PUBLICATIONS DE L'INSTITUT MATHÉMATIQUE
Nouvelle série, tome 100(114)) (2016), 229–241 DOI: 10.2298/PIM1614229K

THE FOURTH POWER MOMENT OF THE DOUBLE ZETA-FUNCTION

Isao Kiuchi

ABSTRACT. We prove the fourth power moment of the Euler–Zagier type double zeta-function $\zeta_2(s_1, s_2)$ and provide an improvement on the Ω results of Kiuchi, Tanigawa, and Zhai. We also calculate the double integral under certain conditions.

1. Introduction

Let $s_j = \sigma_j + it_j$ ($\sigma_j,\ t_j \in \mathbb{R},\ j = 1, 2$) be complex variables, and let $\zeta(s)$ be the Riemann zeta function, which is defined as $\sum_{n=1}^{\infty} n^{-s}$ for $\mathrm{Re}\, s > 1$. The double zeta-function of the Euler–Zagier type is defined by

$$\zeta_2(s_1, s_2) = \sum_{1 \leqslant m < n} \frac{1}{m^{s_1} n^{s_2}}, \tag{1.1}$$

which is absolutely convergent for $\sigma_2 > 1$ and $\sigma_1 + \sigma_2 > 2$. The proof of Atkinson's formula for the mean value theorem of the Riemann zeta-function $\zeta(s)$ (see Atkinson [**2**] or Ivić [**7**]) is applied applies to series (1.1). Double zeta-function (1.1) has many applications to mathematical physics. In particular, some algebraic relations among the values of double zeta-function (1.1) at positive integers have been extensively studied by Ohno [**17**]. Some analytic properties of this function have been obtained by Akiyama, Egami, and Tanigawa [**1**], Ishikawa and Matsumoto [**6**], Kiuchi and Tanigawa [**12**], Kiuchi, Tanigawa, and Zhai [**13**], Matsumoto [**14, 15**], Zhao [**19**], and others.

1.1. Mean square formula. Matsumoto and Tsumura [**16**] were the first to study a new type of the mean value formula for $\int_2^T |\zeta(s_1, s_2)|^2 dt_2$ with a fixed complex number s_1. They conjectured that when $\sigma_1 + \sigma_2 = \frac{3}{2}$, the form of the main term of the mean square formula would not be CT with a constant C, and

2010 *Mathematics Subject Classification:* Primary 11M41; Secondary 11M06.
Key words and phrases: double zeta-functions, fourth power moment, Riemann zeta-function.
Communicated by Žarko Mijajlović.

that, most probably, some log-factor would appear. Their results were considered by Ikeda, Matsuoka, and Nagata [**5**], who showed that

$$\begin{aligned}\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 &= \Bigg(\sum_{m=1}^{\infty}\frac{1}{m^{2\sigma_1}}\bigg|\zeta(s_2)-\sum_{n=1}^{m}\frac{1}{n^{s_2}}\bigg|^2\Bigg)T \\ &\quad + \begin{cases} O(\log^2 T) & \text{if } \sigma_1+\sigma_2=2, \\ O(T^{4-2\sigma_1-2\sigma_2}) & \text{if } \frac{3}{2}<\sigma_1+\sigma_2<2. \end{cases}\end{aligned} \tag{1.2}$$

Here, the coefficient of the main term on the right-hand side of (1.2) converges if $\sigma_1+\sigma_2>\frac{3}{2}$. Ikeda, Matsuoka, and Nagata deduced that the asymptotic formula

$$\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 = \frac{1}{|s_2-1|^2}T\log T + O(T) \tag{1.3}$$

holds on the line $\sigma_1+\sigma_2=\frac{3}{2}$. This result implied that the conjecture of Matsumoto and Tsumura on the line $\sigma_1+\sigma_2=\frac{3}{2}$ was true. Ikeda, Matsuoka, and Nagata made use of the mean value theorem for Dirichlet polynomials and suitable approximations to the Euler–Maclaurin summation formula to obtain formulas (1.2) and (1.3). Assuming that $2\leqslant t_1\leqslant T$, $0<\sigma_1<1$, $0<\sigma_2<1$ and $0<\sigma_1+\sigma_2<\frac{3}{2}$, Kiuchi and Minamide [**11**] recently considered five formulas for the mean square of the double zeta-function $\zeta_2(s_1,s_2)$ concerning the variable t_1 and showed that, for any sufficiently large positive number $T>2$,

$$\begin{aligned}\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 &= (2\pi)^{2\sigma_1+2\sigma_2-3}\frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2}T^{4-2\sigma_1-2\sigma_2} \\ &\quad + O(t_2^{-\frac{1}{2}}T^{\frac{5}{2}-\sigma_1-\sigma_2})\end{aligned} \tag{1.4}$$

with $1<\sigma_1+\sigma_2<\frac{3}{2}$ and $2\leqslant t_2\leqslant T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$,

$$\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 = \frac{\zeta(2)}{4\pi|s_2-1|^2}T^2 + O\big(t_2^{-\frac{1}{2}}(\log t_2)T^{\frac{3}{2}}\big) \tag{1.5}$$

with $\sigma_1+\sigma_2=1$ and $2\leqslant t_2\leqslant \frac{T^{\frac{1}{3}}}{\log T}$,

$$\begin{aligned}\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 &= (2\pi)^{2\sigma_1+2\sigma_2-3}\frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2}T^{4-2\sigma_1-2\sigma_2} \\ &\quad + O\big(t_2^{\frac{1}{2}-\sigma_1-\sigma_2}T^{\frac{5}{2}-\sigma_1-\sigma_2}\big)\end{aligned} \tag{1.6}$$

with $\frac{1}{2}<\sigma_1+\sigma_2<1$, and $2\leqslant t_2\leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$, and

$$\begin{aligned}\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 &= \frac{\zeta(3)}{12\pi^2|s_2-1|^2}T^3 \\ &\quad + \begin{cases} O(T^2) & \text{if } \sigma_1+\sigma_2=\frac{1}{2} \text{ and } \sqrt{\log T}\leqslant t_2\leqslant T^{\frac{1}{2}}, \\ O(t_2^{-1}T^2\sqrt{\log T}) & \text{if } \sigma_1+\sigma_2=\frac{1}{2} \text{ and } 2\leqslant t_2\leqslant\sqrt{\log T}. \end{cases}\end{aligned} \tag{1.7}$$

Furthermore, they derived the formula

$$\int_2^T |\zeta_2(s_1,s_2)|^2 dt_1 = (2\pi)^{2\sigma_1+2\sigma_2-3}\frac{\zeta(4-2\sigma_1-2\sigma_2)}{(4-2\sigma_1-2\sigma_2)|s_2-1|^2}T^{4-2\sigma_1-2\sigma_2} + \begin{cases} O\big(t_2^{\frac{1}{2}-\sigma_1-\sigma_2}T^{\frac{5}{2}-\sigma_1-\sigma_2}\big) & \text{if } T^{\frac{1-2\sigma_1-2\sigma_2}{3-2\sigma_1-2\sigma_2}} \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}, \\ O(t_2^{-1}T^{3-2\sigma_1-2\sigma_2}) & \text{if } 2 \leqslant t_2 \leqslant T^{\frac{1-2\sigma_1-2\sigma_2}{3-2\sigma_1-2\sigma_2}} \end{cases} \tag{1.8}$$

for $0 < \sigma_1 + \sigma_2 < \frac{1}{2}$. They used the mean value formula for $|\zeta(\sigma+it)|$ and a weak form of Kiuchi, Tanigawa, and Zhai's approximate formula for $\zeta_2(s_1,s_2)$ to obtain formulas (1.4)–(1.8). Kiuchi and Minamide also showed that

$$\zeta_2(\sigma_1+it_1,\sigma_2+it_2) = \Omega\Big(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\Big) \tag{1.9}$$

for $1 \leqslant \sigma_1+\sigma_2 < \frac{3}{2}$, $2 \leqslant t_1 \leqslant T$ and $2 \leqslant t_2 \leqslant T^{1-\frac{2}{3}\sigma_1-\frac{2}{3}\sigma_2-\varepsilon}$, and

$$\zeta_2(\sigma_1+it_1,\sigma_2+it_2) = \Omega\Big(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\Big) \tag{1.10}$$

for $0 < \sigma_1+\sigma_2 < 1$, $2 \leqslant t_1 \leqslant T$, and $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}-\varepsilon}$, with ε being any small positive constant. Formulas (1.9) and (1.10) provide a certain improvement on the Ω-results of Kiuchi, Tanigawa, and Zhai [**13**].

1.2. Fourth power moment. Before the introduction of our theorems, let us recall the $2k$-th power moment of the Riemann zeta-function $\zeta(\sigma+it)$. It is well known that

$$\int_2^T |\zeta(\sigma+it)|^{2k} dt = \Big(\sum_{n=1}^{\infty}\frac{d_k^2(n)}{n^{2\sigma}}\Big)T + O(T^{2-\sigma+\varepsilon}) + O(1)$$

holds, when $\sigma > 1$ is fixed and $k \geqslant 1$ is a fixed integer (see Theorem 1.10 in [**7**], or [**8**]), where the divisor function $d_k(n)$ is the number of ways n can be written as a product of k fixed factors. Then, in the case $k = 2$, we have

$$\int_2^T |\zeta(\sigma+it)|^4 dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}T + O(T^{2-\sigma+\varepsilon}) + O(1) \tag{1.11}$$

for $\sigma > 1$ and any small positive number ε. The higher power moment for the Riemann zeta-function $\zeta(1+it)$ was derived by Balasubramanian, Ivić, and Ramachandra (see Theorem 1 in [**3**]), who showed that

$$\int_2^T |(\zeta(1+it))^k|^2 dt = \Big(\sum_{n=1}^{\infty}\frac{|d_k(n)|^2}{n^2}\Big)T + O((\log T)^{|k^2|})$$

holds for any complex number k and $(\zeta(s))^k = \sum_{n=1}^{\infty}\frac{d_k(n)}{n^s}$ in $\sigma \geqslant 2$. For the case of $k = 2$ in the above expression, we have

$$\int_2^T |\zeta(1+it)|^4 dt = \frac{\zeta^4(2)}{\zeta(4)}T + O(\log^4 T). \tag{1.12}$$

Ivić [**9, 10**] obtained the formula

$$\int_2^T |\zeta(\sigma+it)|^4 dt = \frac{\zeta^4(2\sigma)}{\zeta(4\sigma)}T + O(T^{2-2\sigma}\log^3 T) \tag{1.13}$$

for $\frac{1}{2}<\sigma<1$. The fourth power moment for the Riemann zeta-function $\zeta(\frac{1}{2}+it)$ in the critical line was obtained by Heath-Brown [**4**], who showed that

$$\int_2^T \left|\zeta\Big(\frac{1}{2}+it\Big)\right|^4 dt = T\sum_{k=0}^{4} c_k(\log T)^{4-k} + O(T^{\frac{7}{8}+\varepsilon}) \tag{1.14}$$

holds with $c_0=\frac{1}{2\pi^2}$, and the other constants c_k being computable.

The main purpose of this paper is to prove the fourth power moment for the double zeta-function $\zeta_2(s_1,s_2)$ within the region $0<\sigma_1<1$, $0<\sigma_2<1$, and $0<\sigma_1+\sigma_2<2$ without using the mean square formulas (1.4)–(1.8). We use formulas (1.11)–(1.14) for the fourth power moment of $|\zeta(\sigma+it)|$ and Lemma 3 below, as well as Lemma 2 below, which was derived from a weak form of the approximate formula of Kiuchi and Minamide [**11**] for $\zeta_2(s_1,s_2)$ to obtain the following formula.

THEOREM 1.1. *Suppose that $2\leqslant t_1\leqslant T$, $0<\sigma_1<1$, $0<\sigma_2<1$ and $\sigma_1+\sigma_2=1$. Then, for any sufficiently large positive number $T>2$, we have*

$$\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1 = \frac{\zeta^4(2)}{12\pi^2\zeta(4)}\frac{T^3}{|s_2-1|^4} + O(t_2^{-\frac{5}{2}}(\log t_2 T^{\frac{5}{2}}) \tag{1.15}$$

with $2\leqslant t_2\leqslant \frac{T^{\frac{1}{3}}}{\log T}$.

REMARK 1.1. Inserting $t_2=\frac{T^{\frac{1}{3}}}{\log T}$ into (1.15), the right-hand side of the formula (1.15) is estimated by $T^{\frac{5}{3}}\log^4 T$, but if we can take $t_2=\frac{T^{\frac{1}{2}}}{\log T}$, we can estimate that $T\log^4 T$. The main term of this theorem is not $T\log^A T$ $(A>0)$, but T^3 since the analytic behavior of the double zeta-function $\zeta_2(s_1,s_2)$ depends on both s_1 and s_2. This observation in fact supports the result of Kiuchi, Tanigawa, and Zhai.

As an application of (1.15), we consider the evaluation of the double integral

$$\int_2^N\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1 dt_2,$$

and then we deduce the following.

COROLLARY 1.1. *Let $0<\sigma_1<1$, $0<\sigma_2<1$ and $\sigma_1+\sigma_2=1$. Within the region $2\leqslant N\leqslant \frac{T^{\frac{1}{3}}}{\log T}$, we obtain*

$$\frac{1}{T^3}\int_2^N\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1 dt_2 \to \frac{5\pi^2}{1728\sigma_1^3}\Big(\frac{\pi}{2}-\mathrm{Tan}^{-1}\frac{2}{\sigma_1}-\frac{2\sigma_1}{4+\sigma_1^2}\Big)+O\Big(\frac{1}{N}\Big)$$

as $T\to\infty$.

Hence, this observation may be regarded as an average order of magnitude for the double zeta-function, which is

$$\frac{5\pi^2}{1728\sigma_1^3}\Big(\frac{\pi}{2} - \mathrm{Tan}^{-1}\frac{2}{\sigma_1} - \frac{2\sigma_1}{4+\sigma_1^2}\Big)$$

if $0<\sigma_1<1$, $0<\sigma_2<1$, $\sigma_1+\sigma_2=1$, and $2 \leqslant N \leqslant \frac{T^{\frac{1}{3}}}{\log T}$.

THEOREM 1.2. *Suppose that* $2 \leqslant t_1 \leqslant T$, $0<\sigma_1<1$, $0<\sigma_2<1$, $\frac{1}{2} \leqslant \sigma_1+\sigma_2 < \frac{3}{2}$, *and* $\sigma_1+\sigma_2 \neq 1$. *Then, for any sufficiently large positive number* $T>2$, *we have*

$$\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1 = \frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2}\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)}\frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} + O(t_2^{-\frac{5}{2}}T^{\frac{11}{2}-3\sigma_1-3\sigma_2}) \tag{1.16}$$

for $2 \leqslant t_2 \leqslant T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$ *and* $1<\sigma_1+\sigma_2<\frac{3}{2}$, *and*

$$\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1 = \frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2}\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)}\frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} + \begin{cases} O(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\varepsilon}) & \text{if } 2 \leqslant t_2 \leqslant T^{\frac{1}{5-2\sigma_1-2\sigma_2}+\varepsilon}, \\ O(t_2^{-\frac{3}{2}-\sigma_1-\sigma_2}T^{\frac{11}{2}-3\sigma_1-3\sigma_2}) & \text{if } T^{\frac{1}{5-2\sigma_1-2\sigma_2}+\varepsilon} \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}} \end{cases} \tag{1.17}$$

for $\frac{1}{2} \leqslant \sigma_1+\sigma_2 < 1$.

THEOREM 1.3. *Suppose that* $2 \leqslant t_1 \leqslant T$, $0<\sigma_1<1$, $0<\sigma_2<1$, *and* $0<\sigma_1+\sigma_2<\frac{1}{2}$. *Then, for any sufficiently large positive number* $T>2$, *we have,*

$$\begin{aligned}\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1 = &\frac{(2\pi)^{4\sigma_1+4\sigma_2-6}\zeta^4(4-2\sigma_1-2\sigma_2)}{(7-4\sigma_1-4\sigma_2)\zeta(8-4\sigma_1-4\sigma_2)}\frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} \\ &+ O(t_2^{6-4\sigma_1-4\sigma_2}T) + O(t_2^{-3}T^{6-4\sigma_1-4\sigma_2}) \\ &+ O(t_2^{-\frac{3}{2}-\sigma_1-\sigma_2}T^{\frac{11}{2}-3\sigma_1-3\sigma_2}) + O(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\varepsilon})\end{aligned} \tag{1.18}$$

for $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$.

REMARK 1.2. An average order of magnitude for double zeta-function (1.1) derived from the asymptotic behavior of the integrals of the double zeta-function with (1.15)–(1.18) holds when the ratio of the order of t_1 to that of t_2 is small. However, it is often difficult to determine their analytic behavior for a general ratio of the order of t_1 to the order of t_2.

From Theorems 1.1, 1.2, and 1.3, we are able to determine an additional proof for the Ω-result of Kiuchi, Tanigawa, and Zhai [**13**], and Kiuchi and Minamide [**11**].

COROLLARY 1.2. *Let* $0<\sigma_1<1$, $0<\sigma_2<1$, *and* $2 \leqslant t_1 \leqslant T$. *Then, we have*

$$\zeta_2(\sigma_1+it_1,\sigma_2+it_2) = \Omega\Big(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\Big) \tag{1.19}$$

for $1 \leqslant \sigma_1 + \sigma_2 < \frac{3}{2}$ *and* $2 \leqslant t_2 \leqslant T^{1-\frac{2}{3}(\sigma_1+\sigma_2)-\varepsilon}$, *and*

$$\zeta_2(\sigma_1 + it_1, \sigma_2 + it_2) = \Omega\Big(\frac{t_1^{\frac{3}{2}-\sigma_1-\sigma_2}}{t_2}\Big) \tag{1.20}$$

for $0 < \sigma_1 + \sigma_2 < 1$ *and* $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}-\varepsilon}$, *where* ε *is any small positive constant.*

Formulas (1.19) and (1.20) are equivalent to (1.9) and (1.10), respectively, and Corollary 1.2 provides an improvement upon the Ω-result of Kiuchi, Tanigawa, and Zhai [**13**].

THEOREM 1.4. *Suppose that* $2 \leqslant t_1 \leqslant T$, $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, *and* $\frac{3}{2} \leqslant \sigma_1 + \sigma_2 < 2$. *Then, for any sufficiently large positive number* $T > 2$, *we have*

$$\int_2^T |\zeta_2(s_1, s_2)|^4 dt_1 = O(t_2^2 T) \tag{1.21}$$

for $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$, *and*

$$\int_2^T |\zeta_2(s_1, s_2)|^4 dt_1 = \begin{cases} \frac{1}{2\pi^2}\frac{T\log^4 T}{|s_2-1|^4} + O(t_2^{-\frac{5}{2}} T \log^3 T) & \text{if } 2 \leqslant t_2 \leqslant (\log T)^{\frac{2}{3}}, \\ O(t_2^2 T) & \text{if } t_2 \geqslant (\log T)^{\frac{2}{3}} \end{cases} \tag{1.22}$$

for $\sigma_1 + \sigma_2 = \frac{3}{2}$.

To improve our theorems, we must obtain a sharper estimate for the integral $\int_2^T |E(s_1, s_2)|^4 dt_1$ in Lemma 2.2 below, but this is very difficult.

Notations. When $g(x)$ is a positive function of x for $x \geqslant x_0$, $f(x) = \Omega(g(x))$ means that $f(x) = o(g(x))$ does not hold as $x \to \infty$. In what follows, ε denotes any arbitrarily small positive number, not necessarily the same ones at each occurrence.

2. Some lemmas

Let a be any complex number and $\chi(s_2) = 2(2\pi)^{s_2-1}\sin(\frac{\pi}{2}s_2)\Gamma(1-s_2)$. The generalized divisor function $\sigma_a(n)$ is defined by $\sum_{d|n} d^a$. We use a weak form of the approximate formula of Kiuchi, Tanigawa, and Zhai [**13**] to prove our theorems. For our purpose, it is enough to quote the following weak form, which follows from Lemma 1 of Kiuchi and Minamide [**11**].

LEMMA 2.1 ([**11**]). *Suppose that* $0 < \sigma_1 < 1$ *and* $0 < \sigma_2 < 1$. *Then, we have*

$$\zeta_2(s_1, s_2) = \frac{\zeta(s_1 + s_2 - 1)}{s_2 - 1} - \frac{1}{2}\zeta(s_1 + s_2) + E(s_1, s_2), \tag{2.1}$$

where the error term $E(s_1, s_2)$ *is estimated as*

$$E(s_1, s_2) \ll \begin{cases} |t_2|^{\frac{3}{2}-\sigma_1-\sigma_2} & \text{if } 0 < \sigma_1 + \sigma_2 < 1, \\ |t_2|^{\frac{1}{2}} \log |t_2| & \text{if } \sigma_1 + \sigma_2 = 1, \\ |t_2|^{\frac{1}{2}} & \text{if } \sigma_1 + \sigma_2 > 1. \end{cases} \tag{2.2}$$

Note that the error term in this lemma is independent of t_1. We establish a formula for the fourth power moment of $\zeta_2(s_1, s_2)$. Using (2.1), (2.2), and Hölder's inequality, we deduce the following:

LEMMA 2.2. *For $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$ and any sufficiently large number $T > 2$, we have*

$$\begin{aligned}\int_2^T |\zeta_2(s_1, s_2)|^4 dt_1 \\ = I_1 + I_2 + I_3 + O(I_1^{\frac{3}{4}} I_2^{\frac{1}{4}} + I_1^{\frac{3}{4}} I_3^{\frac{1}{4}} + I_2^{\frac{3}{4}} I_3^{\frac{1}{4}} + I_2^{\frac{3}{4}} I_1^{\frac{1}{4}} + I_3^{\frac{3}{4}} I_1^{\frac{1}{4}} + I_3^{\frac{3}{4}} I_2^{\frac{1}{4}}),\end{aligned} \tag{2.3}$$

where

$$I_1 = \frac{1}{|s_2 - 1|^4} \int_2^T |\zeta(s_1 + s_2 - 1)|^4 dt_1, \tag{2.4}$$

$$I_2 = \frac{1}{16} \int_2^T |\zeta(s_1 + s_2)|^4 dt_1 \tag{2.5}$$

$$I_3 = \int_2^T |E(s_1, s_2)|^4 dt_1 \ll, \begin{cases} T|t_2|^{6-4\sigma_1-4\sigma_2} & \text{if } 0 < \sigma_1 + \sigma_2 < 1, \\ T|t_2|^2 \log^4 |t_2| & \text{if } \sigma_1 + \sigma_2 = 1, \\ T|t_2|^2 & \text{if } \sigma_1 + \sigma_2 > 1. \end{cases} \tag{2.6}$$

Kiuchi and Minamide [**11**] suggested that

$$\begin{aligned} E(s_1, s_2) &= \chi(s_2) \sum_{n \leqslant \frac{|t_2|}{2\pi}} \frac{(\mathbf{1} * \mathrm{id}^{1-s_1-s_2})(n)}{n^{1-s_2}} + O(|t_2|^{\max(0, 1-\sigma_1-\sigma_2)+\varepsilon}) \\ &= \chi(s_2) \sum_{ml \leqslant \frac{|t_2|}{2\pi}} \frac{1}{m^{1-s_2}} \cdot \frac{1}{l^{s_1}} + O(|t_2|^{\max(0,1-\sigma_1-\sigma_2)+\varepsilon}) \end{aligned} \tag{2.7}$$

holds, where $*$ denotes the Dirichlet convolution. The first term on the right-hand side of (2.7) can be written as

$$\chi(s_2) \sum_{m \leqslant M} \frac{1}{m^{1-s_2}} \sum_{l \leqslant L} \frac{1}{l^{s_1}} + O(\cdots)$$

with $M \geqslant 1$, $L \geqslant 1$ and $ML = \frac{|t_2|}{2\pi}$. We use the approximate functional equation of the Riemann zeta-function [**18**, Theorem 4.13] and the simplest form of the approximation to the Riemann zeta-function [**18**, Theorem 4.11] to obtain

$$\chi(s_2) \sum_{m \leqslant M} \frac{1}{m^{1-s_2}} = \zeta(s_2) - \sum_{l \leqslant L} \frac{1}{l^{s_2}} + O(L^{-\sigma_2} \log |t_2|) + O(|t_2|^{\frac{1}{2}-\sigma_2} M^{\sigma_2 - 1})$$

with $0 < \sigma_2 < 1$, and

$$\sum_{l \leqslant L} \frac{1}{l^{s_1}} = \zeta(s_1) + \frac{L^{1-s_1}}{1 - s_1} + O(L^{-\sigma_1})$$

with $0 < \sigma_1 < 1$, respectively. Taking the product of the above formulas and using (2.7), we see that the function $E(s_1, s_2)$ is given by

$$E(s_1, s_2) = \zeta(s_2)\zeta(s_1) - \zeta(s_1)\sum_{l \leqslant L}\frac{1}{l^{s_2}} + \frac{L^{1-s_1}}{1-s_1}\zeta(s_2) + \frac{L^{1-s_1}}{1-s_1}\sum_{l \leqslant L}\frac{1}{l^{s_2}} + O(\cdots).$$

Roughly speaking, in the case where $|t_1| \asymp |t_2|$, the true order of magnitude of the function $E(s_1, s_2)$ may be regarded as $|E(s_1, s_2)| \asymp |\zeta(s_1)||\zeta(s_2)|$. Thus, it follows from the above and Lemma 2.1 that $|\zeta_2(s_1, s_2)| \asymp |\zeta(s_1)||\zeta(s_2)|$. However, in the case where $|t_2| \ll |t_1|^{\alpha}$ $(0 < \alpha < \frac{1}{3})$, the order of magnitude of the function $E(s_1, s_2)$ is smaller than that of the first term on the right-hand side of (2.1); hence, (2.7) implies the error term of (2.1).

To deal with the integrals I_j $(j = 1, 2, 3)$, we shall use Lemma 2.3 and formulas (1.11)–(1.14).

LEMMA 2.3. *For any sufficiently large positive number $T > 2$, we have*

$$\int_2^T |\zeta(\sigma+it)|^4 dt = \frac{(2\pi)^{4\sigma-2}}{3-4\sigma}\frac{\zeta^4(2-2\sigma)}{\zeta(4-4\sigma)}T^{3-4\sigma} + O(T^{2-2\sigma}\log^3 T) \tag{2.8}$$

with $0 < \sigma < \frac{1}{2}$,

$$\int_2^T |\zeta(it)|^4 dt = \frac{\zeta^4(2)}{12\pi^2\zeta(4)}T^3 + O(T^2\log^4 T) \tag{2.9}$$

with $\sigma = 0$, and

$$\int_2^T |\zeta(\sigma+it)|^4 dt = \frac{(2\pi)^{4\sigma-2}}{3-4\sigma}\frac{\zeta^4(2-2\sigma)}{\zeta(4-4\sigma)}T^{3-4\sigma} + O(T^{3-3\sigma+\varepsilon}) \tag{2.10}$$

with $-1 < \sigma < 0$.

PROOF. Since the functional equation is $\zeta(\sigma+it) = \chi(\sigma+it)\zeta(1-\sigma-it)$ (see Titchmarsh [**18**] or Ivić [**7**]), the equality $|\zeta(1-\sigma-it)| = |\zeta(1-\sigma+it)|$ and the formula

$$|\chi(\sigma+it)|^4 = \Big(\frac{t}{2\pi}\Big)^{2-4\sigma} + O(t^{1-4\sigma}) \qquad (t \geqslant t_0 > 0),$$

we can integrate by parts and use (1.13) to obtain

$$\begin{aligned}\int_2^T |\zeta(\sigma+it)|^4 dt &= \int_2^T |\chi(\sigma+it)|^4|\zeta(1-\sigma+it)|^4 dt\\ &= \Big(\frac{1}{2\pi}\Big)^{2-4\sigma}\int_2^T t^{2-4\sigma}|\zeta(1-\sigma+it)|^4 dt + O\Big(\int_2^T t^{1-4\sigma}|\zeta(1-\sigma+it)|^4 dt\Big)\\ &= \frac{(2\pi)^{4\sigma-2}}{3-4\sigma}\frac{\zeta^4(2-2\sigma)}{\zeta(4-4\sigma)}T^{3-4\sigma} + O(T^{2-2\sigma}\log^3 T)\end{aligned}$$

for $0 < \sigma < \frac{1}{2}$. Similarly in the case of $\sigma = 0$, we have, by integrating by parts and using (1.12)

$$\int_2^T |\zeta(it)|^4 dt = \int_2^T |\chi(it)|^4|\zeta(1+it)|^4 dt$$

$$= \frac{1}{4\pi^2}\int_2^T t^2|\zeta(1+it)|^4 dt + O\Big(\int_2^T t|\zeta(1+it)|^4 dt\Big)$$
$$= \frac{\zeta^4(2)}{12\pi^2\zeta(4)}T^3 + O(T^2\log^4 T).$$

In a similar manner, we deduce from (1.11) that formula (2.10) holds. □

3. Proofs

3.1. Proofs of Theorem 1.1 and Corollary 1.1. We set $2 \leqslant t_1 \leqslant T$ and $2 \leqslant t_2 \leqslant \frac{T^{\frac{1}{3}}}{\log T}$. We shall evaluate the integral $\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1$ under the condition $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, and $\sigma_1 + \sigma_2 = 1$. From (1.12), we have

$$I_2 = \frac{1}{16}\Big\{\int_2^{T+t_2}|\zeta(1+it)|^4 dt - \int_2^{2+t_2}|\zeta(1+it)|^4 dt\Big\} = O(T). \tag{3.1}$$

Similarly,

$$I_1 = \frac{1}{|s_2-1|^4}\Big\{\int_2^{T+t_2}|\zeta(it)|^4 dt - \int_2^{2+t_2}|\zeta(it)|^4 dt\Big\}. \tag{3.2}$$

Inserting (2.9) into (3.2), we obtain

$$\begin{aligned} I_1 &= \frac{\zeta^4(2)}{12\pi^2\zeta(4)}\frac{(T+t_2)^3-(t_2+2)^3}{|s_2-1|^4} + O\Big(\frac{(T+t_2)^2}{|s_2-1|^4}\log^4 T\Big) \\ &= \frac{\zeta^4(2)}{12\pi^2\zeta(4)|s_2-1|^4}T^3 + O(t_2^{-3}T^2) + O(t_2^{-4}T^2\log^4 T). \end{aligned} \tag{3.3}$$

From (2.6), we have

$$I_3 = O((t_2^2\log^4 t_2)T). \tag{3.4}$$

Substituting (3.1), (3.3), and (3.4) into (2.3), we observe that all error terms on the right-hand side of (2.3) are absorbed into $O(t_2^{-\frac{5}{2}}(\log t_2)T^{\frac{5}{2}})$, completing the proof of (1.15).

As an application of (1.15), we shall evaluate the double integral for the double zeta-function $\zeta_2(s_1,s_2)$:

$$\int_2^N\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1 dt_2.$$

From (1.15), we have

$$\begin{aligned} &\int_2^N\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1 dt_2 \\ &\qquad = \frac{\zeta^4(2)}{12\pi^2\zeta(4)}\Big(\int_2^N \frac{1}{|s_2-1|^4}dt_2\Big)T^3 + O\Big(T^{\frac{5}{2}}\int_2^N t_2^{-\frac{5}{2}}\log t_2 dt_2\Big) \end{aligned}$$

for $2 \leqslant N \leqslant \frac{T^{\frac{1}{3}}}{\log T}$. It follows that

$$\int_2^N \frac{1}{|s_2-1|^4}dt_2 = \frac{1}{2(1-\sigma_2)^3}\Big(\frac{\pi}{2} - \mathrm{Tan}^{-1}\frac{2}{1-\sigma_2} - \frac{2(1-\sigma_2)}{4+(1-\sigma_2)^2}\Big) + O\Big(\frac{1}{N}\Big)$$

and

$$\int_2^N t_2^{-\frac{5}{2}} \log t_2 dt_2 = \int_2^\infty t_2^{-\frac{5}{2}} \log t_2 dt_2 + O(N^{-\frac{3}{2}} \log N).$$

Then, we easily see that

$$\begin{aligned}\int_2^N \int_2^T &|\zeta_2(s_1,s_2)|^4 dt_1 dt_2 \\ &= \frac{5\pi^2}{1728(1-\sigma_2)^3}\Big(\frac{\pi}{2} - \mathrm{Tan}^{-1}\frac{2}{1-\sigma_2} - \frac{2(1-\sigma_2)}{4+(1-\sigma_2)^2}\Big)T^3 \\ &\quad + O(T^{\frac{5}{2}}N^{-\frac{3}{2}}\log N) + O(T^{\frac{5}{2}}) + O(T^3N^{-1}).\end{aligned}$$

Hence, for $0<\sigma_1<1$, $0<\sigma_2<1$, $\sigma_1+\sigma_2=1$, and $2 \leqslant N \leqslant \frac{T^{\frac{1}{3}}}{\log T}$, we obtain, as $T \to \infty$,

$$\frac{1}{T^3}\int_2^N \int_2^T |\zeta_2(s_1,s_2)|^4 dt_1 dt_2 \to \frac{5\pi^2}{1728\sigma_1^3}\Big(\frac{\pi}{2} - \mathrm{Tan}^{-1}\frac{2}{\sigma_1} - \frac{2\sigma_1}{4+\sigma_1^2}\Big) + O\Big(\frac{1}{N}\Big).$$

Therefore, we obtain the assertion of Corollary 2.1.

3.2. Proof of Theorem 1.2. Throughout this section, we assume that $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$. As in the proof of Theorem 1.1, we shall calculate the integral $\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1$ under the condition that $1 < \sigma_1+\sigma_2 < \frac{3}{2}$ with $2 \leqslant t_2 \leqslant T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$. From (1.11) and (2.5), we have

$$I_2 = \frac{1}{16}\Big\{\int_2^{T+t_2} |\zeta(\sigma_1+\sigma_2+it)|^4 dt - \int_2^{2+t_2} |\zeta(\sigma_1+\sigma_2+it)|^4 dt\Big\} = O(T). \tag{3.5}$$

From (2.4) and (2.8), we have

$$\begin{aligned} I_1 &= \frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2}\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)}\frac{(T+t_2)^{7-4\sigma_1-4\sigma_2}-(t_2+2)^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} \\ &\quad + O\Big(\frac{(T+t_2)^{4-2\sigma_1-2\sigma_2}}{|s_2-1|^4}\log^3 T\Big) \\ &= \frac{(2\pi)^{4\sigma_1+4\sigma_2-6}}{7-4\sigma_1-4\sigma_2}\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{\zeta(8-4\sigma_1-4\sigma_2)}\frac{1}{|s_2-1|^4}\ T^{7-4\sigma_1-4\sigma_2} \\ &\quad + O(t_2^{-3}T^{6-4\sigma_1-4\sigma_2}) + O(t_2^{-4}T^{4-2\sigma_1-2\sigma_2}\log^3 T). \end{aligned} \tag{3.6}$$

From (2.6), we have

$$I_3 = O(t_2^2 T). \tag{3.7}$$

Substituting (3.5), (3.6), and (3.7) into (2.3), we observe that all error terms on the right-hand side of (2.3) are absorbed into $O(t_2^{-\frac{5}{2}}T^{\frac{11}{2}-3\sigma_1-3\sigma_2})$. Hence, we derive formula (1.16) with $2 \leqslant t_2 \leqslant T^{1-\frac{2}{3}(\sigma_1+\sigma_2)}$. In a similar manner as above, we shall calculate the integral $\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1$ in the case of $\frac{1}{2} < \sigma_1+\sigma_2 < 1$ with $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. By (1.13) and (2.5), we have

$$I_2 = O(T), \tag{3.8}$$

and by (2.6)

$$I_3 = O(t_2^{6-4\sigma_1-4\sigma_2}T). \tag{3.9}$$

By (2.4) and (2.10) we have

$$\begin{aligned} I_1 &= \frac{1}{|s_2-1|^4}\int_2^T |\zeta(\sigma_1+\sigma_2-1+i(t_1+t_2))|^4 dt_1 \\ &= (2\pi)^{4\sigma_1+4\sigma_2-6}\frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{(7-4\sigma_1-4\sigma_2)\zeta(8-4\sigma_1-4\sigma_2)}\frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} \\ &\quad + O(t_2^{-3}T^{6-4\sigma_1-4\sigma_2}) + O(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\varepsilon}). \end{aligned} \tag{3.10}$$

Substituting (3.8), (3.9), and (3.10) into (2.3), we observe that all error terms on the right-hand side of (2.3) are absorbed into $O(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\varepsilon})$ if $2 \leqslant t_2 \leqslant T^{\frac{1}{5-2\sigma_1-2\sigma_2}+\varepsilon}$, or into $O(t_2^{-\frac{3}{2}-\sigma_1-\sigma_2}T^{\frac{11}{2}-3\sigma_1-3\sigma_2})$ if $T^{\frac{1}{5-2\sigma_1-2\sigma_2}+\varepsilon} \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. We derive formula (1.17) with $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. Next, we shall calculate the integral $\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1$ in the case where $\sigma_1+\sigma_2 = \frac{1}{2}$ under the assumption that $2 \leqslant t_2 \leqslant T^{\frac{1}{2}}$. By (1.14) and (2.5), we have

$$I_2 = O(T\log^4 T), \tag{3.11}$$

and by (2.6), we have

$$I_3 = O(t_2^4 T). \tag{3.12}$$

By (2.4) and (2.10), we have

$$\begin{aligned} I_1 &= \frac{1}{|s_2-1|^4}\int_2^T \left|\zeta\Big(-\frac{1}{2}+i(t_1+t_2)\Big)\right|^4 dt_1 \\ &= \frac{\zeta^4(3)}{80\pi^4\zeta(6)}\frac{T^5}{|s_2-1|^4} + O(t_2^{-3}T^4) + O(t_2^{-4}T^{\frac{9}{2}+\varepsilon}). \end{aligned} \tag{3.13}$$

Substituting (3.11), (3.12), and (3.13) into (2.3), we observe that all error terms on the right-hand side of (2.3) are absorbed into $O(t_2^{-4}T^{\frac{9}{2}+\varepsilon})$ if $2 \leqslant t_2 \leqslant T^{\frac{1}{4}+\varepsilon}$ and $O(t_2^{-2}T^4)$ if $T^{\frac{1}{4}+\varepsilon} \leqslant t_2 \leqslant T^{\frac{1}{2}}$. We derive the formula (1.17) with $\sigma_1+\sigma_2 = \frac{1}{2}$ and $2 \leqslant t_2 \leqslant T^{\frac{1}{2}}$. Thus, we obtain the assertions of Theorem 1.2.

3.3. Proof of Theorem 1.3. Assuming that $0 < \sigma_1 < 1$, $0 < \sigma_2 < 1$, and $0 < \sigma_1+\sigma_2 < \frac{1}{2}$, we shall calculate the integral $\int_2^T |\zeta_2(s_1,s_2)|^4 dt_1$ with $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. Similar to the above theorems, (2.5) and (2.8) provide us

$$I_2 = O(T^{3-4\sigma_1-4\sigma_2}), \tag{3.14}$$

and by (2.6)

$$I_3 = O(t_2^{6-4\sigma_1-4\sigma_2}T). \tag{3.15}$$

By (2.4) and (2.10), we have

$$I_1 = (2\pi)^{4\sigma_1+4\sigma_2-6} \frac{\zeta^4(4-2\sigma_1-2\sigma_2)}{(7-4\sigma_1-4\sigma_2)\zeta(8-4\sigma_1-4\sigma_2)} \frac{T^{7-4\sigma_1-4\sigma_2}}{|s_2-1|^4} + O(t_2^{-3}T^{6-4\sigma_1-4\sigma_2}) + O(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\varepsilon}). \tag{3.16}$$

Substituting (3.14), (3.15), and (3.16) into (2.3), we observe that all error terms on the right-hand side of (2.3) are given by

$$O(t_2^{-\frac{3}{2}-\sigma_1-\sigma_2}T^{\frac{11}{2}-3\sigma_1-3\sigma_2}) + O(t_2^{6-4\sigma_1-4\sigma_2}T) + O(t_2^{-3}T^{6-4\sigma_1-4\sigma_2}) + O(t_2^{-4}T^{6-3\sigma_1-3\sigma_2+\varepsilon})$$

for $2 \leqslant t_2 \leqslant T^{\frac{3-2\sigma_1-2\sigma_2}{5-2\sigma_1-2\sigma_2}}$. Hence, this completes the proof of Theorem 1.3.

3.4. Proof of Theorem 1.4. Throughout this section, we shall assume that $0 < \sigma_1 < 1$ and $0 < \sigma_2 < 1$. We shall calculate the integral $\int_2^T |\zeta_2(s_1, s_2)|^4 dt_1$ with $\frac{3}{2} \leqslant \sigma_1 + \sigma_2 < 2$. When $\frac{3}{2} < \sigma_1 + \sigma_2 < 2$, (1.11), (1.13), and (2.4)–(2.6) tell us that

$$I_2 = O(T), \quad I_1 = O(t_2^{-4}T), \quad I_3 = O(t_2^2 T). \tag{3.17}$$

Substituting (3.17) into (2.3), we derive formula (1.21). Similarly, in the case of $\sigma_1 + \sigma_2 = \frac{3}{2}$, (1.11), (1.14), and (2.4)–(2.6) tell us that

$$I_2 = O(T), \quad I_3 = O(t_2^2 T), \quad I_1 = \frac{1}{2\pi^2|s_2-1|^4} T \log^4 T + O(t_2^{-4} T \log^3 T). \tag{3.18}$$

Substituting (3.18) into (2.3), we observe that all error terms on the right-hand side of (2.3) are absorbed into $O(t_2^{-\frac{5}{2}} T \log^3 T)$ if $2 \leqslant t_2 \leqslant (\log T)^{\frac{2}{3}}$, and that the right-hand side of (2.3) yields $O(t_2^2 T)$ if $t_2 \geqslant (\log T)^{\frac{2}{3}}$. Hence, we derive formula (1.22).

Acknowledgements. The author would like to thank the referee for his/her careful reading of the earlier version of this paper and giving many valuable suggestions.

References

1. S. Akiyama, S. Egami, Y. Tanigawa, *An analytic continuation of multiple zeta-functions and their values at non-positive integers*, Acta Arith. **98** (2001), 107–116.
2. F. V. Atkinson, *The mean-value of the Riemann zeta-function*, Acta Math. **81** (1949), 353–376.
3. R. Balasubramanian, A. Ivić, K. Ramachandra, *The mean square of the Riemann zeta-function on the line* $\sigma = 1$, Enseign. Math. (2) **38** (1992), 13–25.
4. D. R. Heath-Brown, *The fourth power moment of the Riemann zeta-function* Proc. Lond. Math. Soc. **38**(3) (1979), 385–422.
5. S. Ikeda, K. Matsuoka, Y. Nagata, *On certain mean values of the double zeta-function*, Nagoya Math. J. **217** (2015), 161–190.
6. H. Ishikawa, K. Matsumoto, *On the estimation of the order of Euler-Zagier multiple zeta-functions*, Ill. J. Math. **47** (2003), 1151–1166.
7. A. Ivić, *The Riemann Zeta-Function*, John Wiley and Sons, New York, 1985 (2nd ed. Dover, 2003).

8. ______, *Mean Values of the Riemann Zeta Function*, Lect. Note Ser. **82**; Tata Institute of Fundamental Research, Bombay; Berlin–Heidelberg–New York; Springer 1991.
9. ______, *Some problems on mean values of the Riemann zeta-function*, J. Théor. Nombres Bordx. **8** (1996), 101–123.
10. ______, *On mean values of some zeta-functions in the critical strip*, J. Théor. Nombres Bordx. **15** (2003), 163–178.
11. I. Kiuchi, M. Minamide, *Mean square formula for the double zeta-function*, Funct. Approximatio, Comment. Math. **55** (2016), 31–43..
12. I. Kiuchi, Y. Tanigawa, *Bounds for double zeta-functions*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) **5** (2006), 445–464.
13. I. Kiuchi, Y. Tanigawa and W. Zhai, *Analytic properties of double zeta-functions*, Indag. Math., New Ser. **21** (2011), 16–29.
14. K. Matsumoto, *On the analytic continuation of various multiple zeta-functions*, In: "Number Theory for the Millennium, Proc. Millennial Conf. Number Theory", Vol. II, M. A. Bennett et al. (eds.), A K Peters 2002, 417–440.
15. ______, *Functional equations for double zeta-functions*, Math. Proc. Camb. Philos. Soc. **136** (2004), 1–7.
16. K. Matsumoto, H. Tsumura, *Mean value theorems for the double zeta-function*, J. Math. Soc. Japan **67** (2015), 383–406.
17. Y. Ohno, *A generalization of the duality and sum formulas on the multiple zeta values*, J. Number Theory **74** (1999), 39–43.
18. E. C. Titchmarsh, *The Theory of the Riemann Zeta-Function*, Second Edition, Edited and with a preface by D. R. Heath–Brown, The Clarendon Press, Oxford University Press, New York, 1986.
19. J. Q. Zhao, *Analytic continuation of multiple zeta function*, Proc. Am. Math. Soc. **128** (2000), 1275–1283.

Department of Mathematical Sciences
Yamaguchi University
Yamaguchi
Yoshida
Japan
kiuchi@yamaguchi-u.ac.jp

(Received 25 10 2014)
(Revised 05 01 2015)