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A NEW FORMULA FOR THE BERNOULLI NUMBERS OF THE SECOND KIND IN TERMS OF THE STIRLING NUMBERS OF THE FIRST KIND

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Abstract. We find an explicit formula for computing the Bernoulli numbers of the second kind in terms of the signed Stirling numbers of the first kind.

1. Main result

It is well known that the signed Stirling numbers of the first kind $s(n,k)$ for $n \geqslant k \geqslant 1$ may be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n,k)\frac{x^n}{n!}, \quad |x| < 1$$

and that the Bernoulli numbers of the second kind b_n for $n \geqslant 0$ may be generated by $\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} b_n x^n$. In combinatorics, the signed Stirling number of the first kind $s(n,k)$ may be defined such that the number of permutations of n elements which contain exactly k permutation cycles is the nonnegative number $|s(n,k)| = (-1)^{n-k}s(n,k)$. The Bernoulli numbers of the second kind b_n are also called the Cauchy numbers of the first kind, see [**20**, **27**] and closely related references therein.

In [**14**], the following formula for computing the Bernoulli numbers of the second kind in terms of the signed Stirling numbers of the first kind was derived:

$$b_n = \frac{1}{n!}\sum_{k=0}^{n} \frac{s(n,k)}{k+1}.$$

Our main aim is to find a new and explicit formula for computing the Bernoulli numbers of the second kind in terms of the signed Stirling numbers of the first kind. The main result of this paper may be stated as the following theorem.

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THEOREM 1.1. *For $n \geqslant 2$, the Bernoulli numbers of the second kind b_n may be computed in terms of the signed Stirling numbers of the first kind $s(n,k)$ by*

$$b_n = \frac{1}{n!}\sum_{k=1}^{n-1}(-1)^k \frac{s(n-1,k)}{(k+1)(k+2)}. \tag{1.1}$$

2. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on some results elementarily and inductively obtained in [**22**]. These results can be recited as follows.

(1) Corollary 2.3 in [**22**] states that the signed Stirling numbers of the first kind $s(n,k)$ for $1 \leqslant k \leqslant n$ may be computed by

$$s(n,k) = (-1)^{n-k}(n-1)!\sum_{l_1=1}^{n-1}\frac{1}{l_1}\sum_{l_2=1}^{l_1-1}\frac{1}{l_2}\cdots\sum_{l_{k-2}=1}^{l_{k-3}-1}\frac{1}{l_{k-2}}\sum_{l_{k-1}=1}^{l_{k-2}-1}\frac{1}{l_{k-1}}. \tag{2.1}$$

This formula may be reformulated as

$$(-1)^{n-k}\frac{s(n,k)}{(n-1)!} = \sum_{m=k-1}^{n-1}\frac{1}{m}\left[(-1)^{m-(k-1)}\frac{s(m,k-1)}{(m-1)!}\right].$$

(2) Corollary 2.4 in [**22**] reads that for $1 \leqslant k \leqslant n$ the signed Stirling numbers of the first kind $s(n,k)$ satisfy the recursion

$$s(n+1,k) = s(n,k-1) - ns(n,k). \tag{2.2}$$

This is a recovery of the triangular relation for $s(n,k)$.

(3) Theorem 3.1 in [**22**] tells that the Bernoulli numbers of the second kind b_n for $n \geqslant 2$ may be computed by

$$b_n = (-1)^n\frac{1}{n!}\left(\frac{1}{n+1} + \sum_{k=2}^{n}\frac{a_{n,k} - na_{n-1,k}}{k!}\right), \tag{2.3}$$

where

$$a_{n,2} = (n-1)! \tag{2.4}$$

and, for $n+1 \geqslant k \geqslant 3$,

$$a_{n,k} = (k-1)!(n-1)!\sum_{l_1=1}^{n-1}\frac{1}{l_1}\sum_{l_2=1}^{l_1-1}\frac{1}{l_2}\cdots\sum_{l_{k-3}=1}^{l_{k-4}-1}\frac{1}{l_{k-3}}\sum_{l_{k-2}=1}^{l_{k-3}-1}\frac{1}{l_{k-2}}. \tag{2.5}$$

Observing expressions (2.1) and (2.5), we obtain

$$a_{n,k} = (-1)^{n+k-1}(k-1)!s(n,k-1), \quad n+1 \geqslant k \geqslant 2. \tag{2.6}$$

See [**22**, (2.18)] and [**23**, (6.7)]. By this and recursion (2.2), it follows that

$$\begin{aligned} a_{n,k} - na_{n-1,k} &= (-1)^{n+k-1}(k-1)![s(n,k-1) + ns(n-1,k-1)] \\ &= (-1)^{n+k-1}(k-1)![s(n-1,k-1) + s(n-1,k-2)]. \end{aligned}$$

Substituting this into (2.3) reveals that

$$\begin{aligned}
b_n &= \frac{(-1)^n}{n!}\biggl(\frac{1}{n+1}+\sum_{k=2}^{n}\frac{(-1)^{n+k-1}[s(n-1,k-1)+s(n-1,k-2)]}{k}\biggr)\\
&= \frac{(-1)^n}{(n+1)!}+\frac{1}{n!}\sum_{k=2}^{n}\frac{(-1)^{k-1}[s(n-1,k-1)+s(n-1,k-2)]}{k}\\
&= \frac{(-1)^n}{(n+1)!}+\frac{1}{n!}\biggl[\sum_{k=2}^{n}\frac{(-1)^{k-1}s(n-1,k-1)}{k}+\sum_{k=2}^{n}\frac{(-1)^{k-1}s(n-1,k-2)}{k}\biggr]\\
&= \frac{(-1)^n}{(n+1)!}+\frac{1}{n!}\biggl[\sum_{k=2}^{n}\frac{(-1)^{k-1}s(n-1,k-1)}{k}+\sum_{k=1}^{n-1}\frac{(-1)^{k}s(n-1,k-1)}{k+1}\biggr]\\
&= \frac{(-1)^n}{(n+1)!}+\frac{1}{n!}\frac{(-1)^{n-1}}{n}+\frac{1}{n!}\sum_{k=2}^{n-1}(-1)^{k-1}s(n-1,k-1)\Bigl(\frac{1}{k}-\frac{1}{k+1}\Bigr)\\
&= \frac{1}{n!}(-1)^{n-1}\Bigl(\frac{1}{n}-\frac{1}{n+1}\Bigr)+\frac{1}{n!}\sum_{k=2}^{n-1}(-1)^{k-1}s(n-1,k-1)\Bigl(\frac{1}{k}-\frac{1}{k+1}\Bigr)\\
&= \frac{1}{n!}\sum_{k=2}^{n}(-1)^{k-1}s(n-1,k-1)\Bigl(\frac{1}{k}-\frac{1}{k+1}\Bigr)\\
&= \frac{1}{n!}\sum_{k=2}^{n}(-1)^{k-1}\frac{s(n-1,k-1)}{k(k+1)}.
\end{aligned}$$

Notice that in the above argument, we use the convention $s(n,0)=0$ for $n\in\mathbb{N}$ and the fact $s(n,n)=1$ for $n\geqslant 0$. The proof of the formula (1.1) in Theorem 1.1 is complete.

3. Remarks

In this section, we show some new findings by several remarks.

Remark 3.1. The idea in Theorem 1.1 and its proof ever implicitly thrilled through in [**23**, Remark 6.7].

Remark 3.2. Making use of relation (2.6) in [**22**, Theorem 2.1] leads to

$$\Bigl(\frac{1}{\ln x}\Bigr)^{(n)}=\frac{1}{x^n}\sum_{k=1}^{n}(-1)^k k!s(n,k)\Bigl(\frac{1}{\ln x}\Bigr)^{k+1},\quad n\in\mathbb{N}.$$

This recovers the first formula in [**13**, Lemma 2].

By the way, the formulas (3.4) and (3.5) in [**22**, Corollary 3.1] recover the second formula in [**13**, Lemma 2].

Remark 3.3. In [**22**, Remark 2.2], it was conjectured that the sequence $a_{n,k}$ for $n\in\mathbb{N}$ and $2\leqslant k\leqslant n+1$ is increasing with respect to n while it is unimodal with respect to k for given $n\geqslant 4$. This conjecture may be partially confirmed as follows.

From (2.5), the increasing monotonicity of the sequence $a_{n,k}$ with respect to n follows straightforwardly.

It is clear that the sequence $(k-1)!$ is increasing with k and the sequence

$$\sum_{l_1=1}^{n-1}\frac{1}{l_1}\sum_{l_2=1}^{l_1-1}\frac{1}{l_2}\cdots\sum_{l_{k-3}=1}^{l_{k-4}-1}\frac{1}{l_{k-3}}\sum_{l_{k-2}=1}^{l_{k-3}-1}\frac{1}{l_{k-2}}$$

is decreasing with k. Since $a_{n,n+1}=n!$, see the equation (2.4) or [**22**, (2.8)], we obtain that

$$a_{n,2}<a_{n,n+1},\quad n\geqslant 2. \tag{3.1}$$

In [**23**, Theorem 2.1], the integral representation

$$s(n,k)=\binom{n}{k}\lim_{x\to 0}\frac{d^{n-k}}{dx^{n-k}}\left\{\left[\int_0^\infty\left(\int_{1/e}^1 t^{xu-1}dt\right)e^{-u}du\right]^k\right\} \tag{3.2}$$

was created for $1\leqslant k\leqslant n$. Hence,

$$\begin{aligned}
s(n,n-1)&=n\lim_{x\to 0}\frac{d}{dx}\left\{\left[\int_0^\infty\left(\int_{1/e}^1 t^{xu-1}dt\right)e^{-u}du\right]^{n-1}\right\}\\
&=n(n-1)\lim_{x\to 0}\left[\int_0^\infty\left(\int_{1/e}^1 t^{xu-1}dt\right)e^{-u}du\right]^{n-2}\\
&\quad\times\lim_{x\to 0}\left[\int_0^\infty\left(\int_{1/e}^1 t^{xu-1}\ln t\,dt\right)ue^{-u}du\right]\\
&=n(n-1)\left[\int_0^\infty\left(\int_{1/e}^1\frac{1}{t}\,dt\right)e^{-u}du\right]^{n-2}\int_0^\infty\left(\int_{1/e}^1\frac{\ln t}{t}\,dt\right)ue^{-u}du\\
&=-\frac{1}{2}n(n-1).
\end{aligned}$$

As a result, by (2.6), it follows that $a_{n,n}=-(n-1)!s(n,n-1)=\frac{n-1}{2}n!\geqslant a_{n,n+1}$, $n\geqslant 3$. Combining this with (3.1) shows that the sequence $a_{n,k}$ for given $n\geqslant 4$ has at least one maximum with respect to $2<k<n+1$.

Remark 3.4. By integral representation (3.2) and direct computation, we can recover that

$$\begin{aligned}
s(n,1)&=\binom{n}{1}\lim_{x\to 0}\frac{d^{n-1}}{dx^{n-1}}\int_0^\infty\left(\int_{1/e}^1 t^{xu-1}dt\right)e^{-u}du\\
&=n\lim_{x\to 0}\int_0^\infty\left[\int_{1/e}^1 t^{xu-1}(\ln t)^{n-1}dt\right]u^{n-1}e^{-u}du\\
&=n\int_0^\infty\left[\int_{1/e}^1\frac{(\ln t)^{n-1}}{t}\,dt\right]u^{n-1}e^{-u}du\\
&=(-1)^{n+1}\int_0^\infty u^{n-1}e^{-u}du\\
&=(-1)^{n+1}(n-1)!
\end{aligned}$$

and

$$\begin{aligned}
s(n,2) &= \binom{n}{2}\lim_{x\to 0}\frac{d^{n-2}}{dx^{n-2}}\left\{\left[\int_0^\infty\left(\int_{1/e}^1 t^{xu-1}dt\right)e^{-u}du\right]^2\right\}\\
&= \binom{n}{2}\lim_{x\to 0}\sum_{k=0}^{n-2}\binom{n-2}{k}\int_0^\infty\left[\int_{1/e}^1 t^{xu-1}(\ln t)^k dt\right]u^k e^{-u}du\\
&\quad\times\int_0^\infty\left[\int_{1/e}^1 t^{xu-1}(\ln t)^{n-k-2}dt\right]u^{n-k-2}e^{-u}du\\
&= \binom{n}{2}\sum_{k=0}^{n-2}\binom{n-2}{k}\int_0^\infty\left[\int_{1/e}^1\frac{(\ln t)^k}{t}\,dt\right]u^k e^{-u}du\\
&\quad\times\int_0^\infty\left[\int_{1/e}^1\frac{(\ln t)^{n-k-2}}{t}\,dt\right]u^{n-k-2}e^{-u}du\\
&= (-1)^n\binom{n}{2}\sum_{k=0}^{n-2}\binom{n-2}{k}\frac{k!}{k+1}\frac{(n-k-2)!}{n-k-1}\\
&= (-1)^n(n-2)!\binom{n}{2}\sum_{k=0}^{n-2}\frac{1}{(k+1)(n-k-1)}\\
&= (-1)^n\frac{n!}{2}\sum_{k=0}^{n-2}\frac{1}{(k+1)(n-k-1)}\\
&= (-1)^n\frac{(n-1)!}{2}\sum_{k=0}^{n-2}\Big(\frac{1}{k+1}+\frac{1}{n-k-1}\Big)\\
&= (-1)^n(n-1)!H(n-1),
\end{aligned}$$

where $H(n)=\sum_{k=1}^n\frac{1}{k}$ is the n-th harmonic number. Consequently, we find a relation

$$s(n,2)=(-1)^n(n-1)!H(n-1),\quad n\in\mathbb{N} \tag{3.3}$$

or, equivalently,

$$H(n)=\frac{(-1)^{n+1}s(n+1,2)}{n!},\quad n\in\mathbb{N} \tag{3.4}$$

between the n-th harmonic number $H(n)$ and the signed Stirling numbers of the first kind $s(n,2)$. Relation (3.3), or say, (3.4), may also be deduced by considering (2.5) and (2.6).

We point out that relation (3.3), or say, (3.4) recovers [**2**, p. 275, (6.58)].

For more information on the n-th harmonic numbers $H(n)$, please refer to [**1**, **9**, **10**, **12**, **24**, **26**] and closely related references therein.

Remark 3.5. For more information on some new results for the Bernoulli numbers, the Cauchy numbers, and the Stirling numbers of the first and second

kinds, please refer to [**3**–**8**, **11**, **15**–**18**, **20**–**23**, **25**, **27**] and closely related references therein.

REMARK 3.6. This paper is a slightly revised and corrected version of the preprint [**19**].

References

[1] C.-P. Chen, F. Qi, *The best bounds of the n-th harmonic number*, Glob. J. Appl. Math. Math. Sci. **1**(1) (2008), 41–49.

[2] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete Mathematics—A Foundation for Computer Science*, 2nd ed., Addison-Wesley, Reading, MA, 1994.

[3] B.-N. Guo, I. Mező, F. Qi, *An explicit formula for the Bernoulli polynomials in terms of the r-Stirling numbers of the second kind*, Rocky Mountain J. Math. (2016), in press; arXiv: 1402.2340.

[4] B.-N. Guo, F. Qi, *A new explicit formula for the Bernoulli and Genocchi numbers in terms of the Stirling numbers*, Glob. J. Math. Anal. **3**(1) (2015), 33–36; DOI: 10.14419/gjma.v3i1.4168.

[5] ———, *Alternative proofs of a formula for Bernoulli numbers in terms of Stirling numbers*, Analysis, Berlin **34**(2) (2014), 187–193; DOI: 10.1515/anly-2012-1238.

[6] ———, *An explicit formula for Bell numbers in terms of Stirling numbers and hypergeometric functions*, Glob. J. Math. Anal. **2**(4) (2014), 243–248; DOI: 10.14419/gjma.v2i4.3310.

[7] ———, *An explicit formula for Bernoulli numbers in terms of Stirling numbers of the second kind*, J. Anal. Number Theory **3**(1) (2015), 27–30; DOI: 10.12785/jant/030105.

[8] ———, *Explicit formulae for computing Euler polynomials in terms of Stirling numbers of the second kind*, J. Comput. Appl. Math. **272** (2014), 251–257; DOI: 10.1016/j.cam.2014.05.018.

[9] ———*Sharp bounds for harmonic numbers*, Appl. Math. Comput. **218**(3) (2011), 991–995; DOI: 10.1016/j.amc.2011.01.089.

[10] ———, *Sharp inequalities for the psi function and harmonic numbers*, Analysis (Berlin) **34**(2) (2014), 201–208; DOI: 10.1515/anly-2014-0001.

[11] ———, *Some identities and an explicit formula for Bernoulli and Stirling numbers*, J. Comput. Appl. Math. **255** (2014), 568–579; DOI: 10.1016/j.cam.2013.06.020.

[12] ———, *Some properties of the psi and polygamma functions*, Hacet. J. Math. Stat. **39**(2) (2010), 219–231.

[13] H.-M. Liu, S.-H. Qi, S.-Y. Ding, *Some recurrence relations for Cauchy numbers of the first kind*, J. Integer Seq. **13** (2010), Article 10.3.8.

[14] G. Nemes, *An asymptotic expansion for the Bernoulli numbers of the second kind*, J. Integer Seq. **14** (2011), Article 11.4.8.

[15] F. Qi, *A diagonal recurrence formula for Stirling numbers of the first kind*; Contrib. Discrete Math. **11**(1) (2016), 22–30; arXiv: 1310.5920.

[16] ———, *A double inequality for ratios of the Bernoulli numbers*, ResearchGate Dataset, DOI: 10.13140/RG.2.1.3461.2641.

[17] ———, A double inequality for ratios of Bernoulli numbers, RGMIA Res. Rep. Coll. **17** (2014), Article 103, 4 pages; http://rgmia.org/v17.php.

[18] ———, *An explicit formula for the Bell numbers in terms of the Lah and Stirling numbers*, Mediterr. J. Math. **13**(5) (2016), 2795–2800; DOI:10.1007/s00009-015-0655-7; arXiv: 1401.1625.

[19] ———, *An explicit formula for computing Bernoulli numbers of the second kind in terms of Stirling numbers of the first kind*, arXiv: 1401.4934.

[20] ———, *An integral representation, complete monotonicity, and inequalities of Cauchy numbers of the second kind*, J. Number Theory **144** (2014), 244–255; DOI: 10.1016/j.jnt.2014.05.009.

[21] ———, *Diagonal recurrence relations, inequalities, and monotonicity related to the Stirling numbers of the second kind*, Math. Inequal. Appl. **19**(1) (2016), 313–323; DOI: 10.7153/mia-19-23; arXiv: 1402.2040.

[22] ———, *Explicit formulas for computing Bernoulli numbers of the second kind and Stirling numbers of the first kind*, Filomat **28**(2) (2014), 319–327; DOI: 10.2298/FIL1402319O.

[23] ——— *Integral representations and properties of Stirling numbers of the first kind*, J. Number Theory **133**(7) (2013), no. 7, 2307–2319; DOI: 10.1016/j.jnt.2012.12.015.

[24] F. Qi, R.-Q. Cui, C.-P. Chen, B.-N. Guo, *Some completely monotonic functions involving polygamma functions and an application*, J. Math. Anal. Appl. **310**(1) (2005), 303–308; DOI: 10.1016/j.jmaa.2005.02.016.

[25] F. Qi, B.-N. Guo, *Alternative proofs of a formula for Bernoulli numbers in terms of Stirling numbers*, Analysis (Berlin) **34**(3) (2014), 311–317; DOI: 10.1515/anly-2014-0003.

[26] F. Qi, Q.-M. Luo, *Complete monotonicity of a function involving the gamma function and applications*, Period. Math. Hungar. **69**(2) (2014), 159–169; DOI: 10.1007/s10998-014-0056-x.

[27] F. Qi, X.-J. Zhang, *An integral representation, some inequalities, and complete monotonicity of the Bernoulli numbers of the second kind*, Bull. Korean Math. Soc. **52**(3) (2015), 987–998; DOI: 10.4134/BKMS.2015.52.3.987.

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