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HERMITE–HADAMARD INEQUALITIES FOR DIFFERENTIABLE p-CONVEX FUNCTIONS USING HYPERGEOMETRIC FUNCTIONS

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Abstract. We derive some new integral identities for differentiable functions. Then using these auxiliary results, we obtain new Hermite–Hadamard type inequalities for differentiable p-convex functions. Some special cases are also discussed.

1. Introduction

Recently theory of convexity has received much attentions by many researchers. Consequently the classical concepts of convex sets and convex functions have been extended and generalized in several directions using novel and innovative ideas, see [**1**]. Zhang [**11**] introduced the notion of p-convex functions. It is worth to mention here that besides the classical convex functions, the class of p-convex functions also includes the class of harmonically convex functions introduced and studied by Iscan [**5**]. For some recent investigations on p-convex functions, see [**4**].

The interrelationship between theory of convex functions and theory of inequalities has attracted many researchers. One of the most extensively studied inequality for convex functions is the Hermite–Hadamard inequality. This inequality provides the necessary and sufficient condition for a function to be convex. For some recent investigation on Hermite–Hadamard type inequalities, see [**2**–**10**].

In this article, We consider the class of p-convex functions. We derive two new integral identities for differentiable functions. Using these results we establish our main results that are Hermite–Hadamard type inequalities for differentiable p-convex functions. We use hypergeometric functions to solve our integrals. It is expected that the ideas and techniques of this paper may stimulate further research in this area. This is the main motivation of this paper.

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2. Preliminaries and lemmas

In this section, we recall some previously known concepts and derive some new results which play an important role in the development of our main results.

Definition 2.1. [**11**] An interval I is said to be a p-convex set, if

$$M_p(x,y;t) = [tx^p + (1-t)y^p]^{\frac{1}{p}} \in I,$$

for all $x, y \in I, t \in [0,1]$, where $p = 2k+1$ or $p = \frac{n}{m}, n = 2r+1, m = 2t+1$ and $k, r, t \in \mathbb{N}$.

Definition 2.2. [**11**] Let I be a p-convex set. A function $f\colon I \to \mathbb{R}$ is said to be p-convex function or belongs to the class $PC(I)$, if

$$f(M_p(x,y;t)) \leqslant tf(x) + (1-t)f(y), \quad \forall x, y \in I, \ \ t \in [0,1].$$

It is obvious that for $p = 1$, Definition 2.2 reduces to the definition for classical convex functions. Note that for $p = -1$, we have the definition of harmonically convex functions.

Definition 2.3. [**5**] A function $f\colon I \subset \mathbb{R} \setminus \{0\} \to \mathbb{R}$ is said to be harmonically convex function, if

$$f\Big(\frac{xy}{(1-t)x + ty}\Big) \leqslant tf(x) + (1-t)f(y), \quad \forall x, y \in I, \ \ t \in [0,1].$$

Also note that for $t = \frac{1}{2}$ in Definition 2.2, we have Jensen p-convex functions or mid p-convex functions

$$f(M_p(x,y;1/2)) \leqslant \frac{f(x)+f(y)}{2}, \quad \forall x, y \in I, \ \ t \in [0,1].$$

Now we derive some new integral identities; I^0 will denote the interior of I.

Lemma 2.4. Let $f\colon I = [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0 with $a < b$. If $f' \in L[a,b]$, then

$$\frac{f(a)+f(b)}{2} - \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx = \frac{b^p - a^p}{2p}\int_0^1 M_p^{-1}(a,b;t)(1-2t)f'(M_p(a,b;t))\,dt,$$

where $M_p^{-1}(a,b;t) = [ta^p + (1-t)b^p]^{\frac{1}{p}-1}$.

Proof. It suffices to show that

$$\begin{aligned}\int_0^1 M_p^{-1}(a,b;t)(1-2t)f'(M_p(a,b;t))\,dt &= \frac{f(a)+f(b)}{b^p - a^p} - \frac{2p}{b^p - a^p}\int_0^1 f([ta^p + (1-t)b^p]^{\frac{1}{p}})\,dt \\ &= \frac{f(a)+f(b)}{b^p - a^p} - \frac{2p^2}{(b^p - a^p)^2}\int_a^b \frac{f(x)}{x^{1-p}}dx.\end{aligned}$$

Multiplying both sides of the above inequality by $\frac{b^p-a^p}{2p}$, we get the required result. $\square$

Note that for $p=1$, Lemma 2.4 reduces to the following known integral identity by Dragomir et al. [**2**].

LEMMA 2.5. [**2**] Let $f\colon I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$. If $f'\in L[a,b]$, then, we have

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx=\frac{b-a}{2}\int_0^1(1-2t)f'(ta+(1-t)b)\,dt.$$

If $p=-1$, then Lemma 2.4 reduces to the following integral identity mainly due to Iscan [**5**].

LEMMA 2.6. [**5**] Let $f: I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$. If $f'\in L[a,b]$, then

$$\frac{f(a)+f(b)}{2}-\frac{ab}{b-a}\int_a^b\frac{f(x)}{x^2}\,dx$$
$$=\frac{ab(b-a)}{2}\int_0^1\frac{1-2t}{[tb+(1-t)a]^2}f'\Big(\frac{ab}{tb+(1-t)a}\Big)dt.$$

LEMMA 2.7. Let $f\colon I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$. If $f'\in L[a,b]$, then

$$\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx-f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big)$$
$$=\frac{b^p-a^p}{p}\int_0^1 M_p^{-1}(a,b;t)\vartheta(t)f'(M_p(a,b;t))\,dt,$$

where

$$\vartheta(t)=\begin{cases}t, & [0,\frac{1}{2}),\\ t-1, & [\frac{1}{2},1].\end{cases}$$

PROOF. A simple integration by parts completes the proof. $\square$

For $p=1$, Lemma 2.7 reduces to Lemma 2.1 of [**6**]. For the reader's convenience we recall here the definitions of the Gamma and Beta functions

$$\Gamma(x)=\int_0^\infty e^{-x}t^{x-1}dt,\qquad \mathrm{B}(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt.$$

It holds

$$\mathrm{B}(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

The integral form of the hypergeometric function is

$${}_2F_1(x,y;c;z)=\frac{1}{\mathrm{B}(y,c-y)}\int_0^1 t^{y-1}(1-t)^{c-y-1}(1-zt)^{-x}dt$$

for $|z|<1, c>y>0$.

3. Main results

In this section, we derive our main results.

THEOREM 3.1. *Let $f : I = [a, b] \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I^0 with $a < b$ and $f' \in L[a, b]$. If $|f'|$ is p-convex function, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \leqslant b^{1-p} \cdot \frac{b^p - a^p}{2p}\{K_1|f'(a)| + K_2|f'(b)|\}.$$

where

$$K_1 = \frac{2}{3} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 3; 4; 1 - \frac{a^p}{b^p}\Big) - \frac{1}{2} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 2; 3; 1 - \frac{a^p}{b^p}\Big) + \frac{1}{12} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 2; 4; \frac{1}{2}\Big(1 - \frac{a^p}{b^p}\Big)\Big), \tag{3.1}$$

$$K_2 = \frac{1}{3} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 2; 4; 1 - \frac{a^p}{b^p}\Big) - \frac{1}{2} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 1; 3; 1 - \frac{a^p}{b^p}\Big) + \frac{1}{2} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 1; 3; \frac{1}{2}\Big(1 - \frac{a^p}{b^p}\Big)\Big) - \frac{1}{12} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 2; 4; \frac{1}{2}\Big(1 - \frac{a^p}{b^p}\Big)\Big). \tag{3.2}$$

PROOF. Using Lemma 2.4 and the fact that $|f'|$ is a p-convex function, we have

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2} - \frac{p}{b^p - a^p}\int_a^b \frac{f(x)}{x^{1-p}}dx\right| \\ &= \left|\frac{b^p - a^p}{2p}\int_0^1 M_p^{-1}(a, b; t)(1 - 2t)f'(M_p(a, b; t))\,dt\right| \\ &\leqslant \frac{b^p - a^p}{2p}\int_0^1 \frac{|1 - 2t|}{[ta^p + (1 - t)b^p]^{1-\frac{1}{p}}}\left|f'([ta^p + (1 - t)b^p]^{\frac{1}{p}})\right|dt \\ &\leqslant \frac{b^p - a^p}{2p}\int_0^1 \frac{|1 - 2t|}{[ta^p + (1 - t)b^p]^{1-\frac{1}{p}}}\big[t|f'(a)| + (1 - t)|f'(b)|\big]dt \\ &= \frac{b^p - a^p}{2p}(|f'(a)|I_1 + |f'(b)|I_2), \end{aligned} \tag{3.3}$$

where

$$I_1 = \int_0^1 \frac{|1 - 2t|t}{[ta^p + (1 - t)b^p]^{1-1/p}}dt = b^{1-p}\Big[\frac{2}{3} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 3; 4; 1 - \frac{a^p}{b^p}\Big) - \frac{1}{2} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 2; 3; 1 - \frac{a^p}{b^p}\Big) + \frac{1}{12} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 2; 4; \frac{1}{2}\Big(1 - \frac{a^p}{b^p}\Big)\Big)\Big], \tag{3.4}$$

$$\begin{aligned} I_2 &= \int_0^1 \frac{|1 - 2t|(1 - t)}{[ta^p + (1 - t)b^p]^{1-1/p}}dt \\ &= b^{1-p}\Big[\frac{1}{3} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 2; 4; 1 - \frac{a^p}{b^p}\Big) - \frac{1}{2} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 1; 3; 1 - \frac{a^p}{b^p}\Big) \\ &+ \frac{1}{2} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 1; 3; \frac{1}{2}\Big(1 - \frac{a^p}{b^p}\Big)\Big) - \frac{1}{22} \cdot {}_2F_1\Big(1 - \frac{1}{p}, 2; 4; \frac{1}{2}\Big(1 - \frac{a^p}{b^p}\Big)\Big)\Big]. \end{aligned} \tag{3.5}$$

Introducing relations (3.4) and (3.5) in (3.3) completes the proof. □

THEOREM 3.2. *Let $f\colon I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$ and $f'\in L[a,b]$. If $|f'|^q$ is p-convex function where $q\geqslant 1$, then*

$$\Big|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\Big| \leqslant b^{1-p}\cdot\frac{b^p-a^p}{2p}H^{1-\frac{1}{q}}\big\{K_1|f'(a)|^q+K_2|f'(b)|^q\big\}^{\frac{1}{q}}.$$

where K_1, K_2 are given by (3.1) *and* (3.2) *and*

$$H={}_2F_1\Big(1-\frac{1}{p},2;3;1-\frac{a^p}{b^p}\Big)-{}_2F_1\Big(1-\frac{1}{p},1;2;1-\frac{a^p}{b^p}\Big) + {}_2F_1\Big(1-\frac{1}{p},1;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big).$$

PROOF. Using Lemma 2.4, the fact that $|f'|$ is a p-convex function and power mean's inequality, we have

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2}-\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx\Big|\\
&=\Big|\frac{b^p-a^p}{2p}\int_0^1 M_p^{-1}(a,b;t)(1-2t)f'(M_p(a,b;t))\,dt\Big|\\
&\leqslant\frac{b^p-a^p}{2p}\Big(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\Big)^{1-\frac{1}{q}}\\
&\quad\times\Big(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}|f'([ta^p+(1-t)b^p]^{\frac{1}{p}})|^q dt\Big)^{\frac{1}{q}}\\
&\leqslant\frac{b^p-a^p}{2p}\Big(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}dt\Big)^{1-\frac{1}{q}}\\
&\quad\times\Big(\int_0^1\frac{|1-2t|}{[ta^p+(1-t)b^p]^{1-\frac{1}{p}}}[t|f'(a)|^q+(1-t)|f'(b)|^q]dt\Big)^{\frac{1}{q}}\\
&=b^{1-p}\cdot\frac{b^p-a^p}{2p}H^{1-\frac{1}{q}}\big\{K_1|f'(a)|^q+K_2\|f'(b)|^q\big\}^{\frac{1}{q}}.
\end{aligned}$$

□

THEOREM 3.3. *Let $f\colon I=[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a differentiable function on I^0 with $a<b$ and $f'\in L[a,b]$. If $|f'|$ is p-convex function, then*

$$\Big|\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx-f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big)\Big| \leqslant\frac{b^{1-p}(b^p-a^p)}{p}\big[\{C_1+C_2-C_3\}|f'(a)|+\{C_4+C_5-C_6-C_7\}|f'(b)|\big],$$

where

$$C_1=\frac{1}{6}{}_2F_1\Big(1-\frac{1}{p},2;4;1-\frac{a^p}{b^p}\Big),$$

$$
\begin{aligned}
C_2 &= \frac{1}{12}{}_2F_1\Big(1-\frac{1}{p},3;4;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big),\\
C_3 &= \frac{1}{48}{}_2F_1\Big(1-\frac{1}{p},2;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big),\\
C_4 &= \frac{1}{3}{}_2F_1\Big(1-\frac{1}{p},1;4;1-\frac{a^p}{b^p}\Big),\\
C_5 &= \frac{3}{8}{}_2F_1\Big(1-\frac{1}{p},2;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big),\\
C_6 &= \frac{1}{12}{}_2F_1\Big(1-\frac{1}{p},3;4;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big),\\
C_7 &= \frac{1}{2}{}_2F_1\Big(1-\frac{1}{p},1;2;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big).
\end{aligned}
$$

PROOF. Using Lemma 2.7 and the fact that $|f'|$ is a p-convex function, we have

$$
\begin{aligned}
&\Big|\frac{p}{b^p-a^p}\int_a^b\frac{f(x)}{x^{1-p}}dx-f\Big(\Big[\frac{a^p+b^p}{2}\Big]^{\frac{1}{p}}\Big)\Big|\\
&=\Big|\frac{b^p-a^p}{p}\int_0^{\frac{1}{2}}tM_p^{-1}(a,b;t)f'(M_p(a,b;t))\,dt\\
&\quad+\frac{b^p-a^p}{p}\int_{\frac{1}{2}}^{1}(t-1)M_p^{-1}(a,b;t)f'(M_p(a,b;t))\,dt\Big|\\
&\leqslant\frac{b^p-a^p}{p}\Big[\int_0^{\frac{1}{2}}tM_p^{-1}(a,b;t)|f'(M_p(a,b;t))|dt\\
&\quad+\int_{\frac{1}{2}}^{1}|t-1|M_p^{-1}(a,b;t)|f'(M_p(a,b;t))|dt\Big]\\
&\leqslant\frac{b^{1-p}(b^p-a^p)}{p}\Big[\Big\{\frac{1}{6}{}_2F_1\Big(1-\frac{1}{p},2;4;1-\frac{a^p}{b^p}\Big)\\
&\quad+\frac{1}{12}{}_2F_1\Big(1-\frac{1}{p},3;4;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big)\\
&\quad-\frac{1}{48}{}_2F_1\Big(1-\frac{1}{p},2;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big)\Big\}|f'(a)|\\
&\quad+\Big\{\frac{1}{3}{}_2F_1\Big(1-\frac{1}{p},1;4;1-\frac{a^p}{b^p}\Big)+\frac{3}{8}{}_2F_1\Big(1-\frac{1}{p},2;3;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big)\\
&\quad-\frac{1}{12}{}_2F_1\Big(1-\frac{1}{p},3;4;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big)\\
&\quad-\frac{1}{2}{}_2F_1\Big(1-\frac{1}{p},1;2;\frac{1}{2}\Big(1-\frac{a^p}{b^p}\Big)\Big)\Big\}|f'(b)|\Big].
\end{aligned}
$$

This completes the proof. □

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