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POLYNOMIAL INEQUALITIES IN LAVRENTIEV REGIONS WITH INTERIOR AND EXTERIOR ZERO ANGLES IN THE WEIGHTED LEBESGUE SPACE

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Abstract. We study estimation of the modulus of algebraic polynomials in the bounded and unbounded regions with piecewise-quasismooth boundary, having interior and exterior zero angles, in the weighted Lebesgue space.

1. Introduction and main results

Let $\mathbb{C}$ be a complex plane, $\bar{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region, with $0 \in G$ and the boundary $L := \partial G$ be a simple closed rectifiable Jordan curve, $\Omega := \bar{\mathbb{C}} \setminus \bar{G} = \operatorname{ext} L$; $\Delta := \{w : |w| > 1\}$. Let $w = \Phi(z)$ be the univalent conformal mapping of Ω onto the Δ normalized by $\Phi(\infty) = \infty$, $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$. For $t \geqslant 1$, let us set $L_t := \{z : \ |\Phi(z)| = t\}$, $L_1 \equiv L$, $G_t := \operatorname{int} L_t$, $\Omega_t := \operatorname{ext} L_t$. For $z \in \mathbb{C}$ and $S \subset \mathbb{C}$ let $d(z, S) := \operatorname{dist}(z, S) = \inf\{|\zeta - z| : \zeta \in S\}$. Let $h(z)$ be a weight function defined in G_{R_0} for some fixed $R_0 > 1$ and let $\wp_n$ denote the class of arbitrary algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N} := \{1, 2, \ldots\}$. For any $p > 0$ we denote

$$\|P_n\|_p := \|P_n\|_{\mathcal{L}_p(h,L)} := \Big(\int_L h(z)|P_n(z)|^p|dz|\Big)^{1/p}, \quad 0 < p < \infty;$$
$$\|P_n\|_\infty := \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z\in\bar{G}} |P_n(z)|, \quad p = \infty.$$

Here, we investigate the following two problems: for a given region G, find estimates of the following types, for the points $z \in G$ and $z \in \Omega$, respectively:

$$\|P_n\|_\infty \leqslant \operatorname{const} \nu_n \|P_n\|_p, \tag{1.1}$$

$$|P_n(z)| \leqslant \operatorname{const} \eta_n \|P_n\|_p |\Phi(z)|^{n+1}, \tag{1.2}$$

where $\nu_n := \nu_n(G, h, p) \to \infty$ and $\eta_n := \eta_n(G, h, p, z) \to \infty$ in general, as $n \to \infty$, depending on the geometrical properties of the region G and weight function h.

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Inequalities analogously to (1.1) in the literature are often found under the "Nikol'skii-type inequality". One of the first results analogous to (1.1), in the case $h(z) \equiv 1$ for $L = \{z : |z| = 1\}$ and $0 < p < \infty$ was found by Jackson [**12**] as follows:

$$\max_{|z|=1} |P_n(z)| \leqslant 2n^{1/p} \Big(\int_0^{2\pi} |P_n(e^{i\theta})|^p d\theta \Big)^{1/p}.$$

Another classical results similar to (1.1) belong to Szegö and Zigmund [**21**]. Suetin [**23**], [**24**] investigated this problem with a sufficiently smooth Jordan curve and some special weight function. The estimate of type (1.1) for $0 < p < \infty$ when L is a rectifiable Jordan curve, was investigated by Mamedhanov [**14**, **15**], Nikol'skii [**17**, pp. 122–133], Pritsker [**19**] ($h(z) \equiv 1$), Andrievskii [**10**, Theorem 6], authors [**7**] ($h(z) \neq 1$) and others. More general and detailed references regarding the Nikol'skii type inequality, we can find in Milovanovic et al. [**16**, Sect. 5.3].

Results analogous to (1.2) for some different norms and unbounded regions were obtained in [**22**], [**2**–**7**] and others.

In this work, we study problems (1.1) and (1.2) for some general regions having interior and exterior cusps, and finally, we obtain the estimate for $|P_n(z)|$ in the whole complex plane, depending on the given region G and weight function h.

We give the necessary definitions and notations for the formulation of the main results.

Following [**18**, p.163], we say that a bounded Jordan curve L is *λ-quasismooth or Lavrentiev curve*, if for every pair z_1, $z_2 \in L$, where $l(z_1, z_2)$ denote the shorter subarc of L, joining $z_1 \in L$ and $z_2 \in L$ and $|l(z_1, z_2)|$ is the linear measure (length) of $l(z_1, z_2)$, there exists a constant $\lambda := \lambda(L) \geqslant 1$, such that

$$|l(z_1, z_2)| \leqslant \lambda |z_1 - z_2|, \quad z_1, z_2 \in L.$$

In this case, the inner region $\operatorname{int} L$ of a Lavrentiev curve L is called a *Lavrentiev region*. Any subarc of a λ-quasismooth curve is called a *λ-quasismooth arc*. We denote this class of curves and arcs as $QS(\lambda)$ and say that a Jordan region $G \in QS(\lambda)$, if $\partial G \in QS(\lambda)$, $\lambda \geqslant 1$. Furthermore, we denote that $L(\text{or } G) \in QS$, if $L(\text{or } G) \in QS(\lambda)$ for some $\lambda \geqslant 1$.

We say that a bounded Jordan curve or arc L is *locally λ-quasismooth* at the point $z \in L$, if there exists a closed subarc $\ell \subset L$ containig z, such that every open subarc of the ℓ containing z is the λ-quasismooth.

Now, we shall define a new class of regions with piecewise quasismooth boundary, which may have at the boundary points finite number of interior and exterior cusps.

For any $j = 1, 2, \dots$ and sufficiently small $\varepsilon_1 > 0$, we denote by f_j, g_j : $[0, \varepsilon_1] \to \mathbb{R}$ twice continuously differentiable functions such that $f_j(0) = g_j(0) = 0$ and $f_j^{(k)}(x) > 0$, $g_j^{(k)}(x) > 0$, for $x > 0$ and $k = 0, 1, 2$.

Note that, throughout this paper, $c, c_0, c_1, c_2, \dots$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots$ are sufficiently small positive constants (generally, different in different relations), which depend on G in general.

Definition 1.1. We say that a Jordan region $G \in PQS(\lambda; f_i, g_j)$, $\lambda \geqslant 1$, $f_i = f_i(x)$, $i = \overline{1, m_1}$, $g_j = g_j(x)$, $j = \overline{m_1 + 1, m}$, if $L := \partial G = \bigcup_{j=0}^{m} L_j$ is the union of the finite number of λ-quasismooth arcs L_j, connecting at the points $\{z_j\}_{j=0}^{m} \in L$, and such that L is a locally λ-quasismooth arc at the $z_0 \in L \setminus \{z_j\}_{j=1}^{m}$ and, in the (x, y) local coordinate system with its origin at the z_j, $1 \leqslant j \leqslant m$, the following conditions are satisfied

a) for every $z_j \in L,\ j = \overline{1, m_1},\ m_1 \leqslant m$,

$$\{z = x + iy : |z| \leqslant \varepsilon_1,\ c_1 f_i(x) \leqslant y \leqslant c_2 f_i(x)\} \subset \bar{G},$$
$$\{z = x + iy : |z| \leqslant \varepsilon_1,\ |y| \geqslant \varepsilon_2 x\} \subset \bar{\Omega};$$

b) for every $z_j \in L$, $j = \overline{m_1 + 1, m}$,

$$\{z = x + iy : |z| < \varepsilon_3,\ c_3 g_j(x) \leqslant y \leqslant c_3 g_j(x),\ 0 \leqslant x \leqslant \varepsilon_3\} \subset \bar{\Omega},$$
$$\{z = x + iy : |z| < \varepsilon_3,\ |y| \geqslant \varepsilon_3 x,\ 0 \leqslant x \leqslant \varepsilon_3\} \subset \bar{G},$$

for some constants $-\infty < c_1 < c_2 < \infty$, $-\infty < c_3 < c_4 < \infty$, $\varepsilon_i > 0$, $i = \overline{1, 4}$.

It is clear from Definition 1.1 that each region $G \in PQS(\lambda; f_i, g_j)$ may have m_1 interior and $m - m_1$ exterior zero angles (with respect to $\bar{G}$). If a region G does not have interior zero angles ($m_1 = 0$) (exterior zero angles ($m_1 = m$)), then it is written as $G \in PQS(\lambda; 0, g_j)$ ($G \in PQS(\lambda; f_i, 0)$). If a domain G does not have such angles ($m = 0$), then G is bounded by a λ-quasismooth circle and in this case we set $PQS(\lambda, 0, 0) \equiv QS(\lambda)$.

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of distinct points on the curve L located in the positive direction. Consider a so-called generalized Jacobi weight function $h(z)$ being defined by

$$h(z) := \prod_{j=1}^{m} |z - z_j|^{\gamma_j}, \quad z \in G_{R_0}, \tag{1.3}$$

for some $R_0 > 1$, where $\gamma_j > -1$ for all $j = 1, 2, \ldots, m$.

Here and in further, for any $k \geqslant 0$ and $m > k$, the notation $j = \overline{k, m}$ denotes $j = k, k + 1, \ldots, m$.

Throughout this work, we will assume that the points $\{z_j\}_{j=1}^{m} \in L$ defined in (1.3) and Definition 1.1 are identical. Without loss of generality, we assume that the points $\{z_j\}_{j=1}^{m}$ are ordered in the positive direction on the curve L such that, G may have interior zero angles at the points $\{z_j\}_{j=1}^{m_1}$ and exterior zero angles at the points $\{z_j\}_{j=m_1+1}^{m}$, and $w_j := \Phi(z_j)$.

Now we can state our new results.

THEOREM 1.1. *Let $p > 0$; $G \in PQS(\lambda; f_i, g_i)$, for some $\lambda \geqslant 1$, $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \geqslant 0$, $i = \overline{1, m_1}$, and $g_i(x) = c_i x^{1+\beta_i}, \beta_i > 0$, $i = \overline{m_1 + 1, m}$; $h(z)$ defined as in (1.3). Then, for any $\gamma_i > -1$, $i = \overline{1, m}$, and $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_1 = c_1(G, p, \lambda, \gamma_i, \beta_i) > 0$ such that*

$$\|P_n\|_\infty \leqslant c_1 \Big(\sum_{i=1}^{m_1} n^{\frac{\tilde{\mu}(\tilde{\gamma}_i + 1)}{p}} + \sum_{i=m_1+1}^{m} n^{(\frac{\tilde{\gamma}_i}{1+\beta_i} + 1)\frac{\mu}{p}} \Big) \|P_n\|_p, \tag{1.4}$$

where $\mu := 2(1-\frac{1}{\pi}\arcsin\frac{1}{\lambda}), 1<\mu<2, \tilde{\mu} := \begin{cases}\mu, & \text{if } \alpha_1 = 0,\\ 2, & \text{if } \alpha_1 \neq 0.\end{cases}$

We note that, if $\mu = 1$ (i.e. $\lambda = 1$), then the terms $n^{(\frac{\tilde{\gamma}_i}{1+\beta_i}+1)\frac{\mu}{p}}$ in the second sum of (1.4) for any $\gamma_i > -1$, $i = \overline{m_1+1, m}$, can be replaced by $(n^{\frac{\tilde{\gamma}_i}{1+\beta_i}+1}\ln n)^{\frac{1}{p}}$.

Now, for simplicity of our presentations, we assume that $i = 1, 2$; $m_1 = 1$, $m = 2$; i.e., our region G may have one interior zero (or it does not exist) angle having "f_1-touching" with $f_1(x) = C_1x^{1+\alpha_1}$, $\alpha_1 \geqslant 0$, at the point z_1 and exterior zero angle having" g_2-touching" with $g_2(x) = C_2x^{1+\beta_2}$, $\beta_2 > 0$, at the point z_2, for some constants $-\infty < C_1 := C_1(c_1, c_2) < +\infty$, $-\infty < C_2 := C_2(c_3, c_4) < +\infty$, where the constants c_i, $i = \overline{1, 4}$ are taken from Definition 1.1. In this case, combining the terms related to the interior and exterior corners, we obtain the following

THEOREM 1.2. *Let $p > 0$; $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \geqslant 1$, $f_1(x) = C_1x^{1+\alpha_1}$, $\alpha_1 \geqslant 0$, and $g_2(x) = C_2x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ defined as in* (1.3) *for $m = 2$. Then, for any $\gamma_i > -1$, $i = 1, 2$, and $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_2 = c_2(G, p, \lambda, \gamma_1, \gamma_2, \beta_2) > 0$ such that*

$$\|P_n\|_\infty \leqslant c_2 A_n \|P_n\|_p, \tag{1.5}$$

where A_n is defined by

$$A_n := \begin{cases} n^{\frac{1}{p}\max\{\tilde{\mu};\mu\}}, & -1 < \gamma_1 \leqslant 0, & -1 < \gamma_2 \leqslant 0, & \mu > 1,\\ n^{(\frac{\gamma_2}{1+\beta_2}+1)\frac{\mu}{p}}, & 0 < \gamma_1 < \left(\frac{\gamma_2}{1+\beta_2}+1\right)\frac{\mu}{\tilde{\mu}} - 1, & \gamma_2 > 0, & \mu > 1,\\ n^{\frac{\tilde{\mu}(\gamma_1+1)}{p}}, & \gamma_1 \geqslant \left(\frac{\gamma_2}{1+\beta_2}+1\right)\frac{\mu}{\tilde{\mu}} - 1, & \gamma_2 > 0, & \mu > 1,\\ \left(n^{\frac{\gamma_2}{1+\beta_2}+1}\ln n\right)^{\frac{1}{p}}, & 0 < \gamma_1 \leqslant \left(\frac{\gamma_2}{1+\beta_2}+1\right)\frac{1}{\tilde{\mu}} - 1, & \gamma_2 > 0, & \mu = 1,\\ n^{\frac{\tilde{\mu}(\gamma_1+1)}{p}}, & \gamma_1 \geqslant \left(\frac{\gamma_2}{1+\beta_2}+1\right)\frac{1}{\tilde{\mu}} - 1, & \gamma_2 > 0, & \mu = 1. \end{cases} \tag{1.6}$$

Assume that the region G has only exterior zero angles $g_i(x) = c_ix^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{1, m}$ (i.e., $m_1 = 0$). In this case, from Theorem 1.1, we obtain

COROLLARY 1.1. *Let $p > 0$; $G \in PQS(\lambda; 0, g_i)$, for some $\lambda \geqslant 1$, $g_i(x) = c_ix^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{1, m}$; $h(z)$ defined as in* (1.3). *Then, for any $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_3 = c_3(G, p, \lambda, \gamma_i, \beta_i) > 0$ such that*

$$\|P_n\|_\infty \leqslant c_3 \left(\sum_{i=1}^{m} n^{\left(\frac{\tilde{\gamma}_i}{1+\beta_i}+1\right)\frac{\mu}{p}}\right) \cdot \|P_n\|_p, \tag{1.7}$$

and consequently,

$$\|P_n\|_\infty \leqslant c_3 n^{\left(\frac{\tilde{\gamma}^{\max}}{1+\beta_{\min}}+1\right)\frac{\mu}{p}} \|P_n\|_p, \tag{1.8}$$

where $\tilde{\gamma}^{\max} = \max\{0; \gamma_i : i = \overline{1, m}\}$, $\beta_{\min} := \min\{\beta_i : i = \overline{1, m}\}$.

Now, for these regions, we will state our new results corresponding to the second problem, i.e., pointwise estimations for $|P_n(z)|$ for the points $z \in \Omega$.

THEOREM 1.3. *Let $p > 0$; $G \in PQS(\lambda; f_i, g_i)$, for some $\lambda \geqslant 1$, $f_i(x) = C_i x^{1+\alpha_i}$, $\alpha_i \geqslant 0$, $i = \overline{1, m_1}$, and $g_i(x) = C_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{m_1+1, m}$; $h(z)$ defined as in* (1.3). *Then, for any $\gamma_i > -1$, $i = \overline{1, m}$, and $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_4 = c_4(G, p, \lambda, \gamma_i, \beta_i) > 0$ such that*

$$|P_n(z)| \leqslant c_4 \frac{|\Phi(z)|^{n+1}}{d^{2/p}(z, L)} \left(\sum_{i=1}^{m_1} B_{n,1}^i + \sum_{i=m_1+1}^{m} B_{n,2}^i \right) \cdot \|P_n\|_p, \quad z \in \Omega, \tag{1.9}$$

where

$$B_{n,1}^i := \begin{cases} n^{\frac{(\gamma_i - 1)\tilde{\mu}}{p}}, & \text{if } \gamma_i > 1, \\ (\ln n)^{\frac{1}{p}}, & \text{if } \gamma_i = 1, \\ 1, & \text{if } -1 < \gamma_i < 1, \end{cases} \qquad \text{for all } i = \overline{1, m_1},$$

$$B_{n,2}^i := \begin{cases} n^{\frac{\gamma_i - 1}{p(1+\beta_i)}\mu}, & \text{if } \gamma_i > 1, \\ (\ln n)^{\frac{1}{p}}, & \text{if } \gamma_i = 1, \\ 1, & \text{if } -1 < \gamma_i < 1, \end{cases} \qquad \text{for all } i = \overline{m_1+1, m}.$$

Theorem 1.3 is local, that is, each term in the sum on the right-hand side shows the growth of $|P_n(z)|$, depending on the behavior of the weight function $h(z)$ and the boundary L in the neighborhood of a single point $\{z_j\} \in L$ for any $j = \overline{1, m}$.

Comparing the terms in the sum for each point $\{z_j\}$, $j = \overline{1, m}$, and using the above notations, we can obtain the following result of global character (for the simplicity, we assume that $i = 1, 2$; $m_1 = 1$, $m_2 = 2$).

THEOREM 1.4. *Let $p > 0$; $G \in PQS(\lambda; f_1, g_2)$, for some $\lambda \geqslant 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geqslant 0$, and $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ defined as in* (1.3) *for $m = 2$. Then, for any $\gamma_j = 1, 2$, and $P_n \in \wp_n$, $n \in \mathbb{N}$, there exists $c_5 = c_5(G, p, \lambda, \gamma_j, \beta_2) > 0$ such that*

$$|P_n(z)| \leqslant c_5 \frac{|\Phi(z)|^{n+1}}{d^{2/p}(z, L)} B_n \cdot \|P_n\|_p, \quad z \in \Omega, \tag{1.10}$$

where

$$B_n := \begin{cases} n^{\frac{\gamma_2 - 1}{p(1+\beta_2)}\mu}, & 1 < \gamma_1 < \frac{\gamma_2 - 1}{1+\beta_2} \frac{\mu}{\tilde{\mu}} + 1, & \gamma_2 > 1, \\ n^{\frac{(\gamma_1 - 1)\tilde{\mu}}{p}}, & \gamma_1 \geqslant \frac{\gamma_2 - 1}{1+\beta_2} \frac{\mu}{\tilde{\mu}} + 1, & \gamma_2 > 1, \\ (\ln n)^{\frac{1}{p}}, & \gamma_1 = 1, & \gamma_2 = 1, \\ 1, & -1 < \gamma_1 < 1, & -1 < \gamma_2 < 1. \end{cases} \tag{1.11}$$

Therefore, combining estimations (1.4) with (1.9), we obtain an estimate on the growth of $|P_n(z)|$ in the whole complex plane.

Corollary 1.2. *Under the conditions and notations of Theorems* 1.1 *and* 1.3*, we have*

$$(1.12)\qquad |P_n(z)| \leqslant c_6 \left\{ \begin{array}{ll} \sum\limits_{i=1}^{m_1} n^{\frac{\tilde{\mu}(\tilde{\gamma}_i+1)}{p}} + \sum\limits_{i=m_1+1}^{m} n^{(\frac{\tilde{\gamma}_i}{1+\beta_i}+1)\frac{\mu}{p}}, & z \in \bar{G}, \\ \dfrac{|\Phi(z)|^{n+1}}{d^{2/p}(z,L)} (\sum\limits_{i=1}^{m_1} B^i_{n,1} + \sum\limits_{i=m_1+1}^{m} B^i_{n,2}), & z \in \Omega, \end{array} \right\} \cdot \|P_n\|_p,$$

where $c_6 = c_6(G, p, \lambda, \gamma_i, \beta_i) > 0$ *constant, independent of* z *and* n.

In particular, in the case $i = 1, 2$; $m_1 = 1$, $m = 2$, from Corollary 1.2 we have

Corollary 1.3. *Let* $p > 0$; $G \in PQS(\lambda; f_1, g_2)$, *for some* $\lambda \geqslant 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geqslant 0$, *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *defined as in* (1.3) *for* $m = 2$. *Then, for any* γ_j, $j = 1, 2$, *and* $P_n \in \wp_n$, $n \in \mathbb{N}$, *there exists* $c_7 = c_7(G, p, \lambda, \gamma_j, \beta_2) > 0$ *such that*

$$|P_n(z)| \leqslant c_7 \left\{ \begin{array}{ll} A_n, & z \in \bar{G}, \\ \dfrac{|\Phi(z)|^{n+1}}{d^{2/p}(z,L)} B_n, & z \in \Omega, \end{array} \right\} \cdot \|P_n\|_p,$$

where A_n *and* B_n *are taken from* (1.6) *and* (1.11)*, respectively.*

Remark 1.1. We note that, according to the well-known Bernstein–Walsh Lemma [**27**], estimations (1.4)–(1.8) are also true in G_R with another constant for some $R = 1 + \frac{c}{n}$. Therefore, if we choose

$$\tilde{R} := \sup\{R > 1 : \|P_n\|_{C(\bar{G}_R)} \leqslant c_7 \|P_n\|_{C(\bar{G})}\},$$

then estimations (1.9)–(1.10) will be significant for $z \in \Omega_{\tilde{R}}$.

The sharpness of estimations (1.4)–(1.12) for some special cases can be discussed by comparing them with the following results.

Remark 1.2. For any $n \in \mathbb{N}$ and $i = 1, 2$, there exist polynomials $P_n^{(i)} \in \wp_n$ and regions $G^i \subset \mathbb{C}$, such that

$$\|P_n^{(1)}\| \geqslant c_8 n^{\frac{1}{p}} \|P_n^{(1)}\|_{\mathcal{L}_p(L^1)},$$
$$|P_n^{(2)}(z)| \geqslant c_9 |\Phi(z)|^n \|P_n^{(2)}\|_{\mathcal{L}_2(L^2)}, \quad \forall z \in F \Subset \bar{\mathbb{C}} \setminus \overline{G^2},$$

where $c_8 = c_8(G^1) > 0$, $c_9 = c_9(G^2) > 0$ are constants, independent of z and n, $L^i := \partial G^i$, $i = 1, 2$.

2. Some auxiliary results

For $a > 0$ and $b > 0$, we use the notation "$a \preceq b$" (order inequality), if $a \leqslant cb$ and "$a \asymp b$" are equivalent to $c_1 a \leqslant b \leqslant c_2 a$ for some constants c, c_1, c_2 (independent of a and b) respectively.

Since, any quasismooth curve is quasiconformal, then in the proofs of many facts we use some properties of quasiconformal curves. Therefore, we give the corresponding definition of the quasiconformal curves.

Definition 2.1. [**13**, p.97], [**20**] The Jordan arc (or curve) L is called K-quasiconformal ($K \geqslant 1$), if there is a K-quasiconformal mapping f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle).

Let $F(L)$ denote the set of all sense preserving plane homeomorphisms f of the region $D \supset L$ such that $f(L)$ is a line segment (or circle) and let define $K_L := \inf\{K(f) : f \in F(L)\}$, where $K(f)$ is the maximal dilatation of such a mapping f. L is a quasiconformal curve, if $K_L < \infty$, and L is a K-quasiconformal curve, if $K_L \leqslant K$.

Let $B := \{w : |w| < 1\}$, $z = \psi(w)$ be the univalent conformal mapping of B onto the G normalized by $\psi(0) = 0$, $\psi'(0) > 0$ and let $\varphi := \psi^{-1}$. For $0 < t < 1$, let $L_t := \{z : \ |\varphi(z)| = t\}$.

Lemma 2.1. [**1**] *Let L be a K-quasiconformal curve, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \preceq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) *The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. So are $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$.*

b) *If $|z_1 - z_2| \preceq |z_1 - z_3|$, then*

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{\varepsilon} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c},$$

where $\varepsilon < 1$, $c > 1$, $0 < r_0 < 1$ are constants, depending on G.

Lemma 2.2. [**25**], [**26**] *Let $G \in QS(\lambda)$ for some $\lambda \geqslant 1$. Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{\mu},$$

for all $w_1, w_2 \in \bar{\Omega}'$, where $\mu := 2(1 - \frac{1}{\pi}\arcsin\frac{1}{\lambda})$.

For Ψ' the following is true (see, for example, [**9**, Th.2.8]):

$$|\Psi'(\tau)| \asymp \frac{d(\Psi(\tau), L)}{|\tau| - 1}. \tag{2.1}$$

Let $\{z_j\}_{j=1}^{m}$ be a fixed system of the points on L and the weight function $h(z)$ defined as (1.3).

Lemma 2.3. *Let L be a rectifiable* Jordan curve*; $h(z)$ as defined in* (1.3)*. Then, for arbitrary $P_n(z) \in \wp_n$, any $R > 1$ and $n \in \mathbb{N}$, we have*

$$\|P_n\|_{\mathcal{L}_p(h, L_R)} \leqslant R^{n+\frac{1+\gamma*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \quad p > 0, \tag{2.2}$$

where $\gamma^* = \max\{0; \gamma_k, k = \overline{1, m}\}$.

Remark 2.1. In the case of $h(z) \equiv 1$, estimate (2.2) has been proved in [**11**].

3. Proof of theorems

Proof of Theorem 1.1. Let $G \in PQS(\lambda; f_i, g_i)$, for some $\lambda \geqslant 1$, $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \geqslant 0$, $i = \overline{1, m_1}$, and $g_i(x) = c_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{m_1 + 1, m}$, be given. For some $R > 1$, let $w = \varphi_R(z)$ be the univalent conformal mapping of G_R onto the B normalized by $\varphi_R(0)=0$, $\varphi_R'(0)>0$ and let $\{\zeta_j\}$, $1\leqslant j\leqslant m\leqslant n$, be zeros of $P_n(z)$ lying on G_R. Let

$$B_{m,R}(z):=\prod_{j=1}^{m}B_{j,R}(z)=\prod_{j=1}^{m}\frac{\varphi_R(z)-\varphi_R(\zeta_j)}{1-\overline{\varphi_R(\zeta_j)}\varphi_R(z)}$$

denote a Blashke function with respect to zeros $\{\zeta_j\}$, $1\leqslant j\leqslant m\leqslant n$, of $P_n(z)$.

For any $p>0$ and $z\in G_R$ let us set

$$T_n(z):=\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}.$$

The Cauchy integral representation for the region G_R gives

$$T_n(z)=\frac{1}{2\pi i}\int_{L_R}T_n(\zeta)\frac{d\zeta}{\zeta-z},\quad z\in G_R.$$

Then

$$\left|\left[\frac{P_n(z)}{B_{m,R}(z)}\right]^{p/2}\right|\leqslant\frac{1}{2\pi}\int_{L_R}\left|\frac{P_n(\zeta)}{B_{m,R}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta-z|}\leqslant\frac{1}{2\pi}\int_{L_R}|P_n(\zeta)|^{p/2}\frac{|d\zeta|}{|\zeta-z|},$$

since $|B_{m,R}(\zeta)|=1$, for $\zeta\in L_R$. Multiplying the numerator and determinator of the integrand by $h^{1/2}(\zeta)$, according to the Hölder inequality, we obtain

$$\begin{aligned}\left|\frac{P_n(z)}{B_{m,R}(z)}\right|^{p/2}&\leqslant\frac{1}{2\pi}\Big(\int_{L_R}h(\zeta)|P_n(\zeta)|^p|d\zeta|\Big)^{1/2}\times\Big(\int_{L_R}\frac{|d\zeta|}{\prod_{j=1}^m|\zeta-z_j|^{\gamma_j}|\zeta-z|^2}\Big)^{1/2}\\&=:\frac{1}{2\pi}J_{n,1}\times J_{n,2},\end{aligned}$$

where

$$J_{n,1}:=\|P_n\|^{p/2}_{\mathcal{L}_p(h,L_R)},\quad J_{n,2}:=\Big(\int_{L_R}\frac{|d\zeta|}{\prod_{j=1}^m|\zeta-z_j|^{\gamma_j}|\zeta-z|^2}\Big)^{1/2}.$$

Let $z\in L$. Then, since $|B_{m,R}(z)|<1$ for $z\in L$, from Lemma 2.3 we have

$$|P_n(z)|\preceq(J_{n,1}\cdot J_{n,2})^{2/p}\preceq\|P_n\|_p\cdot(J_{n,2})^{2/p}.\tag{3.1}$$

To estimate the integral $J_{n,2}$, we introduce:

$$w_j:=\Phi(z_j),\ \ \varphi_j:=\arg w_j,\ \ L_R^j:=L_R\cap\bar{\Omega}^j,\ \ j=\overline{1,m},$$

where

$$\Omega^j:=\Psi(\Delta_j'),\quad\Delta_1':=\Big\{t=Re^{i\theta}:R>1,\ \frac{\varphi_m+\varphi_1}{2}\leqslant\theta<\frac{\varphi_1+\varphi_2}{2}\Big\},$$

$$\Delta_m':=\Big\{t=Re^{i\theta}:R>1,\ \frac{\varphi_{m-1}+\varphi_m}{2}\leqslant\theta<\frac{\varphi_m+\varphi_1}{2}\Big\},$$

$$\Delta_j':=\Big\{t=Re^{i\theta}:R>1,\ \frac{\varphi_{j-1}+\varphi_j}{2}\leqslant\theta<\frac{\varphi_j+\varphi_{j+1}}{2}\Big\},\text{ for }j=\overline{2,m-1}.$$

Then, we get

$$(J_{n,2})^2=\sum_{i=1}^{m}\int_{L_R^i}\frac{|d\zeta|}{\prod_{j=1}^m|\zeta-z_j|^{\gamma_j}|\zeta-z|^2}\tag{3.2}$$

$$\asymp \sum_{i=1}^{m} \int_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i} |\zeta - z|^2} =: \sum_{i=1}^{m} J_{n,2}^i,$$

where

$$J_{n,2}^i := \int_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i} |\zeta - z|^2}, \quad i = \overline{1,m}, \tag{3.3}$$

since the points $\{z_i\} \in L$ are distinct. It remains to estimate the integrals $J_{n,2}^i$ for each $i = \overline{1,m}$.

For simplicity of our next calculations, we assume that: $i = 1, 2$; $m_1 = 1$, $m = 2$; $z_1 = -1$, $z_2 = 1$; $(-1, 1) \subset G$; $R = 1 + \frac{\varepsilon_0}{n}$, and let the local co-ordinate axis in Definition 1.1 be parallel to OX and OY in the OXY co-ordinate system; $L = L^+ \cup L^-$, where $L^+ := \{z \in L : \operatorname{Im} z \geqslant 0\}$, $L^- := \{z \in L : \operatorname{Im} z < 0\}$. Let $w^\pm := \{w = e^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\}$, $z^\pm \in \Psi(w^\pm)$ and L^i be arcs, connecting the points z^+, z_i, $z^- \in L$; $L^{i,\pm} := L^i \cap L^\pm$, $i = 1, 2$. Let z_0 be taken as an arbitrary point on L^+ (or on L^- subject to the chosen direction). For simplicity, without loss of generality, we assume that $z_0 = z^+$ ($z_0 = z^-$). Analogously, we introduce: $L_R = L_R^+ \cup L_R^-$, where $L_R^+ := \{z \in L_R : \operatorname{Im} z \geqslant 0\}$, $L_R^- := \{z \in L_R : \operatorname{Im} z < 0\}$. Let $w_R^\pm := \{w = Re^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\}$, $z_R^\pm \in \Psi(w_R^\pm)$. We set: $z_{i,R} \in L_R$, such that $d_{i,R} = |z_i - z_{i,R}|$ and $\zeta^\pm \in L^\pm$, such that $d(z_{2,R}, L^2 \cap L^\pm) := d(z_{2,R}, L^\pm)$; $z_i^\pm := \{\zeta \in L^i : |\zeta - z_i| = c_i d(z_i, L_R)\}$, $z_{i,R}^\pm := \{\zeta \in L_R^i : |\zeta - z_{i,R}| = c_i d(z_{i,R}, L_R)\}$, $w_{i,R}^\pm = \Phi(z_{i,R}^\pm)$. Let L_R^i, $i = 1, 2$, denote arcs, connecting the points z_R^+, $z_{i,R}$, $z_R^- \in L_R$, $L_R^{i,\pm} := L_R^i \cap L_R^\pm$ and $l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm)$ denote arcs, connecting the points $z_{i,R}^\pm$ with $z_R^\pm$, respectively and $|l_{i,R}^\pm| := \operatorname{mes} l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm)$, $i = 1, 2$. We denote:

$$\begin{aligned}
&E_{1,R}^{i,\pm} := \left\{\zeta \in L_R^{i,\pm} : \ |\zeta - z_i| < c_i d_{i,R}\right\}, \\
&E_{2,R}^{i,\pm} := \left\{\zeta \in L_R^{i,\pm} : c_i d_{i,R} \leqslant |\zeta - z_i| \leqslant |l_{i,R}^\pm|\right\}, \quad F_{j,R}^{i,\pm} := \Phi(E_{j,R}^{i,\pm}); \\
&E_1^{i,\pm} := \left\{\zeta \in L^{i,\pm} : \ |\zeta - z_i| < c_i d_{i,R}\right\}, \\
&E_2^{i,\pm} := \left\{\zeta \in L^{i,\pm} : c_i d_{i,R} \leqslant |\zeta - z_i| \leqslant |l_{i,R}^\pm|\right\}, \quad F_j^{i,\pm} := \Phi(E_j^{i,\pm}), \quad i, j = 1, 2.
\end{aligned}$$

Taking into consideration these designations, from (3.3), we have

$$J_{n,2}^i \asymp \sum_{i,j=1}^{2} \int_{E_{j,R}^{i,+} \cup E_{j,R}^{i,-}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i} |\zeta - z|^2} =: \sum_{i,j=1}^{2} \left[J(E_{j,R}^{i,+}) + J(E_{j,R}^{i,-})\right].$$

So, we need to evaluate the integrals $J(E_{j,R}^{i,+})$ and $J(E_{j,R}^{i,-})$ for each $i, j = 1, 2$.

Let

$$\|P_n\|_\infty =: |P_n(z')|, \quad z' \in L = L^1 \cup L^2. \tag{3.4}$$

There are two possible cases: the point z' may lie on L^1 or L^2.

1) Suppose first that $z' \in L^1$. Consider the individual cases.

(1.1) If $z' \in E_1^{1,\pm} \cup E_2^{1,\pm}$, then

$$J(E_{1,R}^{1,+}) + J(E_{1,R}^{1,-}) = \int_{E_{1,R}^{1,+} \cup E_{1,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1} |\zeta - z'|^2}$$

$$\preceq \int_{E_{1,R}^{1,+}\cup E_{1,R}^{1,-}} \frac{|d\zeta|}{[\min\{|\zeta - z_1|; |\zeta - z'|\}]^{\gamma_1+2}}$$
$$\preceq \int_{d_{1,R}}^{cd_{1,R}} \frac{ds}{s^{\gamma_1+2}} \preceq \frac{1}{d_{1,R}^{\gamma_1+1}},$$

for $\gamma_1 > 0$, and

$$J(E_{1,R}^{1,+})+J(E_{1,R}^{1,-}) = \int_{E_{1,R}^{1,+}\cup E_{1,R}^{1,-}} \frac{|\zeta - z_1|^{(-\gamma_1)}|d\zeta|}{|\zeta - z'|^2} \preceq (cd_{1,R})^{(-\gamma_1)} \int_{d_{1,R}}^{cd_{1,R}} \frac{ds}{s^2} \preceq \frac{1}{d_{1,R}^{\gamma_1+1}},$$

for $-1 < \gamma_1 \leqslant 0$

(1.2) If $z' \in E_1^{1,\pm}$, then

$$J(E_{2,R}^{1,+}) + J(E_{2,R}^{1,-}) = \int_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}|\zeta - z'|^2}$$
$$\preceq \int_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{[\min\{|\zeta - z_1|; |\zeta - z'|\}]^{\gamma_1+2}}$$
$$\preceq \int_{cd_{1,R}}^{|l_{1,R}^{\pm}|} \frac{ds}{s^{\gamma_1+2}} \preceq \frac{1}{d_{1,R}^{\gamma_1+1}},$$

for $\gamma_1 > 0$ and

$$J(E_{2,R}^{1,+}) + J(E_{2,R}^{1,-}) = \int_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|\zeta - z_1|^{(-\gamma_1)}|d\zeta|}{|\zeta - z'|^2}$$
$$\preceq \int_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z'|^2} \preceq \int_{cd_{1,R}}^{|l_{1,R}^{\pm}|} \frac{ds}{s^2} \preceq \frac{1}{d_{1,R}},$$

for $-1 < \gamma_1 \leqslant 0$;

(1.3) If $z' \in E_2^{1,\pm}$, then

$$J(E_{2,R}^{1,+}) + J(E_{2,R}^{1,-}) = \int_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}|\zeta - z'|^2}$$
$$\preceq \frac{1}{d_{1,R}^{\gamma_1}} \int_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z'|^2} \preceq \frac{1}{d_{1,R}^{\gamma_1}} \int_{cd_{1,R}}^{|l_{1,R}^{\pm}|} \frac{ds}{s^2} \preceq \frac{1}{d_{1,R}^{(\gamma_1+1)}},$$

for $\gamma_1 > 0$, and

$$J(E_{2,R}^{1,+}) + J(E_{2,R}^{1,-}) = \int_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|\zeta - z_1|^{(-\gamma_1)}|d\zeta|}{|\zeta - z'|^2}$$
$$\preceq \int_{E_{2,R}^{1,+}\cup E_{2,R}^{1,-}} \frac{|d\zeta|}{|\zeta - z'|^2} \preceq \int_{cd_{1,R}}^{|l_{1,R}^{\pm}|} \frac{ds}{s^2} \preceq \frac{1}{d_{1,R}},$$

for $-1 < \gamma_1 \leqslant 0$. Combining cases (1.1)–(1.3), we obtain

$$\sum_{i=1}^{2} \left[J(E_{i,R}^{1,+}) + J(E_{i,R}^{1,-})\right] \preceq \frac{1}{d_{1,R}^{\tilde{\gamma}_1+1}}. \tag{3.5}$$

According to Lemma 2.2 and [**8**, p.10], from (3.5), we have

$$d_{1,R} \succeq \begin{cases} n^{-\mu} & \text{if } \alpha_1 = 0; \\ n^{-2} & \text{if } \alpha_1 \neq 0, \end{cases} \tag{3.6}$$

and, consequently, from (3.5) and (3.6), we get

$$\sum_{i=1}^{2} \left[J(E_{i,R}^{1,+}) + J(E_{i,R}^{1,-}) \right] \preceq \begin{cases} n^{2(\tilde{\gamma}_1+1)}, & \text{if } \alpha_1 \neq 0; \\ n^{\mu(\tilde{\gamma}_1+1)}, & \text{if } \alpha_1 = 0. \end{cases} \tag{3.7}$$

2) Now, suppose that $z' \in L^2$. In this case, replacing the variable $\tau = \Phi(\zeta)$, according to (2.1), we have

$$\begin{aligned} J_{n,2}^{i} &\asymp \sum_{i,j=1}^{2} \int_{F_{j,R}^{i,\pm}} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2} |\Psi(\tau)-\Psi(w')|^2} \\ &=: \sum_{i,j=1}^{2} \left[J(F_{j,R}^{i,+}) + J(F_{j,R}^{i,-}) \right]. \end{aligned} \tag{3.8}$$

(2.1) If $z' \in E_1^{2,\pm}$, then

$$\begin{aligned} J(F_{1,R}^{2,+}) + J(F_{1,R}^{2,-}) &= \int_{F_{1,R}^{2,+} \cup F_{1,R}^{2,-}} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2} |\Psi(\tau)-\Psi(w')|^2} \\ &\asymp \int_{F_{1,R}^{2,+} \cup F_{1,R}^{2,-}} \frac{d(\Psi(\tau), L)|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2} |\Psi(\tau)-\Psi(w')|^2 (|\tau|-1)} \\ &\preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2} |\Psi(\tau)-\Psi(w')|} \\ &\quad + n \int_{F_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2} |\Psi(\tau)-\Psi(w')|}, \end{aligned} \tag{3.9}$$

for all $\gamma_2 > -1$. The last two integrals are evaluated identically. Therefore, we evaluate one of them, say the first. When $\tau \in F_{1,R}^{2,+}$ for the $|\Psi(\tau)-\Psi(w')|$, we obtain

$$\begin{aligned} |\Psi(\tau)-\Psi(w')| &\succeq \max\{|\Psi(\tau)-\Psi(w_2)|;\ |\Psi(\tau)-z_2^+|\} \\ &\succeq |\Psi(\tau)-\Psi(w_2)| \succeq |\Psi(\tau)-z_2^+|^{\frac{1}{1+\beta_2}}. \end{aligned}$$

Then, from (3.9), we get

$$\begin{aligned} J(F_{1,R}^{2,+}) &\preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-z_2^+|^{\frac{\gamma_2+1}{1+\beta_2}}} \preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-z_2^+|^{\frac{\gamma_2+1}{1+\beta_2}}} \preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\tau-w_2^+|^{\frac{\gamma_2+1}{1+\beta_2}\mu}} \\ &\preceq \begin{cases} n^{\frac{\gamma_2+1}{1+\beta_2}\mu}, & \frac{\gamma_2+1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{\gamma_2+1}{1+\beta_2}\mu = 1, \\ n, & \frac{\gamma_2+1}{1+\beta_2}\mu < 1, \end{cases} \end{aligned}$$

if $\gamma_2 > 0$, and

$$J(F_{1,R}^{2,+}) \preceq n \int_{F_{1,R}^{2,+}} \frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)}|d\tau|}{|\Psi(\tau)-z_2^+|^{\frac{1}{1+\beta_2}}}$$

$$\preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\tau-w_2^+|^{\frac{\mu}{1+\beta_2}}} \preceq \begin{cases} n^{\frac{\mu}{1+\beta_2}}, & \frac{\mu}{1+\beta_2} > 1, \\ n\ln n, & \frac{\mu}{1+\beta_2} = 1, \\ n, & \frac{\mu}{1+\beta_2} < 1, \end{cases}$$

if $-1 < \gamma_2 \leqslant 0$, and so, in this case we get

$$J(F_{1,R}^{2,+}) + J(F_{1,R}^{2,-}) \preceq \begin{cases} n^{\frac{\tilde{\gamma}_2+1}{1+\beta_2}\mu}, & \frac{\tilde{\gamma}_2+1}{1+\beta_2}\mu > 1, \\ n\ln n, & \frac{\tilde{\gamma}_2+1}{1+\beta_2}\mu = 1, \\ n, & \frac{\tilde{\gamma}_2+1}{1+\beta_2}\mu < 1. \end{cases} \tag{3.10}$$

(2.2) If $z' \in E_2^{2,\pm}$, then

$$J(F_{1,R}^{2,+}) + J(F_{1,R}^{2,-}) = \int_{F_{1,R}^{2,+}\cup F_{1,R}^{2,-}} \frac{d(\Psi(\tau),L)|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}|\Psi(\tau)-\Psi(w')|^2(|\tau|-1)}$$

$$\preceq n \int_{F_{1,R}^{2,+}\cup F_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}|\Psi(\tau)-\Psi(w')|}, \tag{3.11}$$

for all $\gamma_2 > -1$. When $\tau \in F_{1,R}^{2,+}$ for the $|\Psi(\tau)-\Psi(w')|$, we obtain $|\Psi(\tau)-\Psi(w')| \succeq |\Psi(\tau)-z_2^+|$ and, analogously to the previous case, we get

$$J(F_{1,R}^{2,+}) \preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}|\Psi(\tau)-z_2^+|} \preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-z_2^+|^{\frac{\gamma_2}{1+\beta_2}+1}}$$

$$\preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\tau-w_2^+|^{(\frac{\gamma_2}{1+\beta_2}+1)\mu}} \preceq \begin{cases} n^{(\frac{\gamma_2}{1+\beta_2}+1)\mu}, & (\frac{\gamma_2}{1+\beta_2}+1)\mu > 1, \\ n\ln n, & (\frac{\gamma_2}{1+\beta_2}+1)\mu = 1, \end{cases} \tag{3.12}$$

if $\gamma_2 > 0$, and

$$J(F_{1,R}^{2,+}) \preceq n \int_{F_{1,R}^{2,+}} \frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)}|d\tau|}{|\Psi(\tau)-z_2^+|} \preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau)-z_2^+|} \tag{3.13}$$

$$\preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\tau-w_2^+|^{\mu}} \preceq \begin{cases} n^{\mu}, & \mu > 1, \\ n\ln n, & \mu = 1, \\ n, & \mu < 1, \end{cases}$$

if $-1 < \gamma_2 \leqslant 0$. So, in this case, from (3.11)–(3.13), we have

$$J(F_{1,R}^{2,+}) + J(F_{1,R}^{2,-}) \preceq \begin{cases} n^{(\frac{\tilde{\gamma}_2}{1+\beta_2}+1)\mu}, & \left(\frac{\tilde{\gamma}_2}{1+\beta_2}+1\right)\mu > 1, \\ n\ln n, & \left(\frac{\tilde{\gamma}_2}{1+\beta_2}+1\right)\mu = 1. \end{cases} \tag{3.14}$$

(2.3) If $z' \in E_1^{2,\pm}$, then

$$J(F_{2,R}^{2,+}) + J(F_{2,R}^{2,-}) = \int_{F_{2,R}^{2,+}\cup F_{2,R}^{2,-}} \frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}|\Psi(\tau)-\Psi(w')|^2} \tag{3.15}$$

$$\begin{aligned}&\preceq n\int_{F_{2,R}^{2,+}\cup F_{2,R}^{2,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}|\Psi(\tau)-\Psi(w')|}\\&\preceq n\int_{F_{2,R}^{2,+}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}|\Psi(\tau)-\Psi(w')|}\\&\quad+n\int_{F_{2,R}^{2,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}|\Psi(\tau)-\Psi(w')|},\end{aligned}$$

for $\gamma_2>0$. The last two integrals are evaluated in the same way. Let us evaluate the first integral.

For $\tau\in F_{2,R}^{2,+}$ and $z'\in E_1^{2,\pm}$, we have

$$\begin{aligned}&|\Psi(\tau)-\Psi(w')|\succeq|\Psi(\tau)-z_2^+|;\\&|\Psi(\tau)-\Psi(w_2)|\succeq d_{2,R}\succeq|z_{2,R}-z_2^+|^{\frac{1}{1+\beta_2}}\succeq\left(\frac{1}{n}\right)^{\frac{1}{1+\beta_2}\mu}.\end{aligned}$$

Then

$$\begin{aligned}J(F_{2,R}^{2,+})&\preceq n\int_{F_{2,R}^{2,+}}\frac{|d\tau|}{|\Psi(\tau)-z_2^+|^{\gamma_2}|\Psi(\tau)-z_2^+|}\preceq n^{\frac{\gamma_2}{1+\beta_2}\mu+1}\int_{F_{2,R}^{2,+}}\frac{|d\tau|}{|\tau-w_2^+|^{\mu}}\\&\preceq\begin{cases}n^{\frac{\gamma_2}{1+\beta_2}\mu+\mu}, & \mu>1,\\ n^{\frac{\gamma_2}{1+\beta_2}\mu+1}\ln n, & \mu=1,\end{cases}\end{aligned}$$

and so, for $\gamma_2>0$ we obtain

$$J(F_{2,R}^{2,+})+J(F_{2,R}^{2,-})\preceq\begin{cases}n^{\frac{\gamma_2}{1+\beta_2}\mu+\mu}, & \mu>1,\\ n^{\frac{\gamma_2}{1+\beta_2}\mu+1}\ln n, & \mu=1.\end{cases}$$

For $-1<\gamma_2\leqslant 0$ we get

$$\begin{aligned}J(F_{2,R}^{2,+})+J(F_{2,R}^{2,-})&=\int_{F_{2,R}^{2,+}\cup F_{2,R}^{2,-}}\frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)}|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w')|^2}\\&\preceq n\int_{F_{2,R}^{2,+}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|}\preceq n\int_{F_{2,R}^{2,+}}\frac{|d\tau|}{|\Psi(\tau)-z_2^+|}\\&\preceq n\int_{F_{2,R}^{2,+}}\frac{|d\tau|}{|\tau-w_2^+|^{\mu}}\preceq\begin{cases}n^{\mu}, & \mu>1,\\ n\ln n, & \mu=1.\end{cases}\end{aligned}\tag{3.16}$$

Then, in this case, we have

$$J(F_{2,R}^{2,+})+J(F_{2,R}^{2,-})\preceq\begin{cases}n^{\left[\frac{\tilde\gamma_2}{1+\beta_2}+1\right]\mu}, & \mu>1,\\ n^{\frac{\tilde\gamma_2}{1+\beta_2}\mu+1}\ln n, & \mu=1.\end{cases}\tag{3.17}$$

(2.4) If $z'\in E_2^{2,+}$, then for $\gamma_2>0$

$$J(F_{2,R}^{2,+})=\int_{F_{2,R}^{2,+}}\frac{|\Psi(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}|\Psi(\tau)-\Psi(w')|^2}\preceq\frac{n}{d_{2,R}^{\gamma_2}}\int_{F_{2,R}^{2,+}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|}$$

$$\preceq n^{1+\frac{\gamma_2}{1+\beta_2}\mu}\int_{F_{2,R}^{2,+}}\frac{|d\tau|}{|\tau-w'|^{\mu}}\preceq\begin{cases}n^{\left[\frac{\gamma_2}{1+\beta_2}+1\right]\mu}, & \mu>1,\\ n^{\frac{\gamma_2}{1+\beta_2}\mu+1}\ln n, & \mu=1,\end{cases}\tag{3.18}$$

and

$$J(F_{2,R}^{2,-})=\int_{F_{2,R}^{2,-}}\frac{|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2}|\Psi(\tau)-\Psi(w')|^2}\preceq\frac{n}{d_{2,R}^{\gamma_2}}\int_{F_{2,R}^{2,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|}$$
$$\preceq n^{\frac{\gamma_2}{1+\beta_2}\mu+!}\int_{F_{2,R}^{2,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|}\preceq n^{\frac{\gamma_2}{1+\beta_2}\mu+1}\int_{F_{2,R}^{2,-}}\frac{|d\tau|}{|\tau-w'|^{\mu}}$$
$$\preceq\begin{cases}n^{\left[\frac{\gamma_2}{1+\beta_2}+1\right]\mu}, & \mu>1,\\ n^{\frac{\gamma_2}{1+\beta_2}\mu+1}\ln n, & \mu=1.\end{cases}\tag{3.19}$$

The case of $z'\in E_2^{2,-}$ is absolutely identical to the case $z'\in E_2^{2,+}$. If $-1<\gamma_2\leqslant 0$, then

$$J(F_{2,R}^{2,+})=\int_{F_{2,R}^{2,+}}\frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)}|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w')|^2}\tag{3.20}$$
$$\preceq n\int_{F_{2,R}^{2,+}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|}\preceq\begin{cases}n^{\mu}, & \mu>1,\\ n\ln n, & \mu=1,\end{cases}$$

and

$$J(F_{2,R}^{2,-})=\int_{F_{2,R}^{2,-}}\frac{|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)}|\Psi'(\tau)||d\tau|}{|\Psi(\tau)-\Psi(w')|^2}\tag{3.21}$$
$$\preceq n\int_{F_{2,R}^{2,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w')|}\preceq\begin{cases}n^{\mu}, & \mu>1,\\ n\ln n, & \mu=1,\end{cases}$$

Combining relations (3.7)–(3.21), for $m_1=1,\ m_2=1$, and any $p>0$, we get

$$J_{n,2}^1+J_{n,2}^2\preceq\begin{cases}n^2, & \text{if } \alpha_1\neq 0;\\ n^{\mu} & \text{if } \alpha_1=0;\end{cases}+\begin{cases}n^{\mu}, & \mu>1,\\ n\ln n, & \mu=1,\end{cases}\tag{3.22}$$

for each $-1<\gamma_1\leqslant 0$ and $-1<\gamma_2\leqslant 0$;

$$J_{n,2}^1+J_{n,2}^2\preceq n^{\tilde{\mu}(\gamma_1+1)}+\begin{cases}n^{(\frac{\gamma_2}{1+\beta_2}+1)\mu}, & \mu>1,\\ n^{\frac{\gamma_2}{1+\beta_2}+1}\ln n, & \mu=1,\end{cases}\tag{3.23}$$

for each $\gamma_1>0$ and $\gamma_2>0$, where $\tilde{\mu}:=\begin{cases}\mu, & \text{if } \alpha_1=0,\\ 2, & \text{if } \alpha_1\neq 0,\end{cases}$ and $p>0$. Then, from (3.1)–(3.4), (3.22) and (3.23), for all $z\in L$, we obtain

$$|P_n(z)|\preceq\|P_n\|_p\cdot\left[n^{\frac{\tilde{\mu}(\tilde{\gamma}_1+1)}{p}}+\begin{cases}n^{(\frac{\tilde{\gamma}_2}{1+\beta_2}+1)\frac{\mu}{p}}, & \mu>1,\\ \left(n^{\frac{\tilde{\gamma}_2}{1+\beta_2}+1}\ln n\right)^{\frac{1}{p}}, & \mu=1,\end{cases}\right]$$

$$\preceq \|P_n\|_p \cdot \begin{cases} n^{\frac{1}{p}\max\{2,\mu\}}, & \tilde{\mu}=2,\ \alpha_1\neq 0,\ -1<\gamma_1,\ \gamma_2\leqslant 0;\\ n^{\frac{\mu}{p}}, & \tilde{\mu}=\mu,\ \alpha_1=0,\ -1<\gamma_1,\ \gamma_2\leqslant 0;\\ n^{(\frac{\gamma_2}{1+\beta_2}+1)\frac{\mu}{p}}, & \tilde{\mu}=2,\ \alpha_1\neq 0,\ 0<\gamma_1\leqslant\left(\frac{\gamma_2}{1+\beta_2}+1\right)\frac{\mu}{2}-1;\\ n^{\frac{2(\gamma_1+1)}{p}}, & \tilde{\mu}=2,\ \alpha_1\neq 0,\ \gamma_1>\left(\frac{\gamma_2}{1+\beta_2}+1\right)\frac{\mu}{2}-1;\\ n^{(\frac{\gamma_2}{1+\beta_2}+1)\frac{\mu}{p}}, & \tilde{\mu}=\mu,\ \alpha_1=0,\ 0<\gamma_1\leqslant\frac{\gamma_2}{1+\beta_2};\\ n^{\frac{\mu(\gamma_1+1)}{p}}, & \tilde{\mu}=\mu,\ \alpha_1=0,\ \gamma_1>\frac{\gamma_2}{1+\beta_2};\\ \left(n^{\frac{\tilde{\gamma}_2}{1+\beta_2}+1}\ln n\right)^{\frac{1}{p}} & \mu=1,\ -1<\gamma_1,\gamma_2, \end{cases}$$

which completes the proof. □

Proof of Theorem 1.3. Suppose $G\in PQS(\lambda;f_i,g_i)$, for some $\lambda\geqslant 1$, $f_i(x)=c_ix^{1+\alpha_i}$, $\alpha_i\geqslant 0$, $i=\overline{1,m_1}$, and $g_i(x)=c_ix^{1+\beta_i}$, $\beta_i>0$, $i=\overline{m_1+1,m}$; $h(z)$ defined as in (1.3). Let $\{\zeta_j\}$, $1\leqslant j\leqslant m\leqslant n$, zeros of $P_n(z)$ lying on Ω and let

$$B_m(z):=\prod_{j=1}^{m}B_j(z)=\prod_{j=1}^{m}\frac{\Phi(z)-\Phi(\zeta_j)}{1-\overline{\Phi(\zeta_j)}\Phi(z)}$$

denote a Blashke function with respect to zeros $\{\zeta_j\}$, $1\leqslant j\leqslant m\leqslant n$, of $P_n(z)$.

For any $p>0$ and $z\in\Omega$, let us set

$$G_n(z):=\left[\frac{P_n(z)}{B_m(z)\Phi^{n+1}(z)}\right]^{p/2}. \tag{3.24}$$

The Cauchy integral representation for the unbounded region Ω gives

$$G_n(z)=-\frac{1}{2\pi i}\int_L G_n(\zeta)\frac{d\zeta}{\zeta-z},\quad z\in\Omega. \tag{3.25}$$

Since $|B_m(\zeta)|=1$, for $\zeta\in L$, then, for arbitrary $\varepsilon, 0<\varepsilon<\varepsilon_1$, there exists a circle $|w|=1+\frac{\varepsilon_1}{n}$, such that for any $j=\overline{1,m}$ the following is satisfied:

$$|B_j(\Psi(w)|>1-\varepsilon.$$

Then, $|B_m(\zeta)|>(1-\varepsilon)^m\succeq 1$, for each $\varepsilon\leqslant n^{-1}$. On the other hand, $|\Phi(\zeta)|=1$, for $\zeta\in L$. Therefore, for any $z\in\Omega$, from (3.24) and (3.25), we have

$$\begin{aligned}\left|\frac{P_n(z)}{B_m(z)\Phi^{n+1}(z)}\right|^{p/2} &\leqslant \frac{1}{2\pi}\int_L\left|\frac{P_n(\zeta)}{B_m(\zeta)\Phi^{n+1}(\zeta)}\right|^{p/2}\frac{|d\zeta|}{|\zeta-z|}\\ &\preceq \frac{1}{d(z,L)}\int_L|P_n(\zeta)|^{p/2}|d\zeta| =: \frac{1}{d(z,L)}A_n.\end{aligned} \tag{3.26}$$

Similarly to the notations of the previous proof, we have

$$A_n=\sum_{i=1}^{m}\int_{L^i}|P_n(\zeta)|^{p/2}|d\zeta|. \tag{3.27}$$

Multiplying the numerator and denominator of the integrand by $h^{1/2}(\zeta)$, after applying the Hölder inequality, we obtain

$$A_n \leqslant \sum_{i=1}^{m} \Big(\int_{L^i} h(\zeta)|P_n(\zeta)|^p |d\zeta| \Big)^{1/2} \times \Big(\int_{L^i} \frac{|d\zeta|}{\prod_{j=1}^{m} |\zeta - z_j|^{\gamma_j}} \Big)^{1/2} =: \sum_{i=1}^{m} \big(\tilde{J}_{n,1}^{i} \cdot \tilde{J}_{n,2}^{i} \big)^{1/2}. \tag{3.28}$$

According to Lemma 2.3, for the $\tilde{J}_{n,1}^{i}$ we get

$$\tilde{J}_{n,1}^{i} \preceq \|P_n\|_p^{p/2}, \quad i = \overline{1, m}. \tag{3.29}$$

Then, from (3.28) and (3.29), we have

$$A_n \preceq \|P_n\|_p^{p/2} \sum_{i=1}^{m} \big(\tilde{J}_{n,2}^{i} \big)^{1/2}.$$

For the integral $\tilde{J}_{n,2}^{i}$ we obtain

$$\tilde{J}_{n,2}^{i} := \int_{L^i} \frac{|d\zeta|}{\prod_{j=1}^{m} |\zeta - z_j|^{\gamma_i}} \asymp \int_{L^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}}, \quad i = 1, 2, \tag{3.30}$$

since the points $\{z_j\}_{j=1}^{m}$ are distinct on L. Then, from (3.30), we have

$$A_n \preceq \|P_n\|_p^{p/2} \sum_{i=1}^{2} \big(\tilde{J}_{n,2}^{i} \big)^{1/2}, \tag{3.31}$$

where

$$\tilde{J}_{n,2}^{1} = \int_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}}; \quad \tilde{J}_{n,2}^{2} = \int_{L^2} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2}}. \tag{3.32}$$

Taking into consideration the above notations, from (3.31) we get

$$\tilde{J}_{n,2}^{i} =: I_{n,1}^{i}(E_1^{i,\pm}) + I_{n,2}^{i}(E_2^{i,\pm}) =: I_{n,1}^{i,\pm} + I_{n,2}^{i,\pm}, \quad i = 1, 2, \tag{3.33}$$

where

$$I_{n,k}^{i,\pm} := I_{n,k}^{i}(E_k^{i,\pm}) := \int_{E_k^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}} \ ; \quad i, k = 1, 2.$$

According to (3.26) and (3.27), it is sufficient to estimate the integrals $I_{n,k}^{i,\pm}$ for each $i = 1, 2$ and $k = 1, 2$.

Let us start with the evaluation of the integrals $\tilde{J}_{n,2}^{1}$ from (3.32) and (3.33)

$$\tilde{J}_{n,2}^{1} = \int_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}} = \sum_{k=1}^{2} \int_{E_k^{1,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_1}} =: I_{n,1}^{1,\pm} + I_{n,2}^{1,\pm}. \tag{3.34}$$

Given the possible values of $\gamma_i (-1 < \gamma_i < 0, \gamma_i \geqslant 0,\ i = 1, 2)$, we will consider the estimates for the $\tilde{J}_{n,2}^{1}$ separately.

Let $\gamma_1 \geqslant 0$ and $\gamma_2 \geqslant 0$. In this case for the integral $\tilde{J}^1_{n,2}$, we get

$$I^{1,\pm}_{n,1} \preceq \int_{E^{1,\pm}_1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}} \preceq \int_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1}} \preceq \begin{cases} d^{1-\gamma_1}_{1,R}, & \gamma_1 > 1, \\ 1, & \gamma_1 \leqslant 1, \end{cases} c_1 > 1; \tag{3.35}$$

$$I^{1,\pm}_{n,2} \preceq \int_{E^{1,\pm}_2} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}} \preceq \int_{c_1 d_{1,R}}^{|l^{\pm}_1|} \frac{ds}{s^{\gamma_1}} \preceq \begin{cases} d^{1-\gamma_1}_{1,R}, & \gamma_1 > 1, \\ \ln \frac{1}{d_{1,R}}, & \gamma_1 = 1, \\ 1, & \gamma_1 < 1. \end{cases}$$

A similar estimate for the integral $\tilde{J}^2_{n,2}$ is

$$I^{2,\pm}_{n,1} \preceq \int_{E^{2,\pm}_1} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2}} \preceq \int_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2}} \preceq \begin{cases} d^{1-\gamma_2}_{2,R}, & \gamma_2 > 1, \\ 1, & \gamma_2 \leqslant 1, \end{cases} c_2 > 1; \tag{3.36}$$

$$I^{2,\pm}_{n,2} \preceq \int_{E^{2,\pm}_2} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2}} \preceq \int_{c_2 d_{2,R}}^{|l^{\pm}_2|} \frac{ds}{s^{\gamma_2}} \preceq \begin{cases} d^{1-\gamma_2}_{2,R}, & \gamma_2 > 1, \\ \ln \frac{1}{d_{2,R}}, & \gamma_2 = 1, \\ 1, & \gamma_2 < 1. \end{cases}$$

Let $\gamma_1 < 0$ and $\gamma_2 < 0$. Then, analogously to (3.35) and (3.36)

$$I^{1,\pm}_{n,1} \preceq \int_{E^{1,\pm}_1} |\zeta - z_1|^{(-\gamma_1)} |d\zeta| \preceq d^{(-\gamma_1)}_{1,R} \operatorname{mes} E^1_1 \preceq 1, \tag{3.37}$$

$$I^1_{n,2} \preceq \int_{E^{1,\pm}_2} |\zeta - z_1|^{(-\gamma_1)} |d\zeta| \preceq |l^{\pm}_1|^{(-\gamma_1)+1} \preceq 1,$$

and

$$I^{2,\pm}_{n,1} \preceq \int_{E^{2,\pm}_1} |\zeta - z_2|^{(-\gamma_2)} |d\zeta| \preceq d^{(-\gamma_2)}_{2,R} \operatorname{mes} E^{2,\pm}_1 \preceq 1, \tag{3.38}$$

$$I^{2,\pm}_{n,2} \preceq \int_{E^{2,\pm}_2} |\zeta - z_2|^{(-\gamma_2)} |d\zeta| \preceq |l^{\pm}_2|^{(-\gamma_2)+1} \preceq 1.$$

Therefore, in this case, from (3.31)–(3.38), we obtain

$$A_n \preceq \|P_n\|_{\mathcal{L}_p(h,L)} \begin{cases} d^{\frac{1-\gamma_1}{2}}_{1,R} + d^{\frac{1-\gamma_2}{2}}_{2,R}, & \gamma_1, \gamma_2 > 1, \\ \left(\ln \frac{1}{d_{1,R}}\right)^{\frac{1}{2}} + \left(\ln \frac{1}{d_{2,R}}\right)^{\frac{1}{2}}, & \gamma_1 = \gamma_2 = 1, \\ 1, & -1 < \gamma_1, \gamma_2 < 1. \end{cases} \tag{3.39}$$

Comparing (3.26), (3.27) and (3.39), we have

$$|P_n(z)| \leqslant c \frac{B^0_{n,1}}{d^{2/p}(z, L)} \|P_n\|_{\mathcal{L}_p(h,L)} |\Phi(z)|^{n+1}, \tag{3.40}$$

where $c = c(G, p, \gamma_i) > 0$, $i = \overline{1, p}$, is the constant independent of n and z, and

$$B^0_{n,1} := \begin{cases} d^{\frac{1-\gamma_1}{p}}_{1,R} + d^{\frac{1-\gamma_2}{p}}_{2,R}, & \gamma_1, \gamma_2 > 1, \\ \left(\ln \frac{1}{d_{1,R}}\right)^{1/p} + \left(\ln \frac{1}{d_{2,R}}\right)^{1/p}, & \gamma_1 = \gamma_2 = 1, \\ 1, & 0 < \gamma_1, \gamma_2 < 1. \end{cases} \tag{3.41}$$

According to (3.6), for the point z_1, we get

$$d_{1,R} \succeq \begin{cases} n^{-\mu}, & \text{if } \alpha_1 = 0; \\ n^{-2}, & \text{if } \alpha_1 \neq 0 \end{cases} \tag{3.42}$$

For the estimate $d_{2,R}$, let us set $z_R \in L_R$, such that $d_{2,R} = |z_2 - z_R|$; $\zeta^{\pm} \in L^{\pm}$, such that $d(z_R, L^2 \cap L^{\pm}) := d(z_R, L^{+})$; $z_2^{\pm} := \zeta \in L^2 : \ |\zeta - z_2| = c_2 d_{2,R}$. Under these notations, from Lemma 2.1, we obtain

$$d_R^{\pm} := d(z_R, L^2 \cap L^{\pm}) \asymp |z_R - z_2^{\pm}| \asymp d_{2,R}^{1+\beta_2}. \tag{3.43}$$

Hence, $d_{2,R} = (d_R^{\pm})^{\frac{1}{1+\beta_2}}$. On the other hand, according to Lemma 2.2 and [**8**, Corollary 2], we get $d_R^{\pm} \succeq n^{-\mu}$. Therefore,

$$d_{2,R} \succeq n^{\frac{-\mu}{1+\beta_2}}. \tag{3.44}$$

Comparing (3.40)–(3.44), we get

$$|P_n(z)| \preceq \frac{B_{n,1}}{d^{2/p}(z,L)} \|P_n\|_p |\Phi(z)|^{n+1},$$

where

$$B_{n,1} := \begin{cases} n^{\frac{(\gamma_1 - 1)\tilde{\mu}}{p}} + n^{\frac{\gamma_2 - 1}{p(1+\beta_2)}\mu}, & \gamma_1, \gamma_2 > 1, \\ (\ln n)^{\frac{1}{p}}, & \gamma_1 = \gamma_2 = 1, \\ 1, & \gamma_1, \gamma_2 < 1. \end{cases}$$

Given the above mentioned notation, we complete the proof. □

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