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ON A SUBCLASS OF MULTIVALENT CLOSE TO CONVEX FUNCTIONS

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Abstract. We introduce a new subclass of multivalent close to convex functions related with Janowski functions and study some of their properties: coefficient estimates, inclusion and inverse inclusion, distortion problems and sufficiency criteria to be in these subclasses.

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ which are analytic and p-valent in the region $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the condition

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \quad (z \in \mathbb{U}). \tag{1.1}$$

We write $\mathcal{A}(1) = \mathcal{A}$. Robertson introduced in [**9**] the class $\mathcal{S}^*(\alpha)$ of starlike functions of order $\alpha \leqslant 1$, which are defined by

$$\mathcal{S}^*(\alpha) := \Big\{ f \in \mathcal{A} : \mathrm{Re}\Big(\frac{zf'(z)}{f(z)}\Big) > \alpha, \quad z \in \mathbb{U} \Big\}.$$

By $\mathcal{S}^* = \mathcal{S}^*(0)$ we denote the subclasses of $\mathcal{A}$ which consist of univalent starlike functions. An important subclass of analytic functions is the class $\mathcal{K}$ of close-to-convex functions

$$\mathcal{K} = \Big\{ f \in \mathcal{A} : \exists \beta \in \mathbb{R}, \exists g \in \mathcal{S}^* : \mathrm{Re}\Big\{\frac{zf'(z)}{e^{i\beta}g(z)}\Big\} > 0, \quad z \in \mathbb{U} \Big\}.$$

Each close-to-convex function is univalent in the unit disc. For two functions $f(z)$ and $g(z)$ analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, denoted by $f(z) \prec g(z)$, if there is an analytic function $w(z)$ with $|w(z)| \leqslant |z|$ such that $f(z) = g(w(z))$. If $g(z)$ is univalent, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

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In [**11**] Sakaguchi introduced the class $\mathcal{S}_s^*$ of starlike functions with respect to symmetric points; a function $f(z) \in \mathcal{A}$ belongs to the class $\mathcal{S}_s^*$, if and only if

$$\frac{zf'(z)}{f(z) - f(-z)} \prec \frac{1+z}{1-z}, \quad (z \in \mathbb{U}).$$

One can easily obtain that the function $(f(z) - f(-z))/2$ is starlike in $\mathbb{U}$ and therefore the functions in $\mathcal{S}_s^*$ are close-to-convex. Motivated from Sakaguchi's work, Gao and Zhou [**3**] introduced a class $\mathcal{K}_s$. A function $f(z) \in \mathcal{A}$ belongs to the class $\mathcal{K}_s$ if it satisfies the subordination

$$-\frac{z^2 f'(z)}{g(z)g(-z)} \prec \frac{1+z}{1-z}, \quad (z \in \mathbb{U}),$$

for some $g(z) \in \mathcal{S}^*(1/2)$.

In [**6**] it was introduced the class $\mathcal{K}_s(\gamma)$ of functions satisfying

$$\frac{-z^2 f'(z)}{g(z)g(-z)} \prec q_\gamma(z) = \frac{1-(1-2\gamma)z}{1+z}. \tag{1.2}$$

The class $\mathcal{K}_s(\gamma)$ has been generalized in several directions, see the references in [**7**]. Recently, Xu, Srivastava and Li considered in [**15**] the class $\mathcal{K}_s(h)$ of functions satisfying (1.2) with a convex function h instead of q_γ. Şeker introduced in [**12**] the class $\mathcal{K}_s^k(\gamma)$, $k > 1$, of functions defined by (1.2) with

$$-z^{2-k} \prod_{\nu=0}^{k-1} \varepsilon^{-\nu} g(\varepsilon^\nu z), \quad g \in \mathcal{S}^*((k-1)/k),$$

instead of $g(z)g(-z)$. Moreover, Wang, Sun and Xu introduced in [**14**] the class $\mathcal{MK}$ of meromorphic functions satisfying (1.2) with $\gamma = 1$. See also the references in [**14**] for the other papers in this topic.

If $f \in \mathcal{A}(p)$, $\alpha < 1$ and $\operatorname{Re} \frac{zf'(z)}{f(z)} > p\alpha$, $z \in \mathbb{U}$, then we say that f is in the class $\mathcal{S}_p^*(\alpha)$ of p-valent starlike functions of order α. Using the techniques of subordination, we now introduce a subclass of p-valent analytic functions as follows.

Definition 1.1. A function $f(z) \in \mathcal{A}(p)$ is said to be in the class $\mathcal{W}_p(t, \lambda, A, B)$, $(0 < |t| \leqslant 1,\ -1 \leqslant B < A \leqslant 1$ and $\lambda \in (0, 1])$, if it satisfies

$$\frac{t^p z^{p+1} F_\lambda'(z)}{pg(z)g(tz)} \prec q(z) := \frac{1+Az}{1+Bz} \tag{1.3}$$

for some $g(z) \in \mathcal{S}_p^*(1/2)$ and $F_\lambda(z)$ is defined by $F_\lambda(z) = (1-\lambda)f(z) + \frac{\lambda}{p} zf'(z)$.

In the literature, various interesting subclasses of this class have been studied from a number of different view points. For example: if we set $t = -1$, $p = 1$ and $\xi = 0$ in Definition 1.1, we get the class $\mathcal{W}_1(-1, \lambda, A, B) \equiv \mathcal{K}_s(\lambda, A, B)$ which was studied recently by Wang and Chen [**13**] and further for $\lambda = 0$, $A = 1 - 2\gamma$ and $B = -1$, we obtain the class $\mathcal{K}_s(\gamma)$ introduced in [**6**]. For more details of the related work see [**1, 2, 4, 8, 12, 15**].

The main object of the present paper is to introduce a subclass of p-valent analytic functions and then investigate some useful results including the coefficient estimate, sufficiency criteria to be in a class, distortion problem, radius of convexity and inclusion relationship for the new defined class.

To avoid repetition, we shall assume, unless otherwise stated, that $\lambda \in (0,1]$, $-1 \leqslant B < A \leqslant 1$, and $0 < |t| \leqslant 1$.

2. Some properties of the class $\mathcal{W}_p(t, \lambda, A, B)$

Theorem 2.1. *Let $g_i(z) \in \mathcal{S}_p^*(\alpha_i)$ with $\alpha_i < 1$. Then*

$$G(z) = \frac{g_1(t_1 z) g_2(t_2 z)}{t_1^p t_2^p z^p} \in \mathcal{S}_p^*(\gamma), \tag{2.1}$$

where $\gamma = \alpha_1 + \alpha_2 - 1$ and $0 < |t_i| \leqslant 1$, $i = 1, 2$.

Proof. Let $g_i(z) \in \mathcal{S}_p^*(\alpha_i)$. Then by definition we have

$$\operatorname{Re} \frac{t_1 z g_1'(t_1 z)}{g_1(t_1 z)} > p\alpha_1, \quad \operatorname{Re} \frac{t_2 z g_2'(t_2 z)}{g_2(t_2 z)} > p\alpha_2, \quad |z| < 1,\ 0 < |t_i| \leqslant 1.$$

By logarithmic differentiating (2.1), we obtain that

$$\frac{zG'(z)}{G(z)} = \frac{t_1 z g_1'(t_1 z)}{g_1(t_1 z)} + \frac{t_2 z g_2'(t_2 z)}{g_2(t_2 z)} - 1.$$

It follows that

$$\operatorname{Re} \frac{zG'(z)}{G(z)} = \operatorname{Re} \frac{t_1 z g_1'(t_1 z)}{g_1(t_1 z)} + \operatorname{Re} \frac{t_2 z g_2'(t_2 z)}{g_2(t_2 z)} - 1 > p\alpha_1 + p\alpha_2 - p = p\gamma.$$

It implies that $G(z) \in \mathcal{S}_p^*(\gamma)$ and it completes the proof of the theorem. $\square$

Corollary 2.1. *If $g(z) \in \mathcal{S}_p^*(1/2)$ and $0 < |t| < 1$, then*

$$\frac{g(z) g(tz)}{t^p z^p} \in \mathcal{S}_p^*(0) := \mathcal{S}_p^*.$$

Theorem 2.2. *If $-1 < D$, then $\mathcal{W}_p(t, \lambda, A, B) \subset \mathcal{W}_p(t, \lambda, C, D)$. if and only if*

$$\left| \frac{1 - CD}{1 - D^2} - \frac{1 - AB}{1 - B^2} \right| \leqslant \frac{C - D}{1 - D^2} - \frac{A - B}{1 - B^2}.$$

If $-1 = D$, then $\mathcal{W}_p(t, \lambda, A, B) \subset \mathcal{W}_p(t, \lambda, C, D)$ if and only if

$$C \geqslant 1 - \frac{2(1 - A)}{1 - B}. \tag{2.2}$$

Proof. Condition (1.3) means that for $z \in \mathbb{U}$ the values of the function

$$H(z) := \frac{t^p z^{p+1} F_\lambda'(z)}{p g(z) g(tz)}$$

lie in $q(\mathbb{U})$ because $q(z) = (1 + Az)/(1 + Bz)$ is univalent in $\mathbb{U}$. In the case $B \neq -1$, $q(\mathbb{U})$ is a disc $D(A, B)$, with a center $S(A, B)$ and a radius $R(A, B)$

$$S(A, B) = \frac{1 - AB}{1 - B^2}, \quad R(A, B) = \frac{A - B}{1 - B^2}, \tag{2.3}$$

while it is a half-plane for $B=-1$. By simple computation, we can easily obtain that (1.3) is equivalent to

$$\left|\frac{t^p z^{p+1} F_\lambda'(z)}{pg(z)g(tz)} - \frac{1-AB}{1-B^2}\right| < \frac{A-B}{1-B^2}, \quad B \neq -1,$$

or

$$\mathrm{Re}\left\{\frac{t^p z^{p+1} F_\lambda'(z)}{pg(z)g(tz)}\right\} > \frac{1-A}{2}, \quad B=-1. \tag{2.4}$$

Therefore, for the case $B \neq -1$ $D \neq -1$, the inclusion relation $\mathcal{W}_p(t,\lambda,A,B) \subset \mathcal{W}_p(t,\lambda,C,D)$ holds when

$$R(A,B) \leqslant R(C,D), \quad \text{and} \quad |S(C,D)-S(A,B)| \leqslant R(C,D)-R(A,B).$$

This is equivalent to

$$\left|\frac{1-CD}{1-D^2} - \frac{1-AB}{1-B^2}\right| \leqslant \frac{C-D}{1-D^2} - \frac{A-B}{1-B^2}.$$

If $D=-1$, then by (2.4), the inclusion relation $\mathcal{W}_p(t,\lambda,A,B) \subset \mathcal{W}_p(t,\lambda,C,D)$ holds when $\frac{1-C}{2} \leqslant \frac{1-A}{1-B}$. This is equivalent to (2.2). □

LEMMA 2.1. *If $g(z) \in \mathcal{S}_p^*(1/2)$ and it has the form* (1.1), *then*

$$|a_{n+p}t^n + a_{n+p-1}a_{p+1}t^{n-1} + \ldots + a_{p+1}a_{n+p-1}t + a_{n+p}| \leqslant \frac{2p}{n}\prod_{i=1}^{n-1}\Big(1+\frac{2p}{i}\Big).$$

PROOF. By virtue of Corollary 2.1, we have $\frac{g(z)g(tz)}{t^p z^p} \in \mathcal{S}_p^*(0)$, and if

$$G(z) = \frac{g(z)g(tz)}{t^p z^p} = z^p + \sum_{k=p+1}^{\infty} c_k z^k, \tag{2.5}$$

then it is well known that

$$|c_{p+n}| \leqslant \frac{2p}{n}\prod_{i=1}^{n-1}\Big(1+\frac{2p}{i}\Big). \tag{2.6}$$

Substituting the series expansions of $G(z)$ and $g(z)$ in (2.5), we get

$$\frac{\big(z^p + \sum_{k=p+1}^{\infty} a_k z^k\big)\big((tz)^p + \sum_{k=p+1}^{\infty} a_k (tz)^k\big)}{t^p z^p} = z^p + \sum_{k=p+1}^{\infty} c_k z^k,$$

Comparing the coefficients of z^{n+p}, we have

$$a_{p+n}t^n + a_{p+1}a_{p+n-1}t^{n-1} + a_{p+2}a_{p+n-2}t^{n-2} + \ldots + a_{p+n} = c_{p+n}. \tag{2.7}$$

Putting the value from (2.6) in (2.7), we get the required result. □

THEOREM 2.3. *Let $f(z) \in \mathcal{W}_p(t,\lambda,A,B)$ be of the form* (1.1). *Then*

$$|a_{p+n}| \leqslant (A-B)\left[1 + \sum_{i=1}^{n-1}\Big(\frac{2p}{i}\Big)\prod_{j=1}^{i-1}\Big(1+\frac{2p}{j}\Big)\right] + 1 + \frac{2p}{n}\prod_{i=1}^{n-1}\Big(\frac{2p}{i}\Big).$$

PROOF. Let $f(z) \in \mathcal{W}_p(t, \lambda, A, B)$. Then we have

$$\frac{zF'_\lambda(z)}{pG(z)} \prec \frac{1+Az}{1+Bz}, \tag{2.8}$$

where $G(z)$ is given by (2.5). If we put

$$q(z) = \frac{zF'_\lambda(z)}{pG(z)}, \tag{2.9}$$

it follows from (2.8) that

$$q(z) = 1 + \sum_{n=1}^{\infty} q_n z^n \prec \frac{1+Az}{1+Bz} = 1 + \sum_{n=1}^{\infty} A_n z^n, \quad A_n = (A-B)(-B)^{n-1}.$$

The function $(1+Az)/(1+Bz)$ is convex univalent, hence applying the well known Rogosinski result [**10**], we obtain

$$|q_n| \leqslant A_1 = A - B, \quad n = 1, 2, \dots. \tag{2.10}$$

Now by putting the series expansions of $f(z)$, $G(z)$ and $q(z)$ in (2.9) and then comparing the coefficients of z^{n+p}, we obtain

$$\frac{1}{p}(p+n\lambda)(p+n)a_{p+n} = c_{p+n} + q_{p+1}c_{p+n-1} + \dots + q_{p+n-1}c_{p+1} + q_{p+n}.$$

By using (2.10) and (2.6) we have

$$\begin{aligned}
\frac{1}{p}(p+n\lambda)(p+n)|a_{p+n}| &\leqslant (A-B)(|c_{p+n-1}| + \dots + |c_{p+1}| + 1) + |c_{p+n}| \\
&\leqslant (A-B)\Big[1 + \sum_{i=1}^{n-1} |c_{p+i}|\Big] + \Big(\frac{2p}{n}\Big) \prod_{i=1}^{n-1} \Big(1 + \frac{2p}{i}\Big) \\
&\leqslant (A-B)\Big[1 + \sum_{i=1}^{n-1} \Big(\frac{2p}{i}\Big) \prod_{j=1}^{i-1} \Big(1 + \frac{2p}{j}\Big)\Big] + \Big(\frac{2p}{n}\Big) \prod_{i=1}^{n-1} \Big(1 + \frac{2p}{i}\Big).
\end{aligned}$$

This completes the proof. □

THEOREM 2.4. *If $g(z) \in \mathcal{S}^*_p(1/2)$ and $f(z) \in \mathcal{A}(p)$ is of the form* (1.1), *and if it satisfies the condition*

$$\frac{1+A}{2} - \sum_{n=1}^{\infty} \frac{(1+n\lambda/p)(p+n)}{p}|a_{p+n}| - \frac{1-A}{2} \sum_{n=1}^{\infty} |c_{p+n}| > 0, \tag{2.11}$$

where c_{p+n} is given by

$$G(z) = \frac{g(z)g(tz)}{t^p z^p} = z^p + \sum_{n=1}^{\infty} c_{p+n} z^{p+n},$$

then $f(z) \in \mathcal{W}_p(t, \lambda, A, -1)$. Moreover, if it satisfies the condition

$$\frac{A-B}{1+B} - \sum_{n=1}^{\infty} \Big\{\frac{(1+n\lambda/p)(p+n)}{p}|a_{p+n}| + \Big(\frac{A-B}{1-B^2} + \frac{1-AB}{1-B^2}\Big)|c_{p+n}|\Big\} > 0, \tag{2.12}$$

then $f(z) \in \mathcal{W}_p(t, \lambda, A, B)$, $B > -1$.

PROOF. To prove that $f(z) \in \mathcal{W}_p(t, \lambda, A, -1)$, it is enough to show that

$$\left|\frac{zF_\lambda'(z)}{pG(z)}\right| > \frac{1-A}{2}, \quad z \in \mathbb{U}$$

or equivalently to show that

$$\left|\frac{F_\lambda'(z)}{pz^{p-1}}\right| - \left|\frac{(1-A)G(z)}{2z^p}\right| > 0, \quad z \in \mathbb{U}.$$

We have

$$\begin{aligned}
&\left|\frac{F_\lambda'(z)}{pz^{p-1}}\right| - \left|\frac{(1-A)G(z)}{2z^p}\right| \\
&\quad = \left|1 + \sum_{n=1}^{\infty} \frac{(1+n\lambda/p)(p+n)}{p} a_{p+n} z^n\right| - \left|\frac{(1-A)}{2}(1 + \sum_{n=1}^{\infty} c_{p+n} z^n)\right| \\
&\quad \geqslant 1 - \left|\sum_{n=1}^{\infty} \frac{(1+n\lambda/p)(p+n)}{p} a_{p+n} z^n\right| - \left|\frac{(1-A)}{2}(1 + \sum_{n=1}^{\infty} c_{p+n} z^n)\right| \\
&\quad \geqslant 1 - \sum_{n=1}^{\infty} \frac{(1+n\lambda/p)(p+n)}{p} |a_{p+n}| - \frac{(1-A)}{2} - \frac{(1-A)}{2} \sum_{n=1}^{\infty} |c_{p+n}| \\
&\quad = \frac{1+A}{2} - \sum_{n=1}^{\infty} \frac{(1+n\lambda/p)(p+n)}{p} |a_{p+n}| - \frac{1-A}{2} \sum_{n=1}^{\infty} |c_{p+n}| \geqslant 0
\end{aligned}$$

by (2.11). For the case $B > -1$ it suffices to show that

$$\left|z\frac{F_\lambda'(z)}{pG(z)} - S(A,B)\right| < R(A,B),$$

where $S(A,B)$ and $R(A,B)$ are given in (2.3). This is equivalent to

$$\left|\frac{F_\lambda'(z)}{pz^{p-1}} - \frac{G(z)S(A,B)}{z^p}\right| < \left|\frac{G(z)R(A,B)}{z^p}\right|, \quad z \in \mathbb{U}.$$

We have

$$\begin{aligned}
&\left|\frac{G(z)R(A,B)}{z^p}\right| - \left|\frac{F_\lambda'(z)}{pz^{p-1}} - \frac{G(z)S(A,B)}{z^p}\right| \\
&\quad = R(A,B)\left|1 + \sum_{n=1}^{\infty} c_{p+n} z^n\right| \\
&\qquad - \left|1 + \sum_{n=1}^{\infty} \frac{(1+n\lambda/p)(p+n)}{p} a_{p+n} z^n - S(A,B)1 + \sum_{n=1}^{\infty} c_{p+n} z^n\right| \\
&\quad = R(A,B)\left|1 + \sum_{n=1}^{\infty} c_{p+n} z^n\right| \\
&\qquad - \left|\frac{B(A-B)}{1-B^2} + \sum_{n=1}^{\infty} \left\{\frac{(1+n\lambda/p)(p+n)}{p} a_{p+n} - S(A,B) c_k\right\} z^n\right|
\end{aligned}$$

$$
\begin{aligned}
&\geqslant R(A,B) - R(A,B)\sum_{n=1}^{\infty}|c_{p+n}| \\
&\quad - \frac{B(A-B)}{1-B^2} - \sum_{n=1}^{\infty}\Big\{\frac{(1+n\lambda/p)(p+n)}{p}|a_{p+n}| + S(A,B)|c_{p+n}|\Big\} \\
&= \frac{A-B}{1+B} - \sum_{n=1}^{\infty}\Big\{\frac{(1+n\lambda/p)(p+n)}{p}|a_{p+n}| + (R(A,B)+S(A,B))|c_{p+n}|\Big\} > 0
\end{aligned}
$$

by (2.12). □

THEOREM 2.5. *Let $f(z) \in \mathcal{W}_p(t,0,A,B)$. Then*

$$\frac{1-Ar}{1-Br}\frac{(1-r)^{2p}p}{r^{p+1}} \leqslant |f'(z)| \leqslant \frac{1+Ar}{1+Br}\frac{(1+r)^{2p}p}{r^{p+1}}.$$

PROOF. Suppose that $f(z) \in \mathcal{W}_p(t,0,A,B)$. Then by using definition of subordination between analytic functions, we can write

$$\frac{1-Ar}{1-Br} \leqslant \frac{1-A|w(z)|}{1-B|w(z)|} \leqslant \left|\frac{zf'(z)}{pG(z)}\right| \leqslant \frac{1+A|w(z)|}{1+B|w(z)|} \leqslant \frac{1+Ar}{1+Br}, \tag{2.13}$$

where $w(z)$ is the Schwarz function with $w(0) = 0$, $|w(z)| < |z| = r$. Now it is shown in Corollary 2.1 that $G(z) = \frac{g(z)g(tz)}{t^p z^p} \in \mathcal{S}_p^*(0)$, thus we have

$$\frac{r^p}{(1+r)^{2p}} \leqslant |G(z)| \leqslant \frac{r^p}{(1-r)^{2p}}. \tag{2.14}$$

Now by using (2.14) in (2.13), we obtain the required result. □

THEOREM 2.6. *Let $f(z) \in \mathcal{W}_p(t,0,1,B)$. Then*

$$1 + \frac{zf''(z)}{f'(z)} > 0, \quad |z| < r_0,$$

where r_0 is the smallest positive root of the equation

$$p(1-r)^2(1-Br) - r(1+r)(1-B) = 0. \tag{2.15}$$

PROOF. Suppose $f(z) \in \mathcal{W}_p(t,0,1,B)$ and let

$$\frac{zf'(z)}{pG(z)} = q(w(z)) := s(z), \quad |w(z)| \leqslant |z|.$$

where $q(z)$ is given in (1.3). Logarithmic differentiation gives us

$$1 + \frac{zf''(z)}{f'(z)} = \frac{zG'(z)}{G(z)} + \frac{zs'(z)}{s(z)}. \tag{2.16}$$

Applying [**5**, Theorem 3] we have

$$\operatorname{Re}\frac{zs'(z)}{s(z)} > -\frac{(1-B)r}{(1-r)(1-Br)}, \quad |z| < r. \tag{2.17}$$

Moreover, $G(z) \in \mathcal{S}_p^*(0)$, so

$$\operatorname{Re} \frac{zG'(z)}{pG(z)} > \frac{1-r^p}{1+r^p}, \quad |z| < r. \tag{2.18}$$

Applying (2.17) and (2.18) in (2.16), we obtain

$$\begin{aligned}\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} &> p\frac{1-r^p}{1+r^p} - \frac{(1-B)r}{(1-r)(1-Br)} > p\frac{1-r}{1+r} - \frac{(1-B)r}{(1-r)(1-Br)} \\ &= \frac{p(1-r)^2(1-Br) - r(1+r)(1-B)}{(1-r^2)(1-Br)}\end{aligned}$$

and this is positive for $r < r_0$, where r_0 is described in (2.15). $\square$

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