PORTUGALIAE MATHEMATICA
Vol. 51 Fasc. 3 – 1994

NON LOCAL SOLUTIONS OF A NONLINEAR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATION

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Abstract: In this work we prove that the mixed problem for a temporally nonlinear Kirchhoff-Carrier model, for vibrations of a nonhomogeneous stretched string, has unique nonlocal solution for small data. The solution is obtained in S.L. Sobolev spaces.

Introduction

The nonlinear model of Kirchhoff-Carrier, cf. Carrier [5], for vibrations of an elastic string, of lenght L, is given by:

$$\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_o}{\rho.h} + \frac{E}{2L\rho}\int_0^L \left|\frac{\partial u}{\partial s}(s,t)\right|^2 ds\right)\frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

where $0 \le x \le L$ and $t > 0$ represent the string in repose, $u(x,t)$ is the vertical displacement of the point x at the instant t, ρ is the mass density, h is the area of the cross section of the string, L is the lenght of the string, P_o the initial tension on the string and E the Young's modulus of the material.

The natural generalization of the model (1) is given by the following nonlinear mixed problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\Big(\displaystyle\sum_{i=1}^n \int_\Omega \left|\frac{\partial u}{\partial x_i}\right|^2 dx\Big)\Delta u = f \text{ on } \mathbf{Q} = \Omega \times (0,T) \\ u = 0 \text{ on } \Sigma = \Gamma \times (0,T) \\ u(x,0) = \phi_o(x) \text{ on } \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = \phi_1(x) \text{ on } \Omega \end{cases} \tag{2}$$

Received: December 15, 1992.

where Ω is a bounded open set of $\mathbf{R}^n$ with smooth boundary Γ, $M\colon [0,\infty) \to \mathbf{R}$ is a positive real function and $\Delta = \sum\limits_{i=1}^{n} \dfrac{\partial^2}{\partial x_i^2}$ is the Laplace operator.

Remark 1. In the Kirchhoff-Carrier model (1), $M\colon [0,\infty) \to \mathbf{R}$ is $M(\lambda) = \dfrac{P_o}{\rho.h} + \dfrac{E}{2L\rho}\lambda$.

Several authors have investigated the nonlinear problem (2). When $n = 1$ and $\Omega = (0, L)$, it was studied by Dickey [8] and Bernstein [3] whom considered ϕ_o and ϕ_1 analytic functions with some growth conditions. Assuming Ω bounded open set of $\mathbf{R}^n$, ϕ_o and ϕ_1 analytic functions, Pohozaev [18] obtained existence and uniqueness of global solutions for the mixed problem (2). In Lions [12] he formulated the Pohozaev's results in an abstract context obtaining better results and presenting a collection of problems. One of the problems proposed by Lions [12] was the study of the problem (2) with $M\colon \Omega \times [0,\infty) \to \mathbf{R}$, i.e., the problem

$$(3)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\Big(x, \displaystyle\sum_{i=1}^{n} \int_\Omega \Big|\dfrac{\partial u}{\partial x_i}\Big|^2 dx\Big)\Delta u = f \text{ on } \mathbf{Q} \\ u = 0 \text{ on } \Sigma \\ u(x,0) = \phi_o(x) \text{ on } \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = \phi_1(x) \text{ on } \Omega \end{cases}$$

that is, for nonhomogeneous materials. This case has it's origin in the model (1) when the physic elements ρ, h and E are not constants, but depends on the point x in the string. In Rivera Rodrigues [20] the author proved the existence and uniqueness of local solutions for the problem (3).

In a more general context it is correct to consider ρ, h and E changing not only with the point x in the string but with the instant t too, i.e., $\rho = \rho(x,t)$, $h = h(x,t)$ and $E = E(x,t)$. In this case, we have the problem

$$(4)\qquad \begin{cases} \dfrac{\partial^2 u}{\partial t^2} - M\Big(x, t, \displaystyle\sum_{i=1}^{n} \int_\Omega \Big|\dfrac{\partial u}{\partial x_i}\Big|^2 dx\Big)\Delta u = f \text{ on } \mathbf{Q} \\ u = 0 \text{ on } \Sigma \\ u(x,0) = \phi_o(x) \text{ on } \Omega \\ \dfrac{\partial u}{\partial t}(x,0) = \phi_1(x) \text{ on } \Omega \end{cases}$$

where $M\colon \Omega \times [0,T] \times [0,\infty) \to \mathbf{R}$.

In this work we study the problem (4) and making use of the same technique used by Rivera Rodrigues [20], we prove that if ϕ_o, ϕ_1, f and $\dfrac{\partial M}{\partial t}$ are small in some sense, then exist one, and only one, nonlocal solution for the problem (4). It's important to observe that it's a good assumption to consider $\dfrac{\partial M}{\partial t}$ small, because in normal conditions ρ, h and E have a small variation with the time.

For the study of problem (2) with dissipative terms we have, for instance, Brito [4] and Medeiros-Milla Miranda [14]. The problem (2) in the degenerate case can be find in Arosio-Spagnolo [1], Ebihara-Medeiros-Milla Miranda [9], Arosio-Garavaldi [2], Crippa [6], Yamada [21], Nishihara-Yamada [17] and Nishihara [16].

The plan of this paper is the following:

1) Notations and preliminary results;

2) Assumptions and statement of the principal result;

3) Galerkin's approximation and a priori estimates;

4) Proof of the theorem;

5) Uniqueness.

1 – Notation and preliminary results

Let Ω be a bounded open set of $\mathbf{R}^n$ with smooth boundary Γ. By $L^2(\Omega)$ we represent the usual space of Lebesgue square integrable functions on Ω whose inner product and norm will be denoted by $(\cdot\,,\cdot)$ and $|\cdot|$ respectively. In the Sobolev space $H^1_o(\Omega)$ we consider the norm

$$||u||^2 = \sum_{i=1}^{n} \int_\Omega \left|\frac{\partial u}{\partial x_i}(x)\right|^2 dx \tag{5}$$

and inner product

$$((u,v)) = \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i}\,\frac{\partial v}{\partial x_i}\,dx\,. \tag{6}$$

Let $(-\Delta)$ be the operator defined by $\{H^1_o(\Omega), L^2(\Omega), ((\cdot\,,\cdot))\}$. Then as we well known $(-\Delta)$ is an unbounded selfadjoint operator in $L^2(\Omega)$ with domain

$$D(-\Delta) = \Big\{u \in H^1_o(\Omega); \Delta u \in L^2(\Omega)\Big\} = H^1_o(\Omega) \cap H^2(\Omega) \tag{7}$$

and it has the following properties:

(a) There exist $m_o > 0$ such that

(8) $$(-\Delta u, u) \geq m_o\,|u|^2\,,\ \forall\, u \in D(-\Delta)\ ;$$

(b)

(9) $$(-\Delta u, u) = ||u||^2\,,\ \forall\, u \in D(-\Delta)\ ;$$

(c) There exist a sequence $(\lambda_j)_{j\in\mathbf{N}}$ of real numbers and $(w_j)_{j\in\mathbf{N}}$ a sequence of $L^2(\Omega)$ vectors such that

(10) $$m_o \leq \lambda_1 \leq \lambda_2 \leq \ldots$$

(11) $$-\Delta w_j = \lambda_j w_j\,,\ \forall\, j \in \mathbf{N}$$

(12) $$\lim_{j\to\infty} \lambda_j = \infty$$

(13) $\{w_j\}$ is a orthonormal complete set in $L^2(\Omega)$ and orthogonal complete set in $H^1_o(\Omega)$ and in $H^1_o(\Omega) \cap H^2(\Omega)$.

Remark 2. We introduce the equivalent norm

(14) $$||u||_{H^1_o(\Omega)\cap H^2(\Omega)} = |-\Delta u|, \qquad \forall\, u \in H^1_o(\Omega) \cap H^2(\Omega)$$

for smooth boundary Γ.

In order to complete this section we introduce a compactness result. It is a version of Arzela's theorem and it's proof follows the same argument as the usual proof of scalar Arzela's theorem.

Lemma 1. *Let E and F be Banach spaces, $E \hookrightarrow F$ with compact injection. Let $(\sigma_m)_{m\in\mathbf{N}}$ be a sequence of functions from the interval $[a,b] \subset \mathbf{R}$ into E. If $(\sigma_m)_{m\in\mathbf{N}}$ is uniformly bounded in $[a,b]$ with respect to the norm of E and equicontinuous with respect to the norm of F, then there exist a subsequence $(\sigma_{m_\nu})_{\nu\in\mathbf{N}}$ of $(\sigma_m)_{m\in\mathbf{N}}$ and a continuous function $\sigma\colon [a,b] \to F$ such that*

(15) $$\lim_{\nu\to\infty} \sigma_{m_\nu}(t) = \sigma(t) \text{ in } F \text{ uniformly for } t \in [a,b]\ .$$

Moreover, if E is a reflexive Banach space then we find that $\sigma \in L^\infty(a,b;E)$.

2 – Assumptions and principal result

Let Ω be as in section 1, $T > 0$ a real number. We consider a real function

$$M\colon\ \Omega \times [0,T] \times [0,\infty) \longrightarrow \mathbf{R}, \qquad (x,t,\lambda) \longmapsto M(x,t,\lambda)$$

such that the following assumptions are satisfied:

(H.1) $M \in L^\infty_{\rm loc}([0,\infty); W^{1,\infty}(\Omega \times (0,T)))$, i.e., for each $k > 0$ we have $M \in L^\infty(\Omega \times (0,T) \times (0,k))$, $\dfrac{\partial M}{\partial t} \in L^\infty(\Omega \times (0,T) \times (0,k))$ and $\dfrac{\partial M}{\partial x_i} \in L^\infty(\Omega \times (0,T) \times (0,k))$ for $i = 1,\ldots,n$.

(H.2) For each $L > 0$ we have $\dfrac{\partial M}{\partial \lambda} \in L^\infty(\Omega \times (0,T) \times (0,L))$.

(H.3) There exist a real number $m_1 > 0$ such that $m_1 \le M(x,t,\lambda)$, $\forall\, x \in \Omega$, $t \in [0,T]$ and $\lambda \ge 0$.

Now we define

$$\begin{aligned}
&k_o = 4(m_o m_1^3)^{-1/2}, \qquad k_1 = \frac{1}{m_1}\\
&\theta_o = \operatorname*{ess\,sup}_{\substack{\vec{x}\in\Omega\\ 0<t<T}} \left|\frac{\partial M}{\partial t}(x,t,0)\right|\\
&k_2 = \frac{1}{2}\Big[1 + ||M||_{L^\infty(\Omega\times(0,T)\times(0,1))}\Big]\\
&k_3 = \frac{4}{m_o m_1}\Big[\Big(k_2 + \frac{T}{2}\Big)\Big(1 + e^{(1+k_1\theta_o)T}\Big)\Big]\\
&k_4 = \left\|\frac{\partial M}{\partial \lambda}\right\|_{L^\infty(\Omega\times(0,T)\times(0,k_3))}
\end{aligned} \tag{16}$$

$$\delta = \min\left\{1; m_o^{1/2}; \frac{\ln 2}{3T[1 + Tk_ok_4 + Tk_ok_4\, e^{(1+k_1\theta_o)T}]}; \left[\frac{\ln\, 2}{6Tk_ok_2k_4(1 + e^{(1+k_1\theta_o)T})}\right]^{1/2}\right\} \tag{17}$$

$$k_\delta = k_2\delta^2 + \frac{T}{2}\,\delta\ . \tag{18}$$

Theorem. *Let $M\colon \Omega\times[0,T]\times[0,\infty) \to \mathbf{R}$ be a real function satisfying (H.1)–(H.3), $\phi_o \in H^1_o(\Omega)\cap H^2(\Omega)$, $\phi_1 \in H^1_o(\Omega)$ and $f\colon [0,T] \to H^1_o(\Omega)$ a continuous* *function. If*

(19) $$|\Delta\phi_o|^2 + ||\phi_1||^2 + 0 \le t \le T \to \text{Máx}\, ||f(t)||^2 \le \delta^2$$

and

(20) $$\left\| \frac{\partial M}{\partial t} \right\|_{L^\infty(\Omega\times(0,T)\times(0,k_3))} \le \frac{\ln 2}{3Tk_1} \ .$$

Then there exist one, and only one, function $u\colon [0,T] \to H^1_o(\Omega)$ *such that*

(21) $$u \in C([0,T]; H^1_o(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \cap C^2([0,T]lH^{-1}(\Omega)) \ ,$$

(22) $$\begin{cases} u \in L^\infty(0,T; H^1_o(\Omega) \cap H^2(\Omega)) \\ u' \in L^\infty(0,T; H^1_o(\Omega)) \\ u'' \in L^\infty(0,T; L^2(\Omega)) \ , \end{cases}$$

(23) $$\begin{cases} u''(t) - M(t, ||u(t)||^2)\, \Delta u(t) = f(t) \ \textit{in} \ L^2(\Omega), \ \ 0 \le t \le T \\ u(0) = \phi_o \\ u'(0) = \phi_1 \ . \end{cases}$$

Remark 3. In $(23)_1$ we are making use of the following notation: if $\psi\colon \Omega \times (0,T) \to \mathbf{R}$ is a function then $\psi(t)\colon \Omega \to \mathbf{R}$ is defined by $\psi(t)(x) = \psi(x,t)$.

3 – Galerkin's approximation and a priori estimates

We consider $V_o = \{0\}$ and $V_m = [w_1, \ldots, w_m]$ for $m = 1, 2, \ldots$ i.e., V_m is the vector space spanned by $w_1, \ldots, w_m$; where $(w_m)_{m\in\mathbf{N}}$ is as in the section 1. The sequence of Galerkin's approximation is defined by induction as follows: we put

$$\begin{array}{rcl} u_o\colon\ [0,T] & \longrightarrow & V_o \\ t & \longmapsto & u_o(t) = 0 \end{array}$$

and for $m = 1, 2, \ldots$, we consider

$$\begin{array}{rcl} u_m\colon\ [0,T_m] & \longrightarrow & V_m \\ t & \longmapsto & u_m(t) = \sum_{j=1}^m g_{jm}(t) w_j \end{array}$$

the unique solution of the initial value problem, with the coefficient of $-\Delta u_m(t)$ depends on the time t:

(24) $$\begin{cases} u''_m(t) - M(t, ||u_{m-1}(t)||^2)\, \Delta u_m(t) = f_m(t) \ \text{in} \ V_m, \ \ \forall\, t \in [0,T_m] \\ u_m(0) = \varphi_{om} \\ u'_m(0) = \varphi_{1m} \end{cases}$$

where

$$T_m = \sup\Big\{\tau; 0 < \tau \le T_{m-1} \text{ and } u_m\colon [0,\tau] \to V_m \text{ is solution of (24)}\Big\} \ , \tag{25}$$

$$f_m(t) = \sum_{j=1}^{m} (f(t), w_j)\, w_j, \qquad 0 \le t \le T \ , \tag{26}$$

$$\varphi_{om} = \sum_{j=1}^{m} (\phi_o, w_j)\, w_j \ , \tag{27}$$

$$\varphi_{1m} = \sum_{j=1}^{m} (\phi_1, w_j)\, w_j \ . \tag{28}$$

Remark 4. The Galerkin's approximation is well defined. It's sufficient we note that the initial value problem (24) is equivalent to the following system of ordinary differential equations:

$$\begin{cases} g''_{jm}(t) + \displaystyle\sum_{k=1}^{m} \lambda_k g_{km}(t) \Big(M(t, ||u_{m-1}(t)||^2) w_k, w_j\Big) = (f(t), w_j) \\ \qquad\qquad\qquad\qquad 0 \le t \le T_m;\ j = 1, \ldots, m \\ g_{jm}(0) = (\phi_o, w_j) \\ g'_{jm}(0) = (\phi_1, w_j) \ . \end{cases} \tag{29}$$

Estimate (i) From $(24)_1$ we have the approximate equation

$$(u''_m(t), v) - \Big(M(t, ||u_{m-1}(t)||^2\Big) \Delta u_m(t), v\Big) = (f_m(t), v), \quad \forall\, v \in V_m \ . \tag{30}$$

Take $v = -\Delta u'_m(t)$ in (30) we get

$$\frac{1}{2}\frac{d}{dt} ||u'_m(t)||^2 + \int_\Omega M(x, t, ||u_{m-1}||^2)\, \Delta u_m(x,t).\Delta u'_m(x,t)\, dx = ((f_m(t), u'_m(t))),$$

since

$$\begin{aligned} &\int_\Omega M\Big(x, t, ||u_{m-1}(t)||^2\Big)\, \Delta u_m(x,t)\, \Delta u'_m(x,t)\, dx = \\ &\qquad = \frac{1}{2}\frac{d}{dt}\Big(M(t, ||u_{m-1}(t)||^2) \Delta u_m(t), \Delta u_m(t)\Big) \\ &\qquad - \frac{1}{2}\int_\Omega \frac{\partial M}{\partial t}\, (x, t, ||u_{m-1}(t)||^2)\, (\Delta u_m(x,t))^2\, dx \\ &\qquad - ((u_{m-1}(t), u'_{m-1}(t))) \int_\Omega \frac{\partial M}{\partial \lambda}(x, t, ||u_{m-1}(t)||^2)\, (\Delta u_m(x,t))^2\, dx \end{aligned}$$

we have

$$\frac{d}{dt}\left\{\frac{1}{2}\Big[||u_m'(t)||^2 + \Big(M(t,||u_{m-1}(t)||^2)\Delta u_m(t), \Delta u_m(t)\Big)\Big]\right\} = \tag{31}$$

$$= ((f_m(t), u_m'(t))) + \frac{1}{2}\int_\Omega \frac{\partial M}{\partial t}(x,t,||u_{m-1}(t)||^2)\,(\Delta u_m(x,t))^2\,dx$$

$$+ ((u_{m-1}(t), u_{m-1}'(t)))\int_\Omega \frac{\partial M}{\partial \lambda}(x,t,||u_{m-1}(t)||^2)\,(\Delta u_m(x,t))^2\,dx,$$

$$\forall\, t \in [0, T_m],\ m = 1, 2, \ldots$$

Lemma 2. *Let be*

$$\begin{cases} Z_o(t) = 0 \\ Z_m(t) = \dfrac{1}{2}\Big[||u_m(t)||^2 + \Big(M(t,||u_{m-1}(t)||^2)\Delta u_m(t), \Delta u_m(t)\Big)\Big] \\ \qquad\qquad 0 \le t \le T_m,\ m = 1, 2, \ldots\ , \end{cases} \tag{32}$$

$$\alpha = \sup_{0\le t\le T_m} Z_m(t)\,,\ \alpha_m' = \frac{2}{m_o m_1}\,\alpha_m\ ,$$

$$\theta_m = \left\|\frac{\partial M}{\partial t}\right\|_{L^\infty(\Omega\times(0,T)\times(0,\alpha_m'))}, \qquad \beta_m = \left\|\frac{\partial M}{\partial \lambda}\right\|_{L^\infty(\Omega\times(0,T)\times(0,\alpha_m'))}.$$

Then, $T_m = T$, α_m is finite $\forall\, m \in \mathbf{N}$ and

$$Z_m(t) \le \left[Z_m(0) + \frac{1}{2\delta}\int_0^t ||f_m(s)||^2\,ds\right] e^{(\delta + k_1\theta_{m-1} + k_o\alpha_{m-1}\beta_{m-1})t}. \tag{33}$$

Proof: The proof will be done by induction on m. Clearly the solution of the problem

$$\begin{cases} g_{11}''(t) + \lambda_1(M(t,0)w_1, w_1)\, g_{11}(t) = (f(t), w_1) \\ g_{11}(0) = (\phi_o, w_1) \\ g_{11}'(0) = (\phi_1, w_1) \end{cases}$$

is defined in all $[0,T]$. This show us that $T_1 = T$. Moreover if we consider the assumption (H.3) on M we have

$$|\Delta u_1(t)|^2 \le \frac{2}{m_1}\, Z_1(t), \qquad \forall\, t \in [0,T]\ . \tag{34}$$

From (31) and (34) we get

$$Z_1'(t) - (\delta + k_1\theta_o)\, Z_1(t) \le \frac{1}{2\delta}\, ||f_1(t)||^2\ ,$$

where δ is given by (17). By the last inequality we obtain

$$Z_1(t) \leq \left[Z_1(0) + \frac{1}{2\delta} \int_0^t ||f_1(s)||^2 \, ds \right] e^{(\delta + k_1\theta_o)t}$$

and it proves that α_1 is finite and (33) is true when $m = 1$. Now we make the induction assumption, i.e., we assume that for $m \geq 1$ we have $T_m = T$, α_m finite and (34) true for this m. Then (31) for $m+1$ implies

$$\begin{aligned} Z'_{m+1}(t) \leq & \frac{1}{2\delta} ||f_{m+1}(t)||^2 + \delta Z_{m+1}(t) \\ & + \frac{1}{2} \int_\Omega \left| \frac{\partial M}{\partial t}(x,t,||u_m(t)||^2) \right| (\Delta u_{m+1}(x,t))^2 \, dx \\ & + ||u_m(t)|| \; ||u'_m(t)|| \int_\Omega \left| \frac{\partial M}{\partial \lambda}(x,t,||u_m(t)||^2) \right| (\Delta u_{m+1}(x,t))^2 \, dx \; . \end{aligned}$$

By the other hand, we note that

$$\begin{aligned} ||u_m(t)||^2 \leq \frac{1}{m_o} |\Delta u_m(t)|^2 & \leq \frac{2}{m_o m_1} Z_m(t) \\ & \leq \frac{2}{m_o m_1} \alpha_m = \alpha'_m, \quad 0 \leq t \leq T. \end{aligned} \tag{35}$$

It follows that:

$$Z'_{m+1}(t) - (\delta + k_1\theta_m + k_o\alpha_m\beta_m) \, Z_{m+1}(t) \leq \frac{1}{2\delta} \, ||f_{m+1}(t)||^2 \; .$$

The above inequality shows that (33) is true for $(m+1)$, α_{m+1} is finite and $T_{m+1} = T$, i.e., the proof of Lemma 2 is complete. ∎

We denote,

$$\tau_m = Z_m(0) + \frac{1}{2\delta} \int_0^T ||f_m(t)||^2 \, dt \, , \; m = 1, 2, \dots \; , \tag{36}$$

and then the sequence $(\tau_m)_{m \in \mathbf{N}}$ is bounded. In fact, by (26), (27) and (28) we have that

$$\begin{cases} \Delta\varphi_{om} \to \Delta\phi_o \text{ strong in } L^2(\Omega) \\ \varphi_{1m} \to \phi_1 \text{ strong in } H_o^1(\Omega) \\ f_m(t) \to f(t) \text{ strong in } H_o^1(\Omega), \text{ uniformly on } [0,T] \end{cases} \tag{37}$$

and from the hypothesis of small data (17) we obtain

$$|\Delta\varphi_{om}|^2 + ||\varphi_{1m}||^2 + 0 \leq t \leq T \to \text{Máx} \, ||f_m(t)||^2 \leq \delta^2, \quad \forall \, m \in \mathbf{N} \; . \tag{38}$$

Therefore,

$$||\varphi_{om}||^2 \le \frac{1}{m_o}\,|\Delta\varphi_{om}|^2 \le \frac{1}{m_o}\,\delta^2 \le 1, \qquad \forall\, m \in \mathbf{N}\ ,$$

and then,

$$\tau_m = \frac{1}{2}\Big[||\varphi_{1m}||^2 + \int_\Omega M(x,0,||\varphi_{o(m-1)}||^2)\,(\Delta\varphi_{om}(x))^2\,dx\Big] + \frac{1}{2\delta}\int_0^T ||f_m(t)||^2\,dt \le k_2\delta^2 + \frac{T}{2}\,\delta = k_\delta\ .$$

We conclude that:

(39) $$0 \le \tau_m \le k_\delta\,,\ \forall\, m \in \mathbf{N}\ ,$$

and

(40) $$Z_m(t) \le \tau_m\, e^{(\delta + k_1\theta_{m-1} + k_o\alpha_{m-1}\beta_{m-1})t}, \quad \forall\, t \in [0,T], m \in \mathbf{N}\ .$$

Lemma 3. *Exists a constant c_o (independent of $m \in \mathbf{N}$ and $t \in [0,T]$) such that*

(41) $$Z_m(t) \le 2c_o\,,\ \forall\, t \in [0,T],\ \forall\, m \in \mathbf{N}\ .$$

Proof: We consider $c_o = k_\delta[1 + e^{(1+k_1\theta_o)T}]$. Then, we have by (39):

(42) $$\tau_m \le c_o\,,\ \forall\, m \in \mathbf{N}\ ,$$

and by (40)

$$Z_1(t) \le \tau_1\, e^{(\delta+k_1\theta_o)t} \le k_\delta\, e^{(1+k_1\theta_o)T} \le c_o \le 2c_o\ ,$$

it shows that (41) is true for $m = 1$. Now, we do the follows induction assumption: given $m \ge 1$ we assume that (41) is true for this m. In order to prove that (41) is true for $(m+1)$ we have

$$\alpha_m = \sup_{0\le t\le T} Z_m(t) \le 2c_o$$

and

$$\alpha'_m = \frac{2\alpha_m}{m_o m_1} \le \frac{4c_o}{m_o m_1} = \frac{4}{m_o m_1}\Big\{k_\delta[1 + e^{(1+k_1\theta_o)T}]\Big\} = \frac{4}{m_o m_1}\Big\{\Big(k_2\delta^2 + \frac{T}{2}\,\delta\Big)\Big(1 + e^{(1+k_1\theta_o)T}\Big)\Big\} \le k_3\ .$$

Therefore, we can see that

$$\beta_m \le \left\| \frac{\partial M}{\partial \lambda} \right\|_{L^\infty(\Omega\times(0,T)\times(0,k_3))} = k_4 \tag{43}$$

and

$$\theta_m \le \left\| \frac{\partial M}{\partial t} \right\|_{L^\infty(\Omega\times(0,T)\times(0,k_3))} \le \frac{\ln 2}{3Tk_1} \; . \tag{44}$$

By (40), (42), (43) and (44) we get

$$Z_{m+1}(t) \le \tau_{m+1}\, e^{(\delta+k_1\theta_m+k_o\alpha_m\beta_m)t} \le c_o\, e^{(\delta+\frac{\ln 2}{3T}+2k_ok_4c_o)t} \; .$$

We note that, from our choice we have

$$\Big(\delta + \frac{\ln 2}{3T} + 2k_ok_4c_o\Big) = \Big[1 + Tk_ok_4 + Tk_ok_4\, e^{(1+k_1\theta_o)T}\Big]\,\delta + \\ + 2k_ok_2k_4[1+e^{(1+k_1\theta_o)T}]\,\delta^2 + \frac{\ln 2}{3T} \le \frac{\ln 2}{3T} + \frac{\ln 2}{3T} + \frac{\ln 2}{3T} = \frac{\ln 2}{T} \; .$$

Therefore,

$$\Big(\delta + \frac{\ln 2}{3T} + 2k_ok_4c_o\Big)t \le \ln\, 2\,,\ \ \forall\, t \in [0,T]\ , \tag{45}$$

and then

$$Z_{m+1}(t) \le 2c_o\,,\ \ \forall\, t \in [0,T]\ .$$

The above relation complete the proof of lemma 3. ∎

We obtain from (41) the first estimate: There exists a constant c_1 such that

$$||u_m(t)||^2 + ||u'_m(t)||^2 + |\Delta u_m(t)|^2 \le c_1\,,\ \ \forall\, t\in[0,T],\ \ \forall\, m \in \mathbf{N} \tag{46}$$

Estimate (ii) We start observing that

$$\Big|M(t,||u_{m-1}(t)||^2)\,\Delta u_m(t)\Big|^2 = \int_\Omega \Big|M(x,t,||u_{m-1}(t)||^2)\Big|^2\,|\Delta u_m(x,t)|^2\,dx \\ \le ||M||_{L^\infty(\Omega\times(0,T)\times(0,c_1))}\cdot c_1$$

and

$$|f_m(t)|^2 = \sum_{j=1}^m |(f(t),w_j)|^2 \le |f(t)|^2 \le \frac{1}{m_o}||f(t)||^2 \le \frac{\delta^2}{m_o} \le 1\ .$$

Thus, using $(24)_1$ we obtain the existence of a constant c_2 such that

$$|u''_m(t)|^2 \le c_2\,, \ \forall\, t \in [0,T], \ \forall\, m \in \mathbf{N}\ . \tag{47}$$

By (46), (47) and the fundamental theorem of calculus we choose $t, s \in [0,T]$ and we have that

$$||u_m(t) - u_m(s)|| \le \sqrt{c_1}\,|t-s|\ , \tag{48}$$

$$|u'_m(t) - u'_m(s)| \le c_2\,|t-s|\ . \tag{49}$$

In order to obtain an estimate for (u''_m) analogous to (48) and (49) we choose $t, s \in [0,T]$ and by $(24)_1$ we get

$$\begin{aligned} u''_m(t) - u''_m(s) &= M(t, ||u_{m-1}(t)||^2)\,\Delta(u_m(t) - u_m(s)) + \\ &\quad + \Big[M(t, ||u_{m-1}(t)||^2) - M(s, ||u_{m-1}(s)||^2)\Big]\,\Delta u_m(s) + (f_m(t) - f_m(s))\ . \end{aligned}$$

On the other hand, for $v \in H^1_o(\Omega)$ we note that

$$\begin{aligned} &\Big\|M(t, ||u_{m-1}(t)||^2).v\Big\|^2 = \\ &= \sum_{i=1}^{m} \int_\Omega \left|\frac{\partial M}{\partial x_i}(x,t,||u_{m-1}(t)||^2).v(x) + M(x,t,||u_{m-1}(t)||^2)\frac{\partial v}{\partial x_i}(x)\right|^2 dx \\ &\le 2|v|^2 \sum_{i=1}^{n} \left\|\frac{\partial M}{\partial x_i}\right\|^2_{L^\infty(\Omega\times(0,T)\times(0,c_1))} + 2||M||^2_{L^\infty(\Omega\times(0,T)\times(0,c_1))}\cdot \sum_{i=1}^{n}\left|\frac{\partial v}{\partial x_i}\right|^2 \\ &\le 2\left[||M||_{L^\infty(\Omega\times(0,T)\times(0,c_1))} + \sum_{i=1}^{n}\left\|\frac{\partial M}{\partial x_i}\right\|_{L^\infty(\Omega\times(0,T)\times(0,c_1))}\right]\left[|v|^2 + \sum_{i=1}^{n}\left|\frac{\partial v}{\partial x_i}\right|^2\right]. \end{aligned}$$

Whence, there exists a constant c_3 such that

$$\Big\|M(t, ||u_{m-1}(t)||^2).v\Big\|^2 \le c_3\,||v||^2\,, \ \forall\, t \in [0,T], \ \forall\, m \in \mathbf{N}\ . \tag{50}$$

By the above estimate we have

$$\begin{aligned} (M(t, ||u_{m-1}(t)||^2)\,\Delta(u_m(t) - u_m(s)), v) &= \\ &= \Big(\Delta(u_m(t) - u_m(s)),\ M(t, ||u_{m-1}(t)||^2).v\Big) \\ &= \Big((u_m(s) - u_m(t), M(t, ||u_{m-1}(t)||^2)v)\Big) \\ &\le \sqrt{c_3}\,||v||\,||u_m(s) - u_m(t)|| \end{aligned}$$

and using (48) we get

$$\left|\Big(M(t,||u_{m-1}(t)||^2)\,\Delta(u_m(t)-u_m(s)),v\Big)\right| \le \sqrt{c_1c_3}\,||v||\,|t-s| \;. \tag{51}$$

Now, if we consider $g(x,t)=(x,t,||u_{m-1}(t)||^2)$ then we have

$$\begin{aligned} M(x,t,||u_{m-1}(t)||^2)-M(x,s,||u_{m-1}(s)||^2) &= \\ &= \int_s^t \frac{\partial}{\partial\xi}(M\circ g)(x,\xi)\,d\xi \\ &= \int_s^t \frac{\partial M}{\partial\xi}(x,\xi,||u_{m-1}(\xi)||^2)\,d\xi \\ &\quad + 2\int_s^t \frac{\partial M}{\partial\lambda}(x,\xi,||u_{m-1}(\xi)||^2)\,((u_{m-1}(\xi),u'_{m-1}(\xi)))\,d\xi \;. \end{aligned}$$

Then we can see that there exists a constant c_4 such that

$$\left|M(x,t,||u_{m-1}(t)||^2)-M(x,s,||u_{m-1}(s)||^2)\right| \le c_4\,|t-s|$$

and this estimate shows that there exists a constant c_5 such that

$$\left|\Big(\Big[M(t,||u_{m-1}(t)||^2)-M(s,||u_{m-1}(s)||^2)\Big]\Delta u_m(s),v\Big)\right| \le c_5\,||v||\,|t-s| \;. \tag{52}$$

Finally, we note that

$$|(f_m(t)-f_m(s),v)| \le \frac{1}{m_o}\,||f(t)-f(s)||\,||v|| \;. \tag{53}$$

From (51), (52) and (53) we obtain that there exists a constant c_6 such that

$$||u''_m(t)-u''_m(s)||_{H^{-1}(\Omega)} \le c_6\Big(|t-s|+||f(t)-f(s)||\Big) \;. \tag{54}$$

The estimate (ii) is the relations (47), (48), (49) and (54).

4 – Proof of the theorem

By estimates (i) and (ii) we have:

$(u_m)_{m\in\mathbf{N}}$ uniformly bounded in $[0,T]$ with respect to the norm of $H^1_o(\Omega)\cap H^2(\Omega)$ and equicontinuous with respect to the norm of $H^1_o(\Omega)$.

$(u'_m)_{m\in\mathbf{N}}$ uniformly bounded in $[0,T]$ with respect to the norm of $H^1_o(\Omega)$ and equicontinuous with respect to the norm of $L^2(\Omega)$.

$(u''_m)_{m\in\mathbf{N}}$ uniformly bounded in $[0,T]$ with respect to the norm of $L^2(\Omega)$ and equicontinuous with respect to the norm of $H^{-1}(\Omega)$.

Then, by lemma 1, there exists a function $u\colon\Omega\times[0,T]\to\mathbf{R}$ and a subsequence $(u_{m_\nu})_{\nu\in\mathbf{N}}$ extracted from $(u_m)_{m\in\mathbf{N}}$, such that

$$u\in C([0,T];H^1_o(\Omega))\cap C^1([0,T];L^2(\Omega))\cap C^2([0,T];H^{-1}(\Omega)) \;, \tag{55}$$

$$\begin{cases} u_{m_\nu}(t)\to u(t) \text{ strongly in } H^1_o(\Omega), \text{ uniformly in } [0,T] \\ u'_{m_\nu}(t)\to u'(t) \quad \text{strongly in } L^2(\Omega), \text{ uniformly in } [0,T] \\ u''_{m_\nu}(t)\to u''(t) \quad \text{strongly in } H^{-1}(\Omega), \text{ uniformly in } [0,T]\;. \end{cases} \tag{56}$$

Moreover, since $H^1_o(\Omega)\cap H^2(\Omega)$, $H^1_o(\Omega)$ and $L^2(\Omega)$ are reflexive Banach spaces, we still have

$$\begin{cases} u\in L^\infty(0,T;H^1_o(\Omega)\cap H^2(\Omega) \\ u'\in L^\infty(0,T;H^1_o(\Omega)) \\ u''\in L^\infty(0,T;L^2(\Omega))\;. \end{cases} \tag{57}$$

The convergences don't allow us to pass to the limit in the approximate equation. Indeed, the sequence $(u_{m_\nu})_{\nu\in\mathbf{N}}$ have the properties, but we can't say the same for $(u_{m_\nu-1})_{\nu\in\mathbf{N}}$. In order to solve this problem we will prove the following lemma.

Lemma 4. $\lim\limits_{m\to\infty}||u_{m+1}(t)-u_m(t)||^2=0$ *uniformly on* $[0,T]$.

Proof: For each $m\in\mathbf{N}$ we define $w_m=u_{m+1}-u_m$. Then

$$||u_{m+1}(t)-u_m(t)||^2=\sum_{i=1}^n\int_\Omega\left(\frac{\partial w_m}{\partial x_i}(x,t)\right)^2dx$$

and making use of the assumption (H.3) we can see that there exists a constant c_7 such that

$$\begin{aligned} &||u_{m+1}(t)-u_m(t)||^2\le \\ &\qquad\le c_7\left\{\frac{1}{2}\left[|w'_m(t)|^2+\sum_{i=1}^n\left(M(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t),\frac{\partial w_m}{\partial x_i}(t)\right)\right]\right\}. \end{aligned} \tag{58}$$

Hence, we are motivated to put

$$\psi_m(t)=\frac{1}{2}\left[|w'_m(t)|^2+\sum_{i=1}^n\left(M(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t),\frac{\partial w_m}{\partial x_i}(t)\right)\right] \tag{59}$$

and then, we will conclude with the proof of lemma showing that $\psi_m(t) \to 0$ uniformly in $[0,T]$.

Differentiating $\psi_m(t)$, we have

$$\begin{aligned} \psi'_m(t) &= \frac{1}{2}\frac{d}{dt}|w'_m(t)|^2 + \\ &+ \frac{1}{2}\sum_{i=1}^{n}\left(\frac{\partial M}{\partial t}(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), \frac{\partial w_m}{\partial x_i}(t)\right) + \\ &+ ((u_m(t),u'_m(t)))\sum_{i=1}^{n}\left(\frac{\partial M}{\partial \lambda}(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}, \frac{\partial w_m}{\partial x_i}(t)\right) + \\ &+ \sum_{i=1}^{n}\left(M(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), \frac{\partial w'_m}{\partial x_i}\right). \end{aligned} \tag{60}$$

From the approximation equation we find

$$\begin{aligned} w''_m(t) + \Big[M(t,||u_{m-1}(t)||^2) - M(t,||u_{m-1}(t)||^2)\Big]\,\Delta u_m(t) - \\ - M(t,||u_m(t)||^2)\,\Delta w_m(t) = f_{m+1}(t) - f_m(t) \end{aligned}$$

and then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|w'_m(t)|^2 &= \Big(M(t,||u_m(t)||^2)\Delta w_m, w'_m(t)\Big) \\ &+ \Big(\Big[M(t,||u_m(t)||^2) - M(t,||u_{m-1}(t)||^2)\Big]\Delta u_m(t), w'_m(t)\Big) \\ &+ \Big(f_{m+1}(t) - f_m(t), w'_m(t)\Big)\ . \end{aligned}$$

From the above relation and (60) we obtain

$$\psi'_m(t) = A_m(t) + B_m(t) + C_m(t) + D_m(t) + E_m(t) \tag{61}$$

where

$$\begin{cases} A_m(t) = -\displaystyle\sum_{i=1}^{n}\left(\frac{\partial M}{\partial x_i}(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), w'_m(t)\right) \\ B_m(t) = \Big(\Big[M(t,||u_m(t)||^2) - M(t,||u_{m-1}(t)||^2)\Big]\Delta u_m(t), w'_m(t)\Big) \\ C_m(t) = ((u_m(t),u'_m(t)))\displaystyle\sum_{i=1}^{n}\left(\frac{\partial M}{\partial \lambda}(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), \frac{\partial w_m}{\partial x_i}(t)\right) \\ D_m(t) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n}\left(\frac{\partial M}{\partial t}(t,||u_m(t)||^2)\frac{\partial w_m}{\partial x_i}(t), \frac{\partial w_m}{\partial x_i}(t)\right) \\ E_m(t) = (f_{m+1}(t) - f_m(t), w'_m(t))\ . \end{cases} \tag{62}$$

By (59) and the estimates we find constants c_8, c_9, c_{10} and c_{11} such that

$$A_m(t) \le c_8\, \psi_m(t), \qquad B_m(t) \le c_9\, [\psi_{m-1}(t) - \psi_m(t)]$$
$$C_m(t) \le c_{10}\, \psi_m(t), \qquad D_m(t) \le c_{11}\, \psi_m(t)$$

and $E_m(t) \le \frac{1}{2}|f_{m+1}(t) - f_m(t)|^2 + \psi_m(t)$.

Then we prove that there exists a constant c_{12}, independent of m and $t \in [0,T]$, such that

$$\psi'_m(t) - c_{12}\, \psi_m(t) \le \frac{1}{2}\, |f_{m+1}(t) - f_m(t)|^2 + c_{12}\, \psi_{m-1}(t)$$

and then,

$$\psi_m(t) \le e^{c_{12}T}\Big[\psi_m(0) + \frac{1}{2}\int_0^T |f_{m+1}(t) - f_m(t)|^2\, dt\Big] + c_{12}\, e^{c_{12}T}\int_0^t \psi_{m-1}(s)\, ds\ .$$

Now we denote by

$$\gamma_m = \psi_m(0) + \frac{1}{2}\int_0^T |f_{m+1}(t) - f_m(t)|^2\, dt\ ,$$

and choose

$$c_{13} = \text{Máx}\Big\{e^{c_{12}T}, c_{12}\, e^{c_{12}T}, 0 \le t \le T \to \text{Máx}\, \psi_1(t)\Big\}\ .$$

Then, we can see that

$$\text{(63)} \qquad \begin{cases} \psi_1(t) \le c_{13} \\ \psi_m(t) \le c_{13}\, \gamma_m + c_{13}\displaystyle\int_0^t \psi_{m-1}(s)\, ds\ . \end{cases}$$

By induction we find

$$\text{(64)} \qquad \psi_m(t) \le c_{13}\sum_{j=0}^{m-1} \frac{(c_{13}+t)^j}{j!}\, \gamma_{m-j}\ ,\ \forall\, t \in [0,T],\ m = 2,3,\dots$$

If we consider (37) we get

$$\text{(65)} \qquad \lim_{m\to\infty} \gamma_m = 0$$

and, as we well know,

$$\text{(66)} \qquad \sum_{j=1}^{\infty} \frac{(c_{13}T)^j}{j!} = e^{c_{13}T}\ .$$

Therefore, from (64), (65) and (66) we conclude that $\psi_m(t) \to 0$ uniformly in $[0,T]$ and the proof of lemma 4 is complete. ∎

The result of lemma 4 implies that

$$\lim_{\nu\to\infty} ||u_{m_\nu-1}(t)||^2 = ||u(t)||^2 \text{ uniformly in } [0,T] \ . \tag{67}$$

Then, we have the following convergences:

$$M(t,||u_{m_\nu-1}(t)||^2).v \to M(t,||u(t)||^2).v \quad \text{strongly in } L^2(\Omega), \text{ uniformly in } [0,T], \ \forall\, v\in L^2(\Omega) \ , \tag{68}$$

$$\Delta u_{m_\nu}(t) \to \Delta u(t) \text{ weakly in } L^2(\Omega), \ 0\le t\le T \ . \tag{69}$$

The convergences (68) and (69) imply

$$M(t,||u_{m_\nu-1}(t)||^2)\Delta u_{m_\nu}(t) \to M(t,||u(t)||^2)\Delta u(t) \quad \text{weakly in } L^2(\Omega), \ 0\le t\le T \ . \tag{70}$$

We have then by passage to the limit in ν that

$$u''(t) - M(t,||u(t)||^2)\,\Delta u(t) = f(t) \text{ in } L^2(\Omega), \ 0\le t\le T \ .$$

Clearly we also have $u(0)=\phi_o$ and $u'(0)=\phi_2$.

5 – Uniqueness

Let u and v be satisfying (21), (22) and (23). Then, if we define $w=u-v$ we get

$$\begin{cases} w''(t) + M(t,||v(t)||^2)\,\Delta v(t) - M(t,||u(t)||^2)\,\Delta u(t) = 0 \\ w(0)=w'(0)=0 \ . \end{cases} \tag{71}$$

Now we put

$$\psi(t) = \frac{1}{2}\Big[|w'(t)|^2 + \sum_{i=1}^{n}\Big(M(t,||u(t)||^2)\frac{\partial w}{\partial x_i}(t), \frac{\partial w}{\partial x_i}(t)\Big)\Big]. \tag{72}$$

Therefore, using again the same analysis used in the proof of lemma 4, we obtain a constant c_{14} such that

$$\psi'(t) - c_{14}\,\psi(t) \le 0$$

and this imply

$$\psi(t) \le c^{c_{14}t}\,\psi(0)\,,\ \forall\, t \in [0,T]\ . \tag{73}$$

But, from (72) there exists a constant c_{15} such that

$$0 \le \psi(t) \le c_{15}\Big[|w'(t)|^2 + ||w(t)||^2\Big]\,,\ 0 \le t \le T\ .$$

By $(71)_2$, if we take $t = 0$ in the above relation, we have $\psi(0) = 0$. This fact with (73) shows that $\psi(t) = 0$, $0 \le t \le T$; and then we have uniqueness.

REFERENCES

[1] Arosio, A. and Spagnolo, S. – *Global solutions to the Cauchy problem for a nonlinear hyperbolic equation,* Nonlinear Partial Differential Equations and their Applications, Collège de France Seminar, vol. 6, Edited by Brézis H. and Lions J.L., Pitman, London, 1984.

[2] Arosio, A. and Garavaldi – On the mildly degenerate Kirchhoff string, *Mathematical Methods in the Applied Sciences*, 14 (1991), 177–195.

[3] Bernstein, S. – Sur une classe d'équations fonctionelles aux derivées partielles, *Isv. Acad. Nauk. SSSR Ser. Math.*, 4, 17–26.

[4] Brito, E.H. – The damped elastic stretched string equation generalized: existence, uniqueness, regularity and stability, *Applicable Anal.*, 13 (1982), 219–233.

[5] Carrier, G.F. – On the vibration problem of elastic string, *Q.J. Appl. Math.*, 3 (1945), 151–165.

[6] Crippa, H.R. – *On local solutions of some mildly degenerate hyperbolic equations* (to appear).

[7] D'Ancona, P. and Spagnolo, S. – Global solvability for the degenerate Kirchhoff equation with real analytic data, *Inventiones Mathematicae*, 108 (1992), 247–262.

[8] Dickey, R.W. – Infinite systems of nonlinear oscillation equations related to string, *Proc. A.M.S.* (1969), 459–469.

[9] Ebihara, Y.; Medeiros, L.A. and Milla Miranda, M. – Local solutions for a nonlinear degenerated hyperbolic equation, *Nonlinear Analysis*, 10 (1986), 27–40.

[10] Geng Di and Qu Chang Zheng – On nonlinear hyperbolic equation in unbounded domain, *Applied Mathematics and Mechanics* (English Edition), 23(3) (1992), 255–261.

[11] Lions, J.L. – *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, Gauthier-Villars, 1968.

[12] Lions, J.L. – *On some questions in boundary value problem of Mathematical Physics*, Contemporary developments in Continuum Mechanics and Partial Differential Equations, North Holland, Math. Studies, Edited by G.M. de la Penha and L.A. Medeiros, 1977.

[13] Matos, M.P. – Mathematical analysis of the nonlinear model for the vibrations of a string, *Nonlinear Analysis, Theory, Methods & Aplications*, 17(12) (1991), 1125–1137.

[14] Medeiros, L.A. and Milla Miranda, M. – On a nonlinear wave equation with damping, *Revista Matemática de la Universidade Complutense de Madrid*, 3(2,3) (1990).

[15] Menzala, G.P. – On global solutions of a quasilinear hyperbolic equations, *Nonlinear Analysis*, 3(5) (1979), 613–627.

[16] Nishihara, K. – Degenerate quasilinear hyperbolic equation with strong damping, *Funkcialaj Ekvacioj*, 27 (1984), 125–145.

[17] Nishihara, K. and Yamada, Y. – On global solutions of some degenerated quasilinear hyperbolic equations with dissipative terms, *Funkcialaj Ekvacioj*, 33 (1990), 151–159.

[18] Pohozaev, S. – On a class of quasilinear hyperbolic equations, *Math. Sbornik*, 95 (1975), 152–166.

[19] Rivera Rodrigues, P.H. – On local strong solutions of a nonlinear partial differential equation, *Applicable Analysis*, 10 (1980), 93–104.

[20] Rivera Rodrigues, P.H. – On a nonlinear hyperbolic partial differential equation, *Revista de Ciencias, Univ. San Marcos*, 74(1), (1986), 1–16.

[21] Yamada, Y. – Some nonlinear degenerate wave equation, *Nonlinear Analysis, Theory, Methods & Applications*, 11(10) (1987), 1155–1168.

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