(2^{-n})\lambda_n\}\|_{\mathfrak{c}_0}\leqslant C\|F\|_{C_0[0,1]},$$

where $C_0[0,1]=\{u\in C[0,1]:u(1)=0\}$ and $\mathfrak{c}_0$ is the set of the sequences tending to zero.

2° Consider the spaces $\mathfrak{b}_{\log^\alpha}\subset\mathfrak{B}_{\log^\alpha}$ and $X=\{f:\|V_n*f\|=o((n+1)^\alpha)\}$, which is, by assertion (i) and its proof, a subspace of a space Y isomorphic to $\mathfrak{B}_{\log^\alpha}$. It is not hard to show that the polynomials are dense in both $\mathfrak{b}_{\log^\alpha}$ and X. This proves (ii).

CASE (iii). In this case we use the function $\varphi(x)=x\log^\alpha(2/x)$, let $q=1$, and then proceed as in the proof of (i). The details are omitted. This concludes the proof of the theorem. □

REMARK 3.1. By choosing $\phi(x)=x\log\log(\frac{4}{x})$, then we can conclude that Theorem 3.1 remains true if $\log^\alpha$, resp. $(n+1)^\alpha$, are replaced with $\log\log$, resp. $\log(n+2)$.

[1] *Following Shields and Williams* [**20**], *we call such a function* ***normal***.

4. Functions with decreasing coefficients

Proof of Theorem 1.1. Assuming that (1.3) holds, we want to prove that

$$\|f\|_{\mathfrak{B}^1_{\log^\alpha}} \leqslant Ca_0 + C\sum_{n=1}^{\infty} a_{2^{n-1}}(n+1)^\alpha.$$

According to Theorem 3.1 and its proof, we have

$$C^{-1}\|f\|_{\mathfrak{B}^1_{\log^\alpha}} \leqslant a_0 + \sum_{n=1}^{\infty}(n+1)^\alpha\|V_n * f\|_1 \leqslant C\|f\|_{\mathfrak{B}^1_{\log^\alpha}}.$$

Let $n \geqslant 1$, $m = 2^{n-1}$, and $Q_k = \sum_{j=m}^{k}\varphi(j/m)e_j$. Since $Q_{4m-1} = V_n$, we have

$$\begin{aligned} V_n * f = \sum_{k=m}^{4m-1}\varphi(k/m)a_k e_k &= \sum_{k=m}^{4m-1}(a_k - a_{k+1})Q_k + a_{4m}Q_{4m-1} \\ &= \sum_{k=m}^{4m-1}(a_k - a_{k+1})Q_k + a_{4m}V_n. \end{aligned}$$

On the other hand, $Q_k = V_n * \Delta_{n,k}$, where

$$\Delta_{n,k} = \sum_{j=2^{n-1}}^{k} z^k, \quad 2^{n-1} \leqslant k \leqslant 2^{n+1}.$$

By (3.2), with $g = \Delta_{n,k}$, we have

$$\|Q_k\|_1 \leqslant C\|\Delta_{n,k}\|_1 \leqslant C\log(k+1-2^{n-1}) \leqslant C(n+1).$$

Combining these inequalities we get

$$\begin{aligned} \|V_n * f\|_1(n+1)^\alpha &\leqslant C\sum_{k=m}^{4m-1}(a_k - a_{k+1})(n+1)^{\alpha+1} + Ca_{4m}\|V_n\|_1(n+1)^\alpha \\ &\leqslant C(n+1)^{\alpha+1}(a_m - a_{4m}) + Ca_{4m}(n+1)^\alpha \\ &= C(n+1)^{\alpha+1}(a_{2^{n-1}} - a_{2^{n+1}}) + C(n+1)^\alpha a_{2^{n+1}}. \end{aligned}$$

Here we have used the relation $\|V_n\|_1 \leqslant C$ (see (3.3))! Thus

$$\begin{aligned} (n+1)^\alpha\|V_n * f\|_1 \leqslant\ & C(n+1)^{\alpha+1}(a_{2^{n-1}} - a_{2^n}) \\ & + C(n+1)^{\alpha+1}(a_{2^n} - a_{2^{n+1}}) + C(n+1)^\alpha a_{2^{n+1}}, \end{aligned}$$

and therefore it remains to estimate the sums

$$S_1 = \sum_{n=1}^{\infty}(n+1)^{\alpha+1}(a_{2^{n-1}} - a_{2^n}) \text{ and } S_2 = \sum_{n=1}^{\infty}(n+1)^{\alpha+1}(a_{2^n} - a_{2^{n+1}}).$$

If $\alpha > -1$, then

$$(n+1)^{\alpha+1} \leqslant C\sum_{k=1}^{n}(k+1)^\alpha,$$

and hence

$$\begin{aligned} S_1 &\leqslant C\sum_{n=1}^{\infty}(a_{2^{n-1}}-a_{2^n})\sum_{k=1}^{n}(k+1)^{\alpha} = C\sum_{k=1}^{\infty}(k+1)^{\alpha}\sum_{n=k}^{\infty}(a_{2^{n-1}}-a_{2^n}) \\ &= C\sum_{k=1}^{\infty}(k+1)^{\alpha}_{2^{k-1}}. \end{aligned}$$

In the case of S_2 we get

$$S_2 \leqslant C\sum_{k=1}^{\infty}(k+1)^{\alpha}a_{2^k},$$

which completes the proof of "if" part of the theorem in the case $\alpha > -1$. If $\alpha = -1$, then

$$\|V_n * f\|_1(n+1)^{-1} \leqslant C(a_{2^{n-1}} - a_{2^{n+1}}) + C(n+1)^{-1}a_{2^{n+1}},$$

from which we get the desired result in the case $\alpha = -1$.

To prove "only if" part we use Hardy's inequality in the form

$$\pi M_1(r,g) \geqslant \sum_{n=0}^{\infty}\frac{|\hat{g}(n)|}{n+1}r^n.$$

It follows that

$$\begin{aligned} \int_{\mathbb{D}}|f'(z)|\log^{\alpha}\frac{2}{1-|z|}\,dA(z) &= 2\int_0^1 M_1(r,f')\log^{\alpha}\frac{2}{1-r}r\,dr \\ &\geqslant \frac{2}{\pi}\sum_{n=1}^{\infty}a_n\frac{n}{n+1}\int_0^1\log^{\alpha}\frac{2}{1-r}r^n\,dr \end{aligned}$$

Now the desired result follows from the inequality

$$\int_0^1\frac{\varphi(1-r)}{1-r}r^n\,dr \geqslant c\varphi\Big(\frac{1}{n+1}\Big) \quad (c = \text{const.} > 0),$$

valid for any normal function φ (see [**9**, Lemma 4.1]). □

Before proving Theorem 1.2, some remarks are in order. Let $f(z) = \sum_{n=0}^{\infty}a_nz^n$, $a_n \geqslant 0$. In order that $\mathcal{L}f$ be well defined by (1.8) *it is necessary and sufficient* that

$$\sum_{n=0}^{\infty}\frac{a_n}{n+1} < \infty. \tag{4.1}$$

We already have mentioned in Introduction that this condition implies the existence of the integral in (1.8). In fact, this integral converges uniformly on compact subsets of $\mathbb{D}$, which means that the limit

$$\lim_{x\uparrow 1}\int_0^x f(t+(1-t)z)\,dt$$

exists and is uniform in $|z| < \rho$, for every $\rho < 1$. This guarantees that $\mathcal{L}f$ is analytic. On the other hand, if the integral in (1.8) exists, then we take $z = 0$ to conclude that (4.1) holds.

Proof of Theorem 1.2. The Taylor coefficients of $\mathcal{L}f$ are

$$b_n = \sum_{k=n}^{\infty} \frac{a_n}{n+1}.$$

The sequence $\{b_n\}$ is nonincreasing so we can apply Theorem 1.1 to conclude that $\mathcal{L}f \in \mathfrak{B}^1_{\log^\alpha}$ if and only if

$$\sum_{n=0}^{\infty} \frac{\log^\alpha(n+2)}{n+1} \sum_{k=n}^{\infty} \frac{a_k}{k+1} = \sum_{k=0}^{\infty} \frac{a_k}{k+1} \sum_{n=0}^{k} \frac{\log^\alpha(n+2)}{n+1} < \infty.$$

Now the desired result follows from the estimate

$$C^{-1} \log^{\alpha+1}(k+2) \leqslant \sum_{n=0}^{k} \frac{\log^\alpha(n+2)}{n+1} \leqslant C \log^{\alpha+1}(k+2),$$

which holds because $\alpha > -1$. □

Remark 4.1. The above proof shows that $\mathcal{L}f$ belongs to $\mathfrak{B}^1_{\log^{-1}}$ if and only if

$$\sum_{n=0}^{\infty} \frac{a_n}{\log\log(n+4)} < \infty.$$

Now we pass to the proof of Theorem 2.1.

Proof of Theorem 1.1(c). Since $\mathfrak{B}_{\log^\alpha} \subset \mathfrak{B}_{\log^\beta}$ for $\beta < \alpha$, we may assume that $-1 < \alpha < 0$. Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \log^{-\varepsilon-\alpha}(n+2),$$

where $\varepsilon > 1$. Condition (4.1) holds because $\varepsilon > 1$. For every $r \in (0,1)$ the function $f_r(z) = f(rz)$ belongs to $H(\overline{\mathbb{D}})$ and, by Theorem 1.1 and its proof, the set $\{f_r : 0 < r < 1\}$ is bounded in $\mathfrak{B}^1_{\log^\alpha}$. On the other hand,

$$\overline{\mathcal{L}}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1)\log^{\alpha+\varepsilon}(k+2)}.$$

Now choose $\varepsilon = 1 - \alpha > 1$ (because $\alpha < 0$) to get

$$\overline{\mathcal{L}}(f_r)(0) = \sum_{k=0}^{\infty} \frac{r^k}{(k+1)\log(k+2)} \longrightarrow \infty \quad (r \uparrow 1).$$

This contradicts the fact that if a set $X \subset \mathfrak{B}^1_{\log^\alpha}$ is bounded and $\overline{\mathcal{L}}$ is bounded on $\mathfrak{B}^1_{\log^\alpha}$, then the set $\{\overline{\mathcal{L}}f(0) : f \in X\}$ is bounded because the functional $h \mapsto h(0)$ is continuous on $H(\mathbb{D})$. This completes the proof. □

Proof of Theorem 2.1(a). Let $g \in \mathfrak{B}_{\log^\alpha}$. Then

$$M_2(r, g') \leqslant C(1-r)^{-1} \log \frac{2}{1-r}.$$

It follows that

$$2^n\Bigg(\sum_{k=2^n}^{2^{n+1}-1}|\hat{g}(k)|^2\Bigg)^{1/2} r^{2^{n+1}} \leqslant C(1-r)^{-1}\log^\alpha\frac{2}{1-r}.$$

Taking $r=1-2^{-n}$, $n\geqslant 1$, we get

$$\Bigg(\sum_{k=2^n}^{2^{n+1}-1}|\hat{g}(k)|^2\Bigg)^{1/2} \leqslant C\log^\alpha(n+1).$$

Hence,

$$2^{-n}\sum_{k=2^n}^{2^{n+1}-1}|\hat{g}(k)| \leqslant \Bigg(2^{-n}\sum_{k=2^n}^{2^{n+1}-1}|\hat{g}(k)|^2\Bigg)^{1/2} \leqslant 2^{-n/2}\log^\alpha(n+1).$$

This gives the result. □

Proof of Theorem 2.1(b). In this case we use Hardy's inequality as in the proof of Theorem 1.1 to get

$$\|g\|_{\mathfrak{B}^1_{\log^\alpha}} \geqslant c\sum_{n=0}^\infty\frac{|\hat{g}(n)|\log^\alpha(n+2)}{n+1}.$$

This proves the result because $\alpha\geqslant 0$. □

5. Proofs of Theorem 2.2 and 2.3

Define the operator $\mathcal{R}: H(\mathbb{D})\mapsto H(\mathbb{D})$ by

$$\mathcal{R}f(z)=\sum_{n=0}^\infty(n+1)\hat{f}(n)z^n=\frac{d}{dz}(zf(z)).$$

By using Theorem 3.1 and the relation

$$C^{-1}2^n\|V_n*f\|_p\leqslant\|V_n*\mathcal{R}f\|_p\leqslant C2^n\|V_n*f\|_p\quad(n\geqslant 0),\tag{5.1}$$

one proves that the norm in $\mathfrak{B}_{\log^\alpha}$ is equivalent to

$$\sup_{z\in\mathbb{D}}(1-|z|)\log^{-\alpha}\frac{2}{1-|z|}|\mathcal{R}f(z)|.$$

Proof of Theorem 2.2(a). Let $\alpha>-1$ and $f\in\mathfrak{B}_{\log^\alpha}$. Then, by integration,

$$|f(z)|\leqslant\log^{\alpha+1}\frac{1}{1-|z|}.$$

Since

$$\mathcal{R}\mathcal{C}f(z)=\frac{f(z)}{1-z},$$

we see that

$$|\mathcal{R}\mathcal{C}f(z)|\leqslant C(1-|z|)^{-1}\log^{\alpha+1}\frac{1}{1-|z|}.$$

The result follows. □

Proof of Theorem 2.2(b). The function $\varphi(x) = x\log\log(4/x)$ is normal because $\lim_{x\to 0} x\varphi'(x)/\varphi(x) = 1$. Hence, arguing as in the proof of Theorem 3.1 we conclude that $f \in \mathfrak{B}_{\mathrm{logg}}$ if and only if $\sup_{n\geqslant 0} \|V_n * f\|_\infty/\log(n+2) < \infty$. Then using (5.1) we find that $g \in \mathfrak{B}_{\mathrm{logg}}$ if and only if

$$|\mathcal{R}g(z)| \leqslant C(1-|z|)^{-1}\log\log\frac{4}{1-|z|}.$$

The rest of the proof is the same as in the case of (a). □

Remark 5.1. In the case of the small spaces the proofs are similar and therefore omitted.

For the proof of Theorem 2.3 we need the following lemma [**18**]:

Lemma 5.1. *If $f \in \ell^1_{-1}$, then $\mathcal{L}f$ is well defined by* (1.8) *and the inequality*

$$rM_1(r,(\mathcal{L}f)') \leqslant 2(1-r)^{-1}\int_r^1 M_1(s,f')\,ds, \quad 0<r<1, \tag{5.2}$$

holds.

Before passing to the proof observe that $\mathfrak{B}^1_{\log^\alpha} \subset \mathfrak{B}^1$ and $\mathfrak{B}^1_{\mathrm{logg}} \subset \mathfrak{B}^1$, and, since $\mathfrak{B}_1 \subset H^1$, we see that in all cases of Theorem 2.3 the operator $\mathcal{L}$ is well defined.

Proof of Theorem 2.3(a). We have, by (5.2),

$$\begin{aligned}\int_{\mathbb{D}} |(\mathcal{L}f)'(z)|\log^{\alpha-1}\frac{2}{1-|z|}\,dA(z) &= 2\int_0^1 M_1(r,(\mathcal{L}f)')\log^{\alpha-1}\frac{2}{1-r}r\,dr\\ &\leqslant 4\int_0^1 (1-r)^{-1}\log^{\alpha-1}\frac{2}{1-r}\,dr\int_r^1 M_1(s,f')\,ds\\ &= 4\int_0^1 M_1(s,f')\,ds\int_0^s (1-r)^{-1}\log^{\alpha-1}\frac{2}{1-r}\,dr\\ &\leqslant C\int_0^1 M_1(s,f')\log^{\alpha}\frac{2}{1-s}\,ds.\end{aligned}$$

A standard application of the maximum modulus principle shows that the inequality remains valid if we replace ds with $s\,ds$. This gives the result. □

The proofs of Theorem 2.3, (b) and (c), are similar and we omit them.

6. Proof of Theorem 2.4

We consider a more general situation. Let $X \subset H(\mathbb{D})$ (with continuous inclusion) be a Banach space such that the functions $f_w(z) = f(wz)$, $|w| \leqslant 1$, belong to X whenever $f \in X$, and $\sup_{|w|\leqslant 1}\|f_w\|_X \leqslant \|f\|_X$. Such a space is said to be *homogeneous* (see [**2**]). A homogeneous space satisfies the condition

$$\|V_n * f\|_X \leqslant C\|f\|_X, \quad f \in X, \tag{6.1}$$

where C is independent of n and f.

If in addition

$$\lim_{r\uparrow 1} \|f - f_r\|_X = 0, \quad f \in X, \tag{6.2}$$

then the dual of X can be identified with the space, X', of those $g \in H(\mathbb{D})$ for which limit (1.2) exists for all $f \in X$ (see [**1, 2**]). Also, the dual of a homogeneous space X satisfying (6.2) can be realized as the space of coefficient multipliers, $(X, A(\mathbb{D}))$, from X to $A(\mathbb{D})$; in this case we have $(X, A(\mathbb{D})) = (X, H^\infty) =: X^*$ (see [**2**]). The norm in X^* is introduced as

$$\|g\|_{X^*} = \sup \|f * g\|_\infty : f \in X,\ \|f\|_X \leqslant 1\},$$

and, if X is homogeneous and satisfies (6.2), it is equal to

$$\|g\|_{X'} = \sup\{|\langle f, g\rangle| : f \in X,\ \|f\|_X \leqslant 1\}.$$

There is another way to express $\langle f, r\rangle$, when $f \in X$, X satisfies (6.2), and $g \in X'$; namely, in this case,the function $f{*}g$ belongs to $A(\mathbb{D})$, and we have $\langle f, g\rangle = (f{*}g)(1)$ (see [**13, 2**]).

We fix a sequence $\lambda = \{\lambda_n\}_0^\infty$ of positive real numbers such that

$$0 < \inf_{n\geqslant 0} \frac{\lambda_{n+1}}{\lambda_n}, \quad \sup_{n\geqslant 0} \frac{\lambda_{n+1}}{\lambda_n} < \infty. \tag{6.3}$$

It is clear that the spaces H^p $(0 < p \leqslant \infty)$, $A(\mathbb{D})$, $\mathfrak{B}_{\log^\alpha}$, $\mathfrak{b}_{\log^\alpha}$, and $\mathfrak{B}^1_{\log^\alpha}$ are homogeneous. Among them only H^∞ and $\mathfrak{B}_{\log^\alpha}$ do not satisfy condition (6.2).

Consider the following three spaces of sequences $\{f_n\}_0^\infty$, $f_n \in H(\mathbb{D})$:

(a) $\mathfrak{c}_0(\lambda, X) = \{\{f_n\} : \lim_{n\to\infty} \|f_n\|_X/\lambda_n = 0\}$;
(b) $\ell^\infty(\lambda, X) = \{\{f_n\} : \sup_{n\geqslant 0} \lambda_n \|V_n * f\|_X < \infty\}$;
(c) $\ell^1(\lambda, X) = \{\{f_n\} : \sum_{n=0}^\infty \|f_n\|_X/\lambda_n < \infty\}$.

We also define the spaces $v_0(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$ (as subsets of $H(\mathbb{D})$) by replacing f_n with $V_n * f$ in (a), (b), and (c), respectively. The proof of the following lemma is rather easy, and is therefore left to the reader.

Lemma 6.1. *If X is a homogeneous space, then so are $v(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$. The spaces $v_0(\lambda, X)$ and $V^1(\lambda, X)$ satisfy* (6.2). *The space $v_0(\lambda, X)$ is equal to the closure in $V^\infty(\lambda, X)$ of the sets of all polynomials.*

Theorem 2.4 will be deduced from Theorem 3.1 and the following.

Proposition 6.1. *If X is a homogeneous space satisfying* (6.2)*, then the dual of $v_0(\lambda, X)$, resp. $V^1(\lambda, X)$, is isomorphic to $V^1(\lambda, X')$, resp. $V^\infty(\lambda, X')$, with respect to* (1.2).

In proving we use ideas from [**13, 14, 7**]. For the proof we need the following lemma.

Lemma 6.2. *The operator $T(\{f_n\}) = \sum_{n=0}^\infty V_n * f_n$ acts as a bounded operator from Y to Z, where Y is one of the spaces $\mathfrak{c}_0(\lambda, X)$, $\ell^\infty(\lambda, X)$, and $\ell^1(\lambda, X)$, while Z is $v_0(\lambda, X)$, $V^\infty(\lambda, X)$, and $V^1(\lambda, X)$, respectively.*

PROOF. We have $V_n * V_j = 0$ for $|j-n| \geqq 2$ and hence

$$V_n * T(\{f_j\}) = \sum_{j=n-1}^{n+1} V_n * V_j * f_j, \quad n \geqslant 0,$$

where, by definition, $w_j = f_j = 0$ for $j < 0$. It follows that

$$\|V_n * T(\{f_j\}\|_X \leqslant C \sum_{j=n-1}^{n+1} \|f_j\|_X,$$

where we have used (6.1). Now the proof is easily completed by using (6.3). □

LEMMA 6.3. *Let $g \in (v_0(\lambda, X))'$, resp. $g \in (V^1(\lambda, X))'$, and define the operator S on $\mathfrak{c}_0(\lambda, X)$, resp. $\ell^1(\lambda, X)$, by*

$$S(\{f_n\}) = T(\{f_n\}) * g = \sum_{k=0}^{\infty} f_k * V_k * g.$$

Then S maps $\mathfrak{c}_0(\lambda, X)$, resp. $\ell^1(\lambda, X)$, into H^∞ and $\|S\| \leqslant C\|g\|_{(v_0(\lambda,X))'}$, resp. $\|S\| \leqslant C\|g\|_{(V^1(\lambda,X))'}$.

PROOF. By the preceding lemma, we have

$$\|S(\{f_n\})\|_\infty \leqslant \|T(\{f_n\})\|_{v_0(\lambda,X)}\|g\|_{(v_0(\lambda,X))*} \leqslant C\|\{f_n\}\|_{\mathfrak{c}_0(\lambda,X)}\|g\|_{(v_0(\lambda,X))*}.$$

This proves the result in one case. In the other case the proof is the same. □

PROOF OF PROPOSITION 6.1. Define the polynomials P_n $(n \geqslant 0)$ by

$$P_n = V_{n-1} + V_n + V_{n+1}.$$

Hence

$$V_n = \sum_{j=0}^{\infty} V_j * V_n = (V_{n-1} + V_n + V_{n+1}) * V_n = P_n * V_n.$$

Let $f \in v_0(\lambda, X)$ and $g \in V^1(\lambda, X')$. It is easily verified that, when $0 < r < 1$,

$$(f * g)(z) = \sum_{n=0}^{\infty} (f * V_n * g)(z) = \sum_{n=0}^{\infty} (P_n * f * V_n * g)(z), \quad z \in \mathbb{D}.$$

the series being absolutely convergent. Since

$$\|P_n * f_r * V_n * g\|_\infty \leqslant \|P_n * f\|_X \, \|V_n * g\|_{X^*},$$

we have

$$\begin{aligned}\|f * g\|_\infty &\leqslant \sum_{n=0}^{\infty} \|P_n * f\|_X \, \|V_n * g\|_{X^*} \leqslant C \sum_{n=0}^{\infty} \|P_n * f\|_X \, \|V_n * g\|_{X^*} \\ &= C \sum_{n=0}^{\infty} \left(\|P_n * f\|_X/\lambda_n\right) \left(\lambda_n \|V_n * g\|_{X^*}\right) \leqslant C\|f\|_{v_0(\lambda,X)}\|g\|_{V^1(\lambda,X^*)}\end{aligned}$$

This proves the inclusion $V^1(\lambda, X') \subset (v_0(\lambda, X))'$.

To prove the converse, let $g \in (v_0(\lambda, X))^*$. Let S denote the operator defined in Lemma 6.3. By Lemma 6.3, S acts as a bounded operator from $\mathfrak{c}_0(\lambda, X)$ into H^∞ and we have $\|S\| \leqslant C\|g\|_{v_0(\lambda,X)^*}$. Now it suffices to prove that

$$\|S\| \geqslant (1/2)\|\{g_n\}\|_{\ell^1(\lambda,X^*)} = (1/2)\|g\|_{V^1(\lambda,X)}.$$

For each $n \geqslant 0$ choose $f_n \in X$ so that $\|f_n\|_X = 1$ and $\langle f_n, g_n\rangle$ is a real number such that $\langle f_n, g_n\rangle \geqslant (1/2)\|g_n\|_{X^*}$. If $\{a_n\}$ is a finite sequence of nonnegative real numbers, then

$$S(\{a_n f_n\}) = \sum_{n=0}^{\infty} a_n\langle f_n, g_n\rangle \geqslant (1/2)\sum_{n=0}^{\infty} a_n\|g_n\|_{X^*} = (1/2)\sum_{n=0}^{\infty}(a_n/\lambda_n)\lambda_n\|g_n\|_{X^*}.$$

Hence, by taking the supremum over all $\{a_n\}$ such that $0\langle a_n\rangle\lambda_n$, we get $S\{\lambda_n f_n\} \geqslant (1/2)\sum_{n=0}^{\infty}\lambda_n\|g_n\|_{X^*}$. Since $\|\{a_n f_n\}\|_{\mathfrak{c}_0(\lambda,X)} \leqslant 1$, where $a_n = \lambda_n$ for $0 \leqslant n \leqslant N$ ($N \in \mathbb{N}$) and $a_n = 0$ for $n > N$ we see that $\|S\| \geqslant (1/2)\|g\|_{V^1(\lambda,X^*)}$, as desired. This completes the proof that $v_0(\lambda, X)' = V^1(\lambda, X')$. In a similar way one proves that $V^1(\lambda, X)' = V^\infty(\lambda, X')$, which is all what has to be proved. □

Proof of Theorem 2.4. First we prove that $(\mathfrak{b}_{\log^\alpha})' = \mathfrak{B}^1_{\log^\alpha}$. By Theorem 3.1, we have $\mathfrak{b}_{\log^\alpha} = v_0(\lambda, A(\mathbb{D}))$, where $\lambda_n = (n+1)^\alpha$. Hence, by Proposition 6.1, the dual of $\mathfrak{b}_{\log^\alpha}$ is isomorphic to $V^1(\lambda, A(\mathbb{D})')$. In order to estimate $\|V_n * g\|_{A()'}$ first observe that $H^1 \subset A(\mathbb{D})'$ and moreover $\|V_n * g\|_{A(\mathbb{D})'} \leqslant \|V_n * g\|_1$. On the other hand, let Φ be a bounded linear functional on $A(\mathbb{D})$, let Φ_0 be the Hahn/Banach extension of Φ to $hC(\mathbb{D})$, and choose $g \in A(\mathbb{D})^{\mathrm{a}}$ so that $\Phi(f) = \langle f, g\rangle$ for all $f \in A(\mathbb{D})$. By the Riesz representation theorem, we have

$$\begin{aligned}\Phi_0(f) &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{-i\theta})\,d\mu(e^{i\theta})\\ &= \lim_{r\to1^-}\frac{1}{2\pi}\int_0^{2\pi} f(re^{-i\theta})g(re^{i\theta})\,d\theta = \lim_{r\to1^-}\sum_{n=0}^{\infty}\hat f(n)\hat g(n)r^{2n},\end{aligned}$$

and $\|\mu\| = \|\Phi\| = \|\Phi_0\|$. In particular, taking $f(w) = (1 - zw)^{-1}$, where $z \in \mathbb{D}$ is fixed, we get $\frac{1}{2\pi}\int_0^{2\pi}(1 - e^{-i\theta}z)^{-1}d\mu(e^{i\theta}) = g(z)$. Hence

$$\mathcal{R}^1 g(z) = \frac{1}{2\pi}\int_0^{2\pi}(1 - e^{-i\theta}z)^{-2}\,d\mu(e^{i\theta}),$$

and hence, by integration, $M_1(r, \mathcal{R}^1 f) \leqslant \|\mu\|(1 - r^2)^{-1} = \|g\|_{A(\mathbb{D})'}(1 - r^2)^{-1}$. Now we proceed as in the proof of Theorem 3.1 to conclude that $\|V_n * g\|_1 \leqslant C\|V_n * g\|_{A(\mathbb{D})'}$. It follows that $g \in (\mathfrak{b}_{\log^\alpha})'$ if and only if $g \in V^1(\lambda, H^1)$, i.e., by Theorem 3.1, $g \in \mathfrak{B}^1_{\log^\alpha}$.

In proving that $(\mathfrak{B}^1_{\log^\alpha})'$ is isomorphic to $\mathfrak{B}_{\log^\alpha}$, we use the inclusions $H^\infty \subset (H^1)' \subset \mathfrak{B}$, and then proceed as above. □

Remark 6.1. The above proof of Theorem 2.4 certainly is not the simplest one. However, it can be applied to prove some general duality and multipliers theorems (see [**13, 14, 7**]. For instance, the dual of $\mathfrak{b}_{\mathrm{logg}}$ is isomorphic to $\mathfrak{B}^1_{\mathrm{logg}}$, and the dual of $\mathfrak{B}^1_{\mathrm{logg}}$ is isomorphic to $\mathfrak{B}_{\mathrm{logg}}$.

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