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ON CLASSIFICATION PROBLEMS IN THE THEORY OF DIFFERENTIAL EQUATIONS: ALGEBRA + GEOMETRY

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Abstract. We study geometric and algebraic approaches to classification problems of differential equations. We consider the so-called Lie problem: provide the point classification of ODEs $y'' = F(x, y)$. In the first part of the paper we consider the case of smooth right-hand side F. The symmetry group for such equations has infinite dimension, so classical constructions from the theory of differential invariants do not work. Nevertheless, we compute the algebra of differential invariants and obtain a criterion for the local equivalence of two ODEs $y'' = F(x, y)$. In the second part of the paper we develop a new approach to the study of subgroups in the Cremona group. Namely, we consider class of differential equations $y'' = F(x, y)$ with rational right hand sides and its symmetry group. This group is a subgroup in the Cremona group of birational automorphisms of $\mathbb{C}^2$, which makes it possible to apply for their study methods of differential invariants and geometric theory of differential equations. Also, using algebraic methods in the theory of differential equations we obtain a global classification for such equations instead of local classifications for such problems provided by Lie, Tresse and others.

The problem of the classification of ordinary differential equations with respect to the contact or point transformations is one of the most important problems in mathematics, which was studied by many famous mathematicians during the XIX-th and XX-th centuries. The main ideas and approaches in their works belong to the geometry of differential equations, which is actively studied nowadays.

The most simple (and perhaps the most fundamental) case of ordinary differential equations consists of differential equations, which can be solved with respect to the highest derivative:

$$y^{(n)} = F(x, y, y', \dots, y^{(n-1)}).$$

The problem of classifications for such equations has a long and interesting history (for general theory of symmetries of differential equations see [**18**, **31**]).

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The first results in this problem were obtained by Lie (see, for example, [**1**]). He proved that each first order ODE is point equivalent to the trivial ODE $y' = 0$ in the neighborhood of a regular point, and each second order ODE is contact equivalent to the trivial ODE $y'' = 0$ in the neighborhood of a regular point (see [**26**]).

More than 100 years ago Lie and Tresse calculated sub-differential invariants for the action of point pseudogroup on the second order ODEs (see [**36**]). Absolute differential invariants and the classification of non-degenerated ODEs were found by Kruglikov in [**23**]. Some other interesting results in this question are obtained in [**4**, **19**, **29**, **30**, **35**].

Contact classification for third-order equations was studied by Chern [**11**], and for fourth-order equations—by Chen and Li [**10**]. But they did not find all differential invariants for these equations. Moreover, it seems, that the methods they used can not be applied to differential equations of higher order.

Dubrov in [**13**] constructed a set of subdifferential invariants for differential equations of arbitrary order. Using these invariants he solved the problem of the contact trivialization of differential equations. Then in [**14**] he obtained a set of sub-differential invariants for the systems of differential equations of arbitrary order.

Finally, Dubrov, Komrakov and Morimoto constructed [**15**] the Cartan connection associated with every ODE (and, more generally, with every holonomic system of differential equations), and reduced the contact equivalence problem to the classical problem of the equivalence of $\{e\}$-structures.

In [**6**] the first author suggested a new approach to the classification of ODEs and the calculation of their differential invariants. Using this approach (which is based on the ideas of Kushner and Lychagin, see [**25**]), he described the algebra of contact differential invariants for ODEs of arbitrary order and solved the classification problem or nondegenerated ODEs in terms of this algebra.

It is important to note that all these results were obtained only for non-degenerated ODEs (the conditions of non-degeneration are different for different classifications). But many interesting differential equations don't fall into these classifications. For example, in the case of second order ODEs the Kruglikov–Tresse–Lie classification does not contain the equations

$$y'' = a_0(x, y)(y')^3 + a_1(x, y)(y')^2 + a_2(x, y)(y') + a_3(x, y), \tag{0.1}$$

which are closely connected with different geometric problems (for example, with projective geometry; see [**2**, **9**, **34**]). Such equations were studied by Lie, Tresse, Cartan, Liuville, etc. (see, for example, [**27**]). The final solution of the classification problem for the regular ODEs of such type was obtained by Yumaguzhin in [**39**].

While studying the class of differential equations (0.1), Lie set the following problem: find differential invariants and classify differential equations $y'' = F(x, y)$ with respect to the pseudogroup of point transformations. We note that these equations do not fall into Yumaguzhin's classification. Despite considerable efforts, Lie could not solve this problem; moreover, he failed to find even one differential invariant in this problem.

In the present paper we study this problem in different ways. In the first section we present the full solution of the Lie problem: we compute the field of differential invariants and find the criterion for the equivalence of two non-degenerated differential equations of the type $y'' = F(x, y)$.

However, these results of ours as well as many other results in this area (see, for example, [**16**, **17**, **20**–**22**, **33**]) have two significant disadvantages: they are local (i.e., they can be applied only in the neighborhood of some point) and they are not computable (with the help of the computer). On the other hand, during the last years in works [**8**, **24**] there were developed some applications of the theory of differential equations in algebra and algebraic geometry and vice versa. In particular, Kruglikov and Lychagin proved the global version of the Lie–Tresse theorem of the global finite generation for the field of rational differential invariants, and Bibikov and Lychagin classified the homogeneous forms with respect to the linear actions of algebraic groups and also studied the representations of these groups with the help of differential invariants.

In the second part of the paper we introduce the algebraic structure in our differential equations. Namely, we consider differential equations with rational right-hand sides and study the symmetry group only with rational morphisms. Then this symmetry group will become a subgroup of plane Cremona group $\mathrm{Cr}(2)$ (see [**12**]). Such observation makes it possible to provide an effective (i.e., computable) criterion for the local equivalence of differential equations.

Recall that Cremona group $\mathrm{Cr}(n)$ is a group of birational automorphisms of the projective space $\mathbb{C}P^n$. Group $\mathrm{Cr}(1)$ is isomorphic to the projective linear group $\mathrm{PGL}(2)$. In the case $n = 2$ the structure of the group $\mathrm{Cr}_2(\mathbb{C})$ is not well understood in spite of extensive research (see [**12**]). The representation of a Cremona subgroup as a symmetry group of the ODEs $y'' = F(x, y)$ makes it possible to study this subgroup using the geometry of differential equations and the differential-geometric technique. Also, this idea allows us to obtain not a local but a global classification of differential equations with rational coefficients.

1. Differential equations with smooth right hand sides

In this section we present the full solution of the original Lie problem (see also [**7**]). First of all, we recall the necessary notation and definition. Then, we compute the number of independent differential invariants and isotropy subalgebras. Using these computations we describe the field of differential invariants. Finally, with the help of this field we obtain a criterion of local point equivalence of two non-degenerated differential equations.

1.1. Lie algebra $\widehat{\mathfrak{g}}$. First of all let us introduce the important notation (details can be found in [**1**]). Let $J^2\mathbb{R}$ be the 2-jet space of functions $f\colon \mathbb{R} \to \mathbb{R}$ with the canonical coordinates (x, y, p, q). Consider the differential equations $\{q = F(x, y)\} \subset J^2\mathbb{R}^2$, which do not depend on p.

We start from the computation of the point symmetry group and its Lie algebra for such class of equations (it means that the actions of this group and the shifts along the vector fields from this algebra map the equation of such type into another equation of the same type). Note that symmetry group and symmetry algebra were computed by Lie.

PROPOSITION 1.1. *1. Point symmetry pseudogroup G consists of transformations*

$$x \mapsto X(x), \qquad y \mapsto C \cdot \sqrt{|X'(x)|}y + A(x),$$

where $A, X \in C^\infty(\mathbb{R})$ are arbitrary smooth functions and $C \in \mathbb{R}$ is a real constant.

2. Lie algebra $\mathfrak{g}$ of the pseudogroup G consists of vector fields

$$X := a(x)\partial_x + \big((a'(x)/2 + c)y + b(x)\big)\partial_y, \tag{1.1}$$

where $a, b \in C^\infty(\mathbb{R})$ are arbitrary smooth functions and $c \in \mathbb{R}$ is a real constant.

PROOF. To prove Statement 2 we will use the standard constructions from the theory of differential equations (see, for example, [**1**]). Let $X = \alpha(x,y)\partial_x + \beta(x,y)\partial_y$ be an arbitrary vector field on the 0-jet space $J^0\mathbb{R}^2$. Denote by $X^{(2)}$ its prolongation in 2-jet space $J^2\mathbb{R}$. Then the class of equations $\{q = F(x,y)\}$ is preserved after shifts along field $X^{(2)}$ if and only if $X^{(2)}(q - F(x,y))\big|_{q=F(x,y)} = \widetilde{F}(x,y)$ (here F is an arbitrary function).

This condition is the system of differential equations on the coefficients α and β of the vector field X. With the help of `DETools` package of the `Maple` software, we obtain

$$\alpha_y = 0, \quad \beta_{yy} - 2\alpha_{xy} = 0, \quad 2\beta_{xy} - \alpha_{xx} = 0.$$

From the first equation we get $\alpha(x,y) = a(x)$, from the second one—$\beta(x,y) = k(x)y + b(x)$, and from the third one—$k(x) = a'(x)/2 + c$. Hence, an arbitrary vector field X, which preserves the class of equations $\{q = F(x,y)\}$, has form (1.1).

Statement 1 immediately follows from Statement 2. $\square$

For reasons of simplicity we will consider not point symmetry pseudogroup G itself but only its connected component of the identity G_+. Pseudogroup G_+ consists of transformations

$$x \mapsto X(x), \qquad y \mapsto C \cdot \sqrt{X'(x)}\, y + A(x),$$

where $C > 0$ and $X'(x) > 0$. Then differential invariants for the Lie algebra $\mathfrak{g}$ and pseudogroup G_+ coincide (see Theorem 1.2).

The action of the Lie algebra $\widehat{g}$ on differential equations $q = F(x,y)$ induces the action on the right hand sides of these equations. This action is defined by vector fields

$$\widehat{X} := a(x)\partial_x + \big((a'(x)/2 + c)y + b(x)\big)\partial_y + \big(a'''(x)y/2 + b''(x) - (3a'(x)/2 - c)F\big)\partial_F. \tag{1.2}$$

Denote the Lie algebra of these vector fields by $\widehat{\mathfrak{g}}$.

The corresponding action of the pseudogroup G_+ looks as follows:

$$x \mapsto X(x), \;\; y \mapsto C \cdot \sqrt{X'(x)}y + A(x), \tag{1.3}$$

$$F \mapsto \frac{\sqrt{X'}^3}{C}F - \frac{2X'X''' - 3(X'')^2}{4(X')^2}y - \frac{A''X' - A'X''}{C\sqrt{X'}^3}.$$

The pseudogroup of these transformations will be denoted by $\widehat{G}_+$.

Remark 1.1. Note that the pseudogroup $\widehat{G}_+$ preserves the class of ODEs $y'' = F(x, y)$ with the rational right hand side in variable y. Moreover, it preserves the degrees of the numerator and denominator of the right hand side F.

Now we study the action of the group $\widehat{G}_+$ on the right hand sides F of our ODEs $\{q = F(x, y)\}$. We shall count the number of independent differential invariants in each order and after that we calculate the algebra of differential invariants of $\widehat{G}_+$-action.

Remark 1.2. The calculations of differential invariants with the help of Cartan's moving frame method was developed by school of Olver (see, for example, [**16**, **17**, **20**–**22**, **31**–**33**, **37**]). Here we use an alternative technique based on the geometry of jet spaces and theory of differential equations (see, for example, [**1**, **5**, **7**, **23**, **25**]).

1.2. Number of independent differential invariants. Denote by $\mathbf{J}^k$ the k-jet space of functions $F \in C^\infty(\mathbb{R}^2)$. Canonical coordinates on this space will be denoted as $(x, y, u, u_{10}, u_{01}, \ldots, u_{ij})$, where $i + j \leqslant k$. Actions (1.3) and (1.2) of the pseudogroup $\widehat{G}_+$ and Lie algebra $\widehat{\mathfrak{g}}$ on the space $\mathbf{J}^0$ canonically prolong to the actions on k-jet space $\mathbf{J}^k$ for all k and to the actions on the infinite-jet space $\mathbf{J}^\infty := \varprojlim \mathbf{J}^k$. Denote by $\widehat{X}^{(k)}$ the prolongations of the vector fields $\widehat{X} \in \mathfrak{g}$, by $\widehat{\mathfrak{g}}^{(k)}$ the prolongation of Lie algebra $\widehat{\mathfrak{g}}$ and by $\widehat{G}_+^{(k)}$ the prolongation of pseudogroup $\widehat{G}_+$.

Recall the following definition.

Definition 1.1. 1. *A differential invariant of order* $\leqslant k$ for the action of pseudogroup $\widehat{G}_+$ on space $\mathbf{J}^k$ is function I, which is constant along all vector fields $\widehat{X}^{(k)} \in \widehat{\mathfrak{g}}^{(k)}$, i.e., $\widehat{X}^{(k)}(I) = 0$.

2. *An invariant derivation* is derivation $\nabla\colon C^\infty(\mathbf{J}^\infty) \to C^\infty(\mathbf{J}^\infty)$, which commutes with the action of Lie algebra $\widehat{\mathfrak{g}}^{(\infty)}$, i.e., $\nabla \circ \widehat{X}^{(\infty)} = \widehat{X}^{(\infty)} \circ \nabla$ for all $\widehat{X}^{(\infty)} \in \widehat{\mathfrak{g}}^{(\infty)}$.

Remark 1.3. According to the Lie–Tresse theorem, the algebra of differential invariants is locally generated by the finite number of differential invariants and invariant derivations. It is proved in work [**24**] that under certain conditions this theorem is true not locally, but globally. But it is necessary to clarify the notion of a differential invariant. Namely, in [**24**] it is required from invariants to be rational functions on jet space. In this section we do not require this condition, because derivation ∇_2 includes the factor $\sqrt{u_{03}}$ (see Theorem 1.1). Although it is possible to overcome this difficulty, the computations and formulas will become much more complicated. That's why we will consider only a local description in this section (for the global results see section 2).

The first main result in this section is the computation of the numbers of independent differential invariants of pure order k.

THEOREM 1.1. *The numbers of independent differential invariants of pure order k for the action of the pseudogroup $\widehat{G}_+^{(k)}$ on k-jet space $\mathbf{J}^k$ are given in the following table:*

order of invariants	$\leqslant 3$	4	5	6	...	k
number of invariants	0	2	4	5	...	$k-1$

PROOF. To compute the number of independent differential invariants we use the construction from [**5**, **23**]. First of all, we describe the prolongations of the vector field (1.2) in k-jet spaces $\mathbf{J}^k$. Denote

(1.4)
$$\Phi := -a(x)u_{10} - \big((a'(x)/2 + c)y + b(x)\big)u_{01} + \big(a'''(x)y/2 + b''(x) - (3a'(x)/2 - c)u\big)$$

the generating function of this vector field (see [**1**]). Then the prolongation $\widehat{X}^{(k)}$ can be written in the following way:

$$\widehat{X}^{(k)} = \Big[a(x)\frac{d}{dx} + \big((a'(x)/2 + c)y + b(x)\big)\frac{d}{dy}\Big]\Big|_{\mathbf{J}^k} + \sum_{i+j\leqslant k} \frac{d^k\Phi}{dx^i dy^j}\partial u_{ij}.$$

Now consider the canonic projection $\pi_{k,k-1}\colon \mathbf{J}^k \to \mathbf{J}^{k-1}$, the jet sequence $\theta_i := [F]^i_{(0,0)}$ (where $F \in C^\infty(\mathbb{R}^2)$ is an arbitrary smooth function in the general position) and the fiber $V_{\theta_{k-1}}$ of this projection over jet θ_{k-1}. Introduce the isotropy subalgebra $\widehat{\mathfrak{g}}_{\theta_{k-1}} \subset \widehat{\mathfrak{g}}^{(k)}$, which consists of the tangent vectors $\widehat{X}^{(k)}_{\theta_{k-1}}$ vanishing in point θ_{k-1}:

$$\widehat{\mathfrak{g}}_{\theta_{k-1}} = \big\{\widehat{X}^{(k)}_{\theta_{k-1}} : \widehat{X}^{(k-1)}_{\theta_{k-1}} = 0\big\}.$$

It follows from formula (1.4) that the vectors from the isotropy subalgebra $\widehat{\mathfrak{g}}_{\theta_{k-1}}$ depend on the variables $a_i := a^{(i)}(0)$, $b_j := b^{(j)}(0)$, c, where $i, j \leqslant k+2$.

Isotropy subalgebra $\widehat{\mathfrak{g}}_{\theta_{k-1}}$ acts on the fiber $V_{\theta_{k-1}}$. The number of independent differential invariants of pure order k equals the codimension of the general orbit of this action.

The dimension of the fiber $V_{\theta_{k-1}}$ equals $(k+1)$. Now let us compute the isotropy subalgebras.

Put $F_{ij} := \frac{\partial^{i+j}F}{\partial x^i \partial y^j}(0,0)$ (we recall that these numbers are the coordinates of jets θ_{k-1}). We shall prove that these algebras look as follows:

$$\widehat{\mathfrak{g}}_{\theta_0} = \big\{\big[b_3 - 5F_{10}a_1/2 - 3Fa_2/2 - F_{01}b_1 + F_{10}c\big]\partial u_{10} + \big[a_3/2 - 2F_{01}a_1\big]\partial u_{01}\big\},$$

$$\begin{aligned}\widehat{\mathfrak{g}}_{\theta_1} = \big\{&\big[b_4 - 4F_{10}a_2 - (7F_{20} + 15FF_{01})a_1/2 - 2F_{11}b_1 + (F_{20} + FF_{01})c\big]\partial u_{20}\\ &+ \big[a_4/2 - 2F_{01}a_2 - 3F_{11}a_1 - F_{02}b_1\big]\partial u_{11} - (F_{02}/2)\big[5a_1 + 2c\big]\partial u_{02}\big\},\end{aligned}$$

$$\begin{aligned}\widehat{\mathfrak{g}}_{\theta_2} = \big\{&\big[b_5 - 15(F_{20} + FF_{01})a_2/2 - (7F_{30} + 21FF_{11} + 27F_{10}F_{01})a_1\\ &\qquad - (3F_{21} + 3FF_{02} + F_{01}^2)b_1\big]\partial u_{30}\\ &+ \big[a_5/2 - 5F_{11}a_2 - (4F_{21} + 4FF_{02} + 8F_{01}^2)a_1 - 2F_{12}b_1\big]\partial u_{21}\\ &\qquad + \big[5F_{02}a_2/2 + F_{12}a_1 + F_{03}b_1\big]\partial u_{12} - \big[2F_{03}a_1\big]\partial u_{03}\big\},\end{aligned}$$

$$\widehat{\mathfrak{g}}_{\theta_3} = \Big\{\Big[b_6 - \frac{2}{5F_{02}}\big(10F_{31}F_{02} - 12F_{30}F_{03} + 32F_{10}F_{01}F_{03} - 21FF_{11}F_{03}$$

$$
\begin{aligned}
&\qquad\qquad + 15FF_{02}F_{12} + 15F_{01}F_{11}F_{02} + 35F_{10}F_{02}^2)b_1\Big]\partial u_{40}\\
&+ \Big[a_6/2 - \frac{1}{5F_{02}}\big(15F_{02}F_{22} - 18F_{03}F_{21} - 3FF_{02}F_{03}\\
&\qquad\qquad - 16F_{01}^2F_{03} + 25F_{01}F_{02}^2\big)b_1\Big]\partial u_{31}\\
&- \Big[\frac{2}{5F_{02}}(5F_{02}F_{13} - 6F_{12}F_{03})b_1\Big]\partial u_{22} - \Big[\frac{1}{5F_{02}}(5F_{02}F_{04} - 6F_{03}^2)b_1\Big]\partial u_{13}\Big\},
\end{aligned}
$$

$$\widehat{\mathfrak{g}}_{\theta_{k-1}} = \big\{\big[b_{k+2}\big]\partial u_{k,0} + \big[a_{k+2}/2\big]\partial u_{k-1,1}\big\}$$

(here $k \geqslant 5$; vector coefficients are in the square brackets).

Isotropy subalgebras $\widehat{\mathfrak{g}}_{\theta_0}, \ldots, \widehat{\mathfrak{g}}_{\theta_3}$ were found by direct computations with the help of `DETools` package of the `Maple` software. To prove the general formula for $\widehat{\mathfrak{g}}_{\theta_{k-1}}$ we use the induction by k. For $k = 5$ this formula is true. From the conditions $\widehat{X}^{(k-1)}_{\theta_{k-1}} = 0$, formula (1.1) and the induction hypothesis, we can find $a_i = b_j = c = 0$ for all i, $j \leqslant k+1$. Then we get $\frac{d^{i+j}\Phi}{dx^i dy^j}(\theta_{k-1}) = 0$ for all i, j such that $i + j \leqslant k$, except cases $(i,j) = (k,0)$ and $(i,j) = (k-1,1)$ (in the first case the derivative of function Φ contains b_{k+2}, and in the second case it contains $a_{k+2}/2$). Hence, according to formula (1.1) the vectors from isotropy subalgebras $\widehat{\mathfrak{g}}_{\theta_{k-1}}$ equal

$$
\begin{aligned}
\widehat{X}^{(k)}_{\theta_{k-1}} &= \frac{d^{k+2}\Phi}{dx^{k+2}}(\theta_{k-1})\partial u_{k,0} + \frac{d^{k+2}\Phi}{dx^{k+1}dy}(\theta_{k-1})\partial u_{k-1,1}\\
&= b_{k+2}\partial u_{k,0} + (a_{k+2}/2)\partial u_{k-1,1}.
\end{aligned}
$$

Finally, we obtain that co-dimensions of the orbits for the action of isotropy subalgebras $\widehat{\mathfrak{g}}_{\theta_{k-1}}$ on the fibers $V_{\theta_{k-1}}$ equal 0 if $k \leqslant 3$, 2 if $k = 4$ and $(k-1)$ if $k \geqslant 5$. □

Remark 1.4. The description of the singular k-jets (i.e., k-jets with non-maximal dimension of the $\widehat{\mathfrak{g}}_{\theta_{k-1}}$-orbits) follow from the proof of Theorem 1.1. Namely, the singular jets are

- 2-jets and 3-jets in set $\{u_{02} = 0\}$,
- 4-jets in sets

$$\{u_{02} = 0\}, \quad \{5u_{02}u_{04} - 6u_{03}^2 = 0\}, \quad \{5u_{02}u_{13} - 6u_{12}u_{03} = 0\},$$

- k-jets ($k \geqslant 5$), which project to the singular 4-jets.

The right hand sides of ODEs $q = F(x,y)$ in these singular cases are equal to

$$
\begin{aligned}
F(x,y) &= C_0(x) + C_1(x)y + \frac{C_2(x)}{(y + C_3(x))^3} \quad \text{or}\\
F(x,y) &= C_0(x) + C_1(x)y + \int\bigg(\int \frac{dy}{\big(C_2(x) + C_3(y)\big)^5}\bigg)dy.
\end{aligned}
$$

Also note that according to the classical result of Cartan [**9**], all second order differential equations with the right-hand side $F(x,y) = C_1(x)y + C_2(x)$ are point-equivalent.

1.3. Algebra of differential invariants. Using Theorem 1.1, we now solve the first part of the original Lie problem. Namely, we describe the algebra of differential invariants for the action (1.3) of pseudogroup $\widehat{G}_+$ on the space of smooth functions $C^\infty(\mathbb{R}^2)$.

THEOREM 1.2. *1. The algebra of differential invariants for the action of pseudogroup $\widehat{G}_+$ on the space of smooth functions $C^\infty(\mathbb{R}^2)$ is locally generated by differential invariants*

$$J := \frac{u_{02}u_{04}}{u_{03}^2},$$

$$K := \frac{u_{03}}{u_{02}^3}\Big((u_{22} + 5u_{01}u_{02} + uu_{03}) + \frac{12u_{13}u_{03}u_{12} - 6u_{04}u_{12}^2 - 5u_{13}^2u_{02}}{5u_{04}u_{02} - 6u_{03}^2}\Big)$$

of order 4 and by invariant derivations

$$\nabla_1 := \frac{u_{02}}{u_{03}} \cdot \frac{d}{dy}, \nabla_2 := \frac{\sqrt{u_{03}}}{u_{02}}\Big(\frac{d}{dx} - \frac{5u_{13}u_{02} - 6u_{12}u_{03}}{5u_{04}u_{02} - 6u_{03}^2} \cdot \frac{d}{dy}\Big).$$

This algebra locally separates the $\widehat{G}_+$-orbits of non-singular jets.

2. Derivations ∇_1 and ∇_2 have the following commutation relation:

$$[\nabla_1, \nabla_2] = \frac{5J_{01}}{6 - 5J} \cdot \nabla_1 + (J/2 - 1) \cdot \nabla_2. \tag{1.5}$$

3. Syzygies of the algebra of differential invariants are generated by a unique relation between the invariants of order 6:

$$\begin{aligned}(5J - 6)\big((J_{02} - K_{20}) + (3J - 5)K_{10} - (2J^2K - 7JK + 10J + 6K - 16)\big)\\ +\big(5J_{10}K_{10} + (3K - 25)J_{10} - 10J_{01}^2\big) = 0\end{aligned} \tag{1.6}$$

and by the commutative relation (1.5) *(here $L_{ij} := \nabla_2^j \nabla_1^i L$).*

REMARK 1.5. Differential invariant J and invariant derivation ∇_1 have a simple geometric sense. Consider an arbitrary point $x_0 \in \mathbb{R}$ in the general position. Isotropy subalgebra $\widehat{\mathfrak{g}}_{x_0}$ is the Lie subalgebra in affine algebra $\mathfrak{ga}(2)$ and it acts on the two-dimensional space with coordinates (y, u). Function J and derivation ∇_1 are the invariants for this affine action.

Unfortunately, the geometric sense of invariant K and derivation ∇_2 is not clear. They were obtained by direct computations with the help of `DETools` package of the `Maple` software.

PROOF. Statement 2 and the invariance of the objects from Statement 1 can be checked by direct computations.

Now let us prove Statements 1 and 3. We compute the number of independent differential invariants of pure order k, which are obtained from the basic invariants J and K by applying the derivations ∇_1 and ∇_2.

For $k \leqslant 3$ there are no differential invariants according to Theorem 1.1.

For $k = 4$ according to Theorem 1.1 there are only two independent differential invariants. These invariants are J and K.

For $k = 5$ according to Theorem 1.1 there exist four independent differential invariants. Let us prove that invariants J_{10}, J_{01}, K_{10} and K_{01} are independent. Indeed, denote by d_1 and d_2 symbols of the total derivations d/dx and d/dy respectively. Then the symbols of invariants J_{10}, J_{01}, K_{10} and K_{01} up to the coefficients are equal to

$$d_2^5, \quad d_2^5 + d_2^4 d_1, \quad d_2^5 + d_2^4 d_1 + d_2^3 d_1^2, \quad d_2^5 + d_2^4 d_1 + d_2^3 d_1^2 + d_2^2 d_1^3.$$

It is clear that these symbols are independent. Hence, the invariants J_{10}, J_{01}, K_{10} and K_{01} are also independent.

Now consider the case $k = 6$. According to Theorem 1.1, in this case there are five independent differential invariants. On the other hand, after the differentiation of functions J and K we get six differential invariants. Hence, there is a unique syzygy (1.6) between them (this syzygy was also found with the help of `Maple`).

Now consider the case of arbitrary $k \geqslant 7$. Let us prove that there exist $k - 1$ independent invariants of order k among invariants J_{ij}, K_{ij} (where $i + j = k - 4$).

Let us take the invariants K_{ij} (where $i + j = k - 4$), $J_{k-4,0}$ and $J_{k-3,1}$. These invariants are functionally independent, because the symbols of invariants $J_{k-4,0}$ and $J_{k-3,1}$ up to coefficients equal d_2^k and $d_2^k + d_2^{k-1} d_1$, whereas symbols of invariants K_{ij} up to coefficients equal

$$d_2^{i+2}(d_1^{j+2} + d_1^{j+1} d_2 + \cdots + d_1 d_2^{j+1} + d_2^{j+2}).$$

So we find $(k - 1)$ independent differential invariants of pure order k.

Finally, after the differentiation of syzygy relation (1.6) and using commutative relation (1.5) one can express all other invariants through K_{ij}, $J_{k-4,0}$ and $J_{k-3,1}$. Hence, all syzygies of our algebra of differential invariants are generated by the relation (1.6).

Theorem 1.2 is proved. □

Using the commutator trick introduced in [**32**], we get the following corollary.

COROLLARY 1.1. *The algebra of differential invariants for the action of pseudogroup $\widehat{G}_+$ on the space of smooth functions $C^\infty(\mathbb{R}^2)$ is locally generated by differential invariant K and invariant derivations ∇_1, ∇_2.*

1.4. Classification. Now we are ready to solve the second part of the original Lie problem and classify ODEs $q = F(x, y)$ with respect to the action of point symmetry pseudogroup $\widehat{G}_+$. However, here we have some problems because our symmetry group $\widehat{G}_+$ has an infinite dimension. So, if we get the equivalence criterion on the level of jet with arbitrary order, we can not claim that this criterion will work for smooth and even analytic functions. That's why the ideas and methods that we used for the finite-dimensional groups (see, for example, [**8**]) do not work here. We suggest another approach.

First of all, we need the following

DEFINITION 1.2. Function $F \in C^\infty(\mathbb{R}^2)$ is said to be *regular*, if the restrictions of invariants K and K_{10} on the graph L_F^4 are functionally independent and $K_{10}K_{11} - K_{01}K_{20} \neq 0$ on L_F^5.

For a regular function F one can express the restrictions of the invariants K_{01} and K_{20} through the restrictions of the invariants K and K_{10}:

$$K_{01}(F) = \mathcal{K}_{01}(K(F), K_{10}(F)), \quad K_{20}(F) = \mathcal{K}_{20}(K(F), K_{10}(F)). \tag{1.7}$$

The following theorem holds.

THEOREM 1.3. *Let F and $\widetilde{F}$ be regular smooth functions, which are analytic in variable y. Then they are locally $\widehat{G}_+$-equivalent if and only if the corresponding sets of dependencies* (1.7) *coincide.*

PROOF. If functions F and $\widetilde{F}$ are $\widehat{G}_+$-equivalent, then obviously sets (1.7) for them coincide. Let us prove the converse statement.

We will follow the ideas of Lychagin from [**28**]. Functions $K(F)$ and $K_{10}(F)$ can be chosen as coordinate functions in the neighborhood $U \subset L^5_F$ and functions $K(\widetilde{F})$ and $K_{10}(\widetilde{F})$ can be chosen as coordinate functions in the neighborhood $\widetilde{U} \subset L^5_{\widetilde{F}}$. As pseudogroup $\widehat{G}_+$ acts transitively on base $\mathbf{J}^0$, then without loss of generality one can assume that the neighborhoods U and $\widetilde{U}$ have a nonempty intersection, which consists an open subset. Hence, there exist such points a and $b \in \mathbb{R}^2$ such that 6-jets $[F]^6_a$ and $[\widetilde{F}]^6_b$ from this intersection are non-singular and have the same coordinates.

It follows from system (1.7), corollary 1.1 and regularity condition that the values of all differential invariants of the fifth order in 6-jets $[F]^6_a$ and $[\widetilde{F}]^6_b$ coincide. So, according to Theorem 1.1 these 6-jets are $\widehat{G}_+$-equivalent, i.e., there exists an element $\widehat{g}^6_{(a,b)} \in \widehat{G}^{(6)}_+$ such that $\widehat{g}^6_{(a,b)} \circ [F]^6_a = [\widetilde{F}]^6_b$.

Taking the prolongations of system (1.7) and applying the same ideas in each order $k > 6$ we get that for all k there exists an element $\widehat{g}^k_{(a,b)} \in \widehat{G}^{(k)}_+$ such that $\widehat{g}^k_{(a,b)} \circ [F]^k_a = [\widetilde{F}]^k_b$. Put $\widehat{g}^\infty_{(a,b)} := \{\widehat{g}^k_{(a,b)}\} \in \widehat{G}^{(\infty)}_+$, hence $\widehat{g}^\infty_{(a,b)} \circ [F]^\infty_a = [\widetilde{F}]^\infty_b$.

Now let us prove that there exists an element$\widehat{g} \in \widehat{G}_+$ such that $\widehat{g} \circ F = \widetilde{F}$, i.e., that

$$\begin{aligned}\frac{\sqrt{X'}^3}{C} F\big(X, C\sqrt{X'}y + A\big) - \frac{2X'X''' - 3(X'')^2}{4(X')^2} y \\ - \frac{A''X' - A'X''}{C\sqrt{X'}^3} - \widetilde{F}(x, y) = 0.\end{aligned} \tag{1.8}$$

First of all, note that constant C in (1.8) can be taken from the element $\widehat{g}^k_{(a,b)}$. Then we need to find functions X and A.

Denote by $H = H(x, y)$ the left part of equation (1.8). Let $a = (x_0, y_0) \in \mathbb{R}^2$ be such a point, that there exist point $b \in \mathbb{R}^2$ and element $\widehat{g}^\infty_{(a,b)} \in \widehat{G}^{(\infty)}_+$ such that $\widehat{g}^\infty_{(a,b)} \circ [F]^\infty_a = [\widetilde{F}]^\infty_b$. Consider the system of equations $H(x, y_0) = H_y(x, y_0) = 0$. This is a system of ordinary differential equations in the unknown functions X and A.

As infinite jets $[F]^\infty_{\widetilde{a}}$ and $[\widetilde{F}]^\infty_{\widetilde{b}}$ are $\widehat{G}^{(\infty)}_+$ (here $\widetilde{a}$ and $\widetilde{b}$ are taken from the small neighborhoods of points a and b correspondingly) this system is formally integrable in each point from a small neighborhood of point a. Hence, this system also has a smooth solution.

Now we only have to prove that this solution satisfies equation (1.8). Note that equation (1.8) is formally solvable and functions X and A are found from the relations $H(x, y_0) = H_y(x, y_0) = 0$. Then, all derivatives of function H by variable y vanish in point $y = y_0$ (because if some derivative of function H in point $\widetilde{a} = (x, y_0)$ is non-zero, then it means that infinite jets $[F]^\infty_{\widetilde{a}}$ and $[\widetilde{F}]^\infty_{\widetilde{b}}$ are not equivalent). Finally, as functions F and $\widetilde{F}$ are analytic in variable y, then function H is also analytic in variable y. Hence, $H(x, y) = 0$, because all its partial derivatives in variable y vanish in point $y = y_0$. □

2. Differential equations with algebraic right-hand sides

In this section we solve the so–called algebraic Lie problem. Namely, now we consider the ODEs $y'' = F(x, y)$ with rational right hand sides over the field $\mathbb{C}$ of complex numbers. Moreover, we also consider the symmetry group only for this class of equations. It appears that this group has only one functional parameter instead of the group $\widehat{G}_+$ from the previous section, and it is a subgroup of the plane Cremone group $\mathrm{Cr}(2)$ of birational automorphisms of $\mathbb{C}P^2$. In this section we provide the analogs of the results from the smooth case.

2.1. Action of symmetry group. First of all, we have to find the symmetry group G for the class of ODEs $q = F(x, y)$ with the rational right hand sides.

PROPOSITION 2.1. *1. Symmetry group G consists of the following transformations:*

$$x \mapsto \frac{Ax + B}{Cx + D}, \quad y \mapsto \frac{\lambda}{Cx + D} y + R(x),$$

where R is a rational function and A, B, C, D, $\lambda \in \mathbb{C}$ are arbitrary constants and $\lambda \neq 0$.

2. Lie algebra $\mathfrak{g}$ associated with G consists of the following vector fields:

$$X := (ax^2 + 2bx + c)\partial_x + ((ax + b + d)y + r(x))\partial_y,$$

where r is a rational function and a, b, c, $d \in \mathbb{C}$ are arbitrary constants.

PROOF. It follows from proposition 1.1 that transformations from our group G look as follows:

$$x \mapsto X(x), \quad y \mapsto \lambda \cdot \sqrt{|X'(x)|} y + R(x).$$

As function X is rational, then it is a linear fraction: $X(x) = \frac{Ax+B}{Cx+D}$. Finally, function R should also be rational.

Statement 2 follows from Statement 1. □

REMARK 2.1. Symmetry group G is connected. Therefore, invariants for the action of $\mathfrak{g}$ coincide with the invariants of the action of G.

Symmetry group G and Lie algebra $\mathfrak{g}$ act on the class of differential equations $\{q = F(x, y)\}$. This action induces the action on the right-hand sides of these equations. The latter is given by the vector fields

$$\widehat{X} := (ax^2 + 2bx + c)\partial_x + ((ax + b + d)y + r(x))\partial_y + (r''(x) - (3ax + 3b - d)F)\partial_F.$$

By $\widehat{\mathfrak{g}}$ we denote Lie algebra of such vector fields.

Corresponding action of G is given by the following transformations:

$$\begin{aligned} &x \mapsto \frac{Ax + B}{Cx + D}, \quad y \mapsto \frac{\lambda}{Cx + D}y + R(x), \\ &F \mapsto \frac{F}{\lambda(Cx + D)^3} - \frac{R'(x)}{\lambda} - \frac{(Cx + D)R''(x)}{\lambda}. \end{aligned} \tag{2.1}$$

We denote the group of these transformations by $\widehat{G}$.

Remark 2.2. Consider theta-function of two complex variables z and τ that is given by the series $\theta(z, \tau) = \sum_{n\in\mathbb{Z}} \exp(\pi i n^2\tau + 2\pi i n z)$. The substitutions in the variables z, τ for which θ is quasi-periodic form a group $\mathrm{SL}(2, \mathbb{Z}) \subset \widehat{G}$ which acts by transformations

$$(z, \tau) \mapsto \Big(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d}\Big).$$

This action is similar to action (2.1) of the symmetry group G on the right hand sides of the equations $q = F(x, y)$.

2.2. Rational differential invariants. In this section we introduce the so-called rational differential invariants and invariant derivations in order to describe the field of differential invariants.

Now by a *differential invariant* of order $\leqslant k$ we understand a *rational* function I on $\mathbf{J}^k$ which is invariant under the prolonged action of $\widehat{\mathfrak{g}}^{(k)}$.

One defines an *invariant derivation* as derivation $\nabla : C^\infty(\mathbf{J}^\infty) \to C^\infty(\mathbf{J}^\infty)$ which is invariant with respect to prolongations of $\mathfrak{g}$ and has *rational* coefficients.

Now let us compute the number of independent differential invariants of order k.

Theorem 2.1. *The number m_k of independent differential invariants of order k for the action of group $\widehat{G}$ on $\mathbf{J}^k$ is presented in the following table:*

order	0	1	2	3	4	...	k
m_k	0	0	0	2	4	...	k

The growth of the dimensions of the space of differential invariants is an important invariant characterizing the freedom in the equivalence problem.

Consider the Poincare series

$$P(t) = \sum_{k=0}^{\infty} m_k t^k = 2t^3 + \sum_{k=4}^{\infty} k t^k = \frac{t^3(2 - t^2)}{(1 - t)^2}.$$

Poincare function $P(t)$ appears to be rational. One of the problems in [**3**] is stated as follows: are the Poincare series of numbers of moduli in jet spaces rational functions in the majority of local problems in the analysis? For a sufficiently broad class of problems, the answer to this question is affirmative (see [**24**]).

Proof. The proof of this Theorem is similar to the proof of Theorem 1.1. We just give the description of isotropy subalgebras $\widehat{\mathfrak{g}}_{\theta_{k-1}}$ and find the singular orbits.

Isotropy algebras have the following form

$$\begin{aligned}
\widehat{\mathfrak{g}}_{\theta_0} &= \{[r_2 + (d-3b)F]\partial u\},\\
\widehat{\mathfrak{g}}_{\theta_1} &= \{[r_3 - 3Fa - F_{01}r_1 - 5F_{10}b]\partial u_{10} - [4F_{01}b]\partial u_{01}\},\\
\widehat{\mathfrak{g}}_{\theta_2} &= \big\{[r_4 - 2F_{11}r_1 - 8F_{10}a + F_{20}d + FF_{01}d]\partial u_{20}\\
&\qquad - [F_{02}r_1 + 4F_{01}a]\partial u_{11} - [F_{02}d]\partial u_{02}\big\},\\
\widehat{\mathfrak{g}}_{\theta_3} &= \Big\{\Big[r_5 - \frac{1}{4F_{01}}(4F_{01}^3 - 3FF_{01}F_{02} - 15F_{02}F_{20} + 12F_{12}F_{01})r_1\Big]\partial u_{30}\\
&\quad - \Big[\frac{1}{2F_{01}}(4F_{12}F_{01} - 5F_{02}F_{11})r_1\Big]\partial u_{21} - \Big[\frac{1}{4F_{01}}(4F_{03}F_{01} - 5F_{02}^2)r_1\Big]\partial u_{12}\Big\},\\
\widehat{\mathfrak{g}}_{\theta_{k-1}} &= \{[r_{k+2}]\partial u_{k0}\},
\end{aligned}$$

where $F_{ij} := \frac{\partial^{i+j}F}{\partial x^i \partial y^j}(0,0), r_i := r^{(i)}(0)$ and $k \geqslant 4$.

The description of singular orbits in space $\mathbf{J}^k$ follows from the explicit form of the isotropy algebras.

There is one orbit $\{F_{01} = 0\}$ of incomplete dimension in space $\mathbf{J}^1$.

There are two types of orbits of incomplete dimension in space $\mathbf{J}^2$: orbit $\{F_{02} = 0\}$ and orbits, whose projections in $\mathbf{J}^1$ are singular.

In a similar way, the orbits of incomplete dimension in space $\mathbf{J}^3$ are orbits lying in $\{4F_{12}F_{01} - 5F_{02}F_{11} = 0\}$, $\{4F_{03}F_{01} - 5F_{02}^2 = 0\}$ and orbits, whose projections in $\mathbf{J}^2$ have an incomplete dimension. All these orbits are called singular. Finally, $\widehat{G}$-orbits in spaces $\mathbf{J}^k$ are called singular, if their projections in $\mathbf{J}^3$ are singular.

Now we write down the equations that belong to singular orbits:

$$\begin{aligned}
F_{01} = 0 &\Leftrightarrow F(x,y) = a(x);\\
F_{02} = 0 &\Leftrightarrow F(x,y) = a(x)y + b(x);\\
4F_{12}F_{01} - 5F_{02}F_{11} = 0 &\Leftrightarrow F(x,y) = c_3\sqrt{2c_1x + c_2}\cdot y + a(x);\\
4F_{03}F_{01} - 5F_{02}^2 = 0 &\Leftrightarrow F(x,y) = \frac{a(x)}{(y+b(x))^3} + c(x).
\end{aligned}$$ □

2.3. Field of differential invariants. We are ready now to describe the field of differential invariants.

Theorem 2.2. 1. *The field of differential invariants for the $\widehat{G}$ action is generated by the differential invariants*

$$J := \frac{u_{01}u_{03}}{u_{02}^2},$$

$$K := \frac{1}{u_{01}^2}\cdot\Big(\frac{4u_{01}u_{12}^2 - 10u_{02}u_{11}u_{12} + 5u_{11}^2u_{03}}{4u_{03}u_{01} - 5u_{02}^2} - (uu_{02} + u_{21}\Big)$$

and by invariant derivations

$$\nabla_1 := \frac{u_{02}}{u_{03}} \cdot \frac{d}{dy},$$

$$\nabla_2 := \frac{1}{u_{02}^2} \cdot \Big(u_{13} - \frac{4u_{01}u_{12}u_{04} + 6u_{11}u_{03}^2 - 6u_{02}u_{03}u_{12} - 5u_{02}u_{11}u_{04}}{4u_{01}u_{03} - 5u_{02}^2}\Big) \cdot \Big(\frac{d}{dx} - \frac{4u_{01}u_{12} - 5u_{02}u_{11}}{4u_{01}u_{03} - 5u_{02}^2} \cdot \frac{d}{dy}\Big).$$

2. *Derivations ∇_1 and ∇_2 satisfy the following commutation relation:*

$$[\nabla_1, \nabla_2] = \frac{5J_{01}}{J(5-4J)}\nabla_1 + \frac{J_{11}J - J_{01}}{2J_{01}J}\nabla_2 \tag{2.2}$$

3. *Syzygies between the invariants are generated by one relation between the invariants of order 5:*

$$(4J-5)\big(J_{02} + 2J^2J_{01}K_{20} - 2(J-3)JK_{10}J_{01} - 2(KJ - 2K - 1)J_{01}\big) - \big((10K_{10} + 6K + 8)JJ_{10} - 16J_{01}\big)J_{01} = 0$$

and also by (2.2) *(where $L_{ij} := \nabla_2^j\nabla_1^i L$).*

Remark 2.3. Commutative relation (2.2) is more complicated than relation (1.5) in the smooth case, and the commutator trick from [**32**] does not work.

Proof. The proof of this Theorem is similar to the proof of Theorem 1.2. We just need to show that each rational differential invariant can be represented as a rational function in basic invariants J and K and their invariant derivations.

We have $m_k = k$ independent invariants in the field of differential invariants of order k. Thus, the transcendence degree of this field is equal to k. This implies that the field of differential invariants of order k is generated by K_{ij} for $i+j = k-3$ and $J_{k-3,0}, J_{k-4,1}$ due to the corollary of the Rosenlicht theorem (see [**38**]). $\square$

2.4. Classification theorem. In this subsection we solve the equivalence problem for the ODEs of the form $y'' = F(x,y)$ such that F is a rational function.

Equations $y'' = F_1(x,y)$ and $y'' = F_2(x,y)$ are said to be $\widehat{G}$*-equivalent* if there exists such an element $g \in \widehat{G}$ that $g \circ F_1 = F_2$. Note that this equation can be interpreted as a differential equation in function $R(x)$ (see (2.1))

$$\frac{1}{\lambda(Cx+D)^3}F_1\Big(\frac{Ax+B}{Cx+D}, \frac{\lambda}{Cx+D}y + R(x)\Big) - \frac{R'(x)}{\lambda} + \frac{(Cx+D)R''(x)}{\lambda} = F_2(x,y).$$

Consider $J(F), K(F), J_{10}(F), J_{01}(F), K_{10}(F), K_{01}(F)$, where $I(F) = I|_{L_F^4}$ is a restriction of invariant I on the graph $L_F^4 \subset \mathbf{J}^4$ of F. Thus, any rational function $F(x,y)$ defines a rational morphism

$$\pi_F\colon \mathbb{C}^2 \to \mathbb{C}^6, \quad \pi_F\colon a \mapsto \big(J([F]_a^4), K([F]_a^4), \dots, K_{01}([F]_a^4)\big).$$

Denote by $\mathcal{D}_F$ a set of dependencies between rational functions $J(F), K(F), \dots,$ $K_{01}(F)$ and by Σ_F a closure of the image of morphism π_F in the Zariski topology. Note that $\mathcal{D}_F$ is zero ideal of Σ_F.

We say that F is *regular* if invariants $J(F), K(F)$ and the restrictions of invariant derivations ∇_1, ∇_2 on the graph $L_F^4 \subset \mathbf{J}^4$ of F are well defined.

THEOREM 2.3. *1. Equations $y'' = F_1(x,y)$ and $y'' = F_2(x,y)$ with regular right-hand sides are $\widehat{G}$-equivalent if and only if $\Sigma_{F_1} = \Sigma_{F_2}$.*

2. Equations $y'' = F_1(x,y)$ and $y'' = F_2(x,y)$ with regular right-hand sides are $\widehat{G}$-equivalent if and only if $\mathcal{D}_{F_1} = \mathcal{D}_{F_2}$.

PROOF. If F_1 and F_2 are $\widehat{G}$-equivalent, then obviously $\mathcal{D}_{F_1}$ and $\mathcal{D}_{F_2}$ coincide. The same is true for Σ_{F_1} and Σ_{F_2}.

Let us prove the converse statements. Assume that for two functions F_1 and F_2 one has $\Sigma_{F_1} = \Sigma_{F_2} = \Sigma$. Following [**8**] we can say that $\mathcal{D}_F$ defines Σ_F. Then $\mathcal{D} = \mathcal{D}_{F_1} = \mathcal{D}_{F_2} \Leftrightarrow \Sigma_{F_1} = \Sigma_{F_2}$.

For any generic point $a_1 \in \mathbb{C}^2$ there exists a point $a_2 \in \mathbb{C}^2$ such that $\pi_{F_1}(a_1) = \pi_{F_2}(a_2)$. Hence, the values of basic differential invariants of order 4 in 4-jets $[F_1]^4_{a_1}$ and $[F_1]^4_{a_2}$ coincide. It follows from Theorem 2.2 that there exists $g^4_{(a_1,a_2)} \in \widehat{G}^{(4)}_{(a_1,a_2)}$ such that $g^4_{(a_1,a_2)} \circ [F_1]^4_{a_1} = [F_2]^4_{a_2}$, where $\widehat{G}^{(4)}_{(a_1,a_2)} \subset \widehat{G}^{(n)}$ is a subgroup of diffeomorphisms taking a_1 to a_2.

The values of basic differential invariants of order 4 in 4-jets $[F_1]^4_{a_1}$ and $[F_1]^4_{a_2}$ coincide in all generic points $a_1 \in \mathbb{C}^2$. Consider the elements of the ideal $\mathcal{D}$ and their derivatives. Invariants of order 5 are included linearly in the obtained expressions. Therefore values of these differential invariants in 5-jets $[F_1]^5_{a_1}$ and $[F_1]^5_{a_2}$ coincide in all generic points $a_1 \in \mathbb{C}^2$. In a similar way differential invariants of order k coincide for $[F_1]^k_{a_1}$ and $[F_2]^k_{a_2}$. As these invariants separate non-singular orbits, then, according to Theorem 2.2, we get $g^k_{(a_1,a_2)} \in \widehat{G}^{(k)}_{(a_1,a_2)}$ such that $g^k_{(a_1,a_2)} \circ [F_1]^k_{a_1} = [F_2]^k_{a_2}$. Then for any generic point $a_1 \in \mathbb{C}^2$ there exists $g^\infty_{(a_1,a_2)} = \{g^k_{(a_1,a_2)}\} \in \widehat{G}^{(\infty)}_{(a_1,a_2)}$ such that $g^\infty_{(a_1,a_2)} \circ [F_1]^\infty_{a_1} = [F_2]^\infty_{a_2}$.

Now we construct $g \in \widehat{G}$ such that $g \circ F_1 = F_2$. Following Lychagin [**28**], we consider the equation

$$H(x,y) = g \circ F_1 - F_2 = 0.$$

Also consider equation $H(x, y_0) = 0$, where $(x_0, y_0) = a_1$ is a generic point in $\mathbb{C}^2$ and $x \in U(x_0)$ for some neighborhood of x_0. Constants A, B, C, D can be found from the explicit form of g^∞. So it takes only to find $R(x)$ to establish required $g \in \widehat{G}$. Condition $H(x, y_0) = 0$ is a differential equation on $R(x)$. So it has a solution that gives us element $g \in \widehat{G}$ such that $[g]^0_{a_1} \circ [F_1]^0_{a_1} = [F_2]^0_{g(a_1)}$ for any generic point $a_1 \in \mathbb{C}^2$.

Now we prove that the constructed element $g \in \widehat{G}$ transforms F_1 into F_2, i.e., $H(x,y) = 0$. Since $[F_1]^k_{a_1}$ and $[F_2]^k_{g(a_1)}$ are $\widehat{G}^k$-equivalent, it follows that $\partial^k_y H(x, y_0) = 0$ for every k and any generic point $a_1 \in \mathbb{C}^2$. It implies that $H(x,y) = 0$ if H is holomorphic.

Finally, we claim that $R(x)$ is rational. $\widehat{G}$-action on $F(x,y)$ is linear in the variable y (see (2.1)). So function $R(x)$ can be found explicitly by equating the coefficients of y^{n-1} for the denominators of $g \circ F_1$ and F_2. $\square$

2.5. Examples. Now let us give some examples of dependencies between the invariants. Due to the computational complexity it is difficult to describe the set $\mathcal{D}_F$, even if the function F is a polynomial. So we calculate the dependencies only between some of the invariants. Let us put $j := J(F)$, $k := K(F)$, $j_1 := J_{10}(F)$, $k_1 := K_{10}(F)$, $j_2 := J_{01}(F)$ and $k_2 := K_{01}(F)$.

Consider the function $F = x^2y - x + y^3$.

1. Dependence between j and j_1: $j = \frac{1}{2}(1 - j_1)$.
2. Dependence between j, k and k_1:

$$\begin{aligned} 576k_1^2 j^{11} + \big((-1152 - 576k_1)k - 1440k_1^2 + 3456k_1\big)j^{10} \\ +\big(144k^2 + (-720k_1 + 8928)k - 16800k_1 + 3456 + 900k_1^2\big)j^9 \\ +\big(720k^2 + (-33000 + 1800k_1)k + 28200k_1 - 20064\big)j^8 \\ +\big(900k^2 + 58890k - 21450k_1 + 39784\big)j^7 + \big(-33690 - 45525k + 8025k_1\big)j^6 \\ +\big(-1125k_1 + 16575k + 13585\big)j^5 + \big(-2325 - 2250k\big)j^4 + 100\,j^3 = 0. \end{aligned}$$

3. Dependence between j, k and j_2.

$$\begin{aligned} 288j^{10}k^2 + (-288k^2 + 576k)j^9 + (288 + 72k^2 + (-768 - 192j_2)k)j^8 \\ +((336 + 336j_2)k - 480 - 384j_2)j^7 + ((-120j_2 - 48)k + 296 + 32j_2^2 + 1072j_2)j^6 \\ + (-80j_2^2 - 1060j_2 - 80)j^5 + (8 + 50j_2^2 + 460j_2)j^4 - 75j_2j^3 = 0. \end{aligned}$$

4. Dependence between j, k and k_2.

$$\begin{aligned} 2654208k^4j^{16} + (-1327104k_2k^2 - 14266368k^4 + 15178752k^3)j^{15} \\ +(24976512k^4 - 83849472k^3 + (34587648 + 6884352k_2)k^2 \\ -8460288kk_2 + 165888k_2^2)j^{14} \\ +(-15603840k^4 + 160519104k^3 + (-202818816 - 12441600k_2)k^2 \\ +(34255872 + 54134784k_2)k - 10450944k_2 - 829440k_2^2)j^{13} \\ +(3175200k^4 - 127219248k^3 + (442974528 + 9201600k_2)k^2 \\ +(-214472448 - 136926720k_2)k + 76806144k_2 + 1555200k_2^2 + 12192768)j^{12} \\ +(41576760k^3 + (-2268000k_2 - 461963952)k^2 + (527356224 + 176126400k_2)k \\ -1296000k_2^2 - 235915776k_2 - 81236736)j^{11} \\ +(-2986200k^3 + 254684088k^2 + (-670334928 - 122142600k_2)k \\ +405000k_2^2 + 397399680k_2 + 220477248)j^{10} \\ +(-648000k^3 - 75315960k^2 + (43591500k_2 + 495200712)k \\ -323744976 - 403607880k_2)j^9 \end{aligned}$$

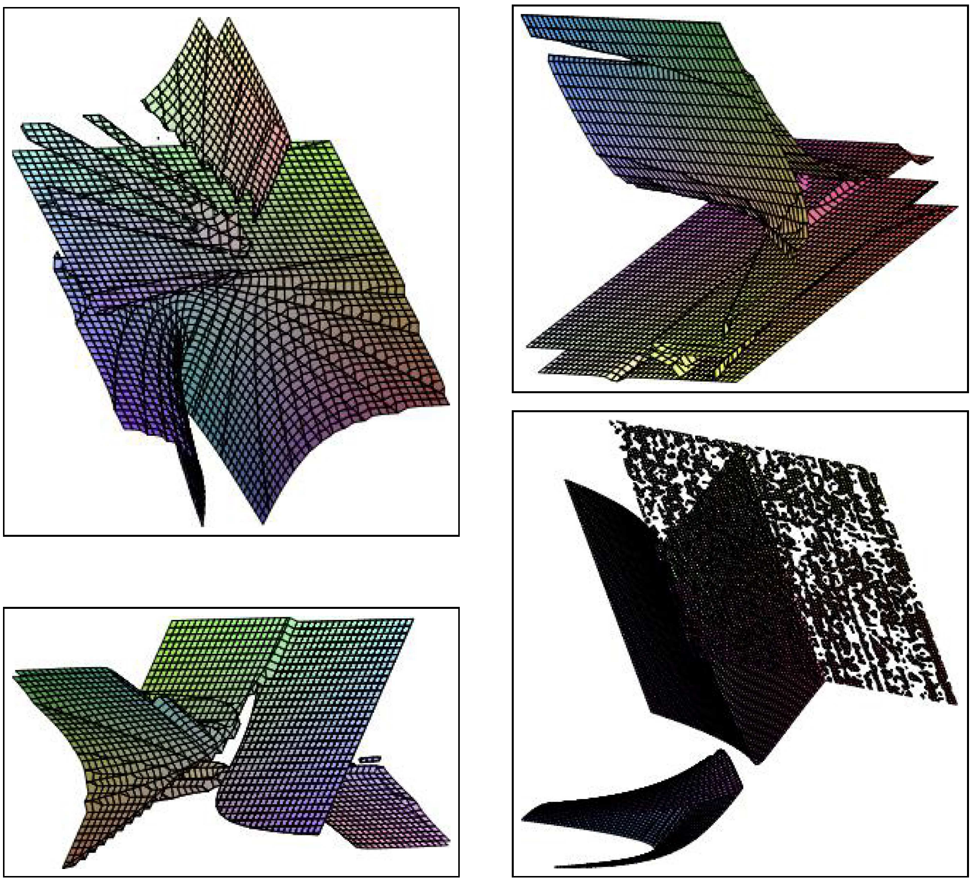

FIGURE 1. The graph defined relationship between:
j, k, k_1 (left-up), j, k, j_2 (right-up),
j, k, k_2 (left-down), j_1, k, j_2 (right-down)

$$+(11140200k^2 + (-6277500k_2 - 223108920)k + 254698200k_2 + 289396520)j^8$$
$$+(-166143160 - 98275500k_2 - 648000k^2 + 61149000k)j^7$$
$$+(-9493800k + 21380625k_2 + 62041680)j^6$$
$$+(-14657800 + 648000k - 2025000k_2)j^5 + 1998200j^4 - 120000j^3 = 0.$$

Now consider the function $F = \dfrac{x^2 + y^2}{x - y}$.

1. Dependence between j and j_1: $j = \frac{3}{2}(1 - j_1)$.
2. Dependence between j, k and j_2:

$$2048j^{14}k^2 + (-21504k^2 - 12288k)j^{13} + (96768k^2 + 165888k + 18432)j^{12}$$

$$
\begin{aligned}
&+ (-248832 - 241920k^2 + (-9216j_2 - 967680)k)j^{11} \\
&+ (362880k^2 + (80640j_2 + 3193344)k + 27648j_2 + 1534464)j^{10} \\
&+ (-326592k^2 + (-293760j_2 - 6531840)k - 324864j_2 - 5660928)j^9 \\
&+ (163296k^2 + (570240j_2 + 8491392)k + 10368j_2^2 + 1607040j_2 + 13716864)j^8 \\
&+ (-34992k^2 + (-622080j_2 - 6858432)k - 72576j_2^2 - 4354560j_2 - 22534848)j^7 \\
&+ ((361584j_2 + 3149280)k + 202824j_2^2 + 6998400j_2 + 24984288)j^6 \\
&+ ((-87480j_2 - 629856)k - 282852j_2^2 - 6683472j_2 - 17950896)j^5 \\
&+ (196830j_2^2 + 3516696j_2 + 7558272)j^4 + (-54675j_2^2 - 787320j_2 - 1417176)j^3 = 0.
\end{aligned}
$$

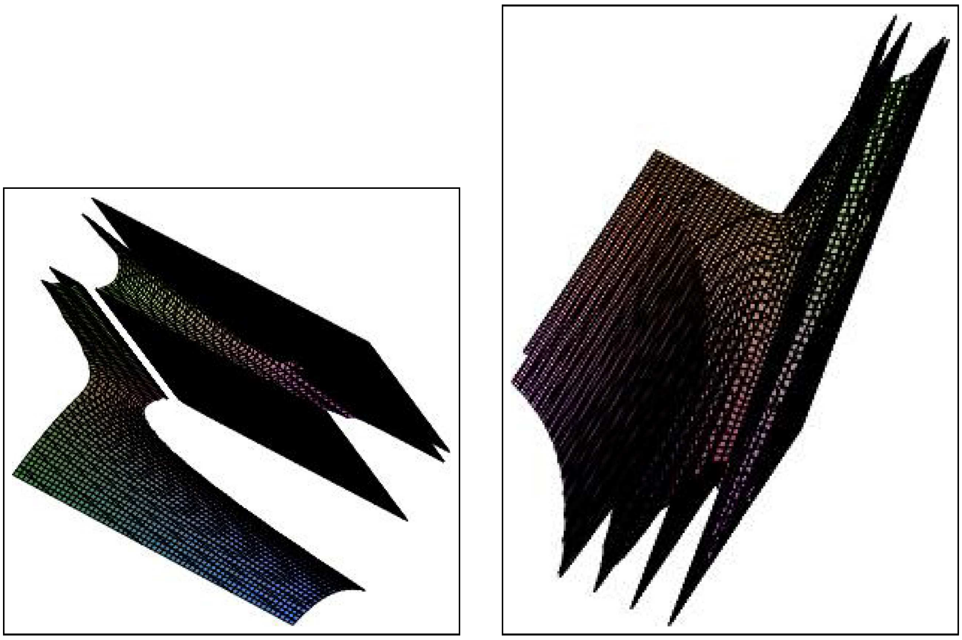

FIGURE 2. The graph defined relationship between: j, k, j_2 (left), j_1, k, j_2 (right)

3. Dependence between j_1, k and j_2

$$
\begin{aligned}
&81j_1^{14}k^2 + (-567k^2 + 324k)j_1^{13} + (1701k^2 - 1296k + 324)j_1^{12} \\
&+(-972 - 2835k^2 + (108j_2 + 1620)k)j_1^{11} \\
&+(2835k^2 - 558kj_2 + 216j_2 + 1296)j_1^{10} \\
&+(-1701k^2 + (1170j_2 - 1620)k - 468j_2 - 1296)j_1^9 \\
&+(567k^2 + (-1260j_2 + 1296)k + 36j_2^2 + 72j_2 + 972)j_1^8 \\
&+(-81k^2 + (720j_2 - 324)k - 120j_2^2 + 432j_2 - 324)j_1^7
\end{aligned}
$$

$$+(-198kj_2 + 145j_2^2 - 288j_2)j_1^6$$
$$+(18kj_2 - 75j_2^2 + 36j_2)j_1^5 + 15j_2^2 j_1^4 - j_1^3 j_2^2 = 0$$

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