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POSITIVE SOLUTIONS OF ELLIPTIC EQUATIONS IN TWO-DIMENSIONAL EXTERIOR DOMAINS

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Abstract: We consider the semilinear elliptic equation $\Delta u + f(x,u) = 0$ in a two-dimensional exterior domain. Sufficient conditions for the existence of a positive solution are given.

1. We consider the semilinear elliptic equation

$$Lu = \Delta u + f(x,u) = 0\,, \qquad x \in G_a\ , \tag{1}$$

in an exterior domain $G_a = \{x \in R^2\colon |x| > a\}$ (here $a > 0$) where f is nonnegative and locally Hölder continuous in $G_a \times R$.

Let us introduce the class $\Re$ of nondecreasing functions $w \in C^1(R_+, R_+)$ with $w(t) > 0$ for $t > 0$ satisfying $\lim_{t\to\infty} w(t) = \infty$ and $\int_1^\infty \frac{dt}{w(t)} = \infty$.

Equation (1) is considered subject to the assumptions:

(**A**) $f \in C^\lambda_{\rm loc}(G_a \times R)$ for some $\lambda \in (0,1)$ (locally Hölder continuous);

(**B**) $0 \le f(x,t) \le \alpha(|x|)\, w(\frac{t}{|x|})$ for all $x \in G_a$ and all $t \ge 0$ where $\alpha \in C(R_+, R_+)$ and $w \in \Re$ with $w(0) = 0$.

We intend to give sufficient conditions for the existence of a positive solution of (1) — a C^2-function satisfying (1) — in $G_b = \{x \in R^2 : |x| > b\}$ for some $b \ge a$.

2. Denote $S_b = \{x \in R^2 : |x| = b\}$ for $b \ge a$. We will make use of the following.

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Lemma [2]. *Let L be the operator defined by (1) where f is nonnegative and satisfies assumption (A) in G_a. If there exists a positive solution u_1 and a nonnegative solution u_2 of $Lu_1 \le 0$ and $Lu_2 \ge 0$, respectively, in G_b $(b \ge a)$ such that $u_2(x) \le u_1(x)$ throughout $G_b \cup S_b$, then equation (1) has at least one solution $u(x)$ satisfying $u(x) = u_1(x)$ on S_b and $u_2(x) \le u(x) \le u_1(x)$ throughout G_b.*

We prove now

Theorem. *Assume that (A), (B) hold and that*

$$\int_a^\infty r\,\alpha(r)\,dr < \infty\ . \tag{2}$$

Then there is a $b \ge a$ such that (1) has a positive solution in G_b.

Proof: We consider the nonlinear differential equation

$$\frac{d}{dr}\left\{r\,\frac{dy}{dr}\right\} + r\,\alpha(r)\,w\left(\frac{y}{l\,n(r)}\right) = 0\,, \qquad r \ge e\ , \tag{3}$$

where we define $w(-y) = -w(y)$ for $y \ge 0$ (we can extend w this way since $w(0) = 0$). As one can easily check, the so-defined w belongs to $C^1(R, R)$.

Liouville's transformation $r = e^s$, $h(s) = y(e^s)$ changes (3) into

$$h''(s) + e^{2s}\,\alpha(e^s)\,w\left(\frac{h(s)}{s}\right) = 0\,, \qquad s \ge 1\ . \tag{4}$$

Let us show that equation (4) has a solution $h(s)$ which is positive in $[c, \infty)$ for some $c \ge 1$.

Hypothesis (2) guarantees (see [1]) that for every solution $h(s)$ of (4) there exist real constants m, l such that $h(s) = m\,s + l + o(s)$ as $s \to \infty$ ($m = \lim_{s\to\infty} h'(s)$). We will show that any nontrivial solution $h(s)$ of (4) is of constant sign for s in a neighbourhood of ∞ and since w is odd on R, this gives a solution of (4) which is positive in $[c, \infty)$ for some $c \ge 1$.

Assume that there is a nontrivial solution $h(s)$ of (4) which has a strictly increasing sequence of zeros $\{s_n\}_{n\ge1}$ accumulating at ∞. Then we have that the corresponding m, l are both equal to 0, i.e. $\lim_{s\to\infty} h(s) = \lim_{s\to\infty} h'(s) = 0$. Denote

$$K = \sup_{s\ge1}\{|h(s)|\} > 0\,, \qquad M = \sup_{|u|\le K}\{|w'(u)|\} > 0$$

and observe that $|w(u)| \le M|u|$ for $|u| \le K$ (by the mean-value theorem since $w(0) = 0$).

Since $\lim_{n\to\infty} s_n = \infty$ and $\int_a^\infty r\,\alpha(r)\,dr < \infty$, there exists an n_0 such that $\int_{s_{n_0}}^\infty e^{2s}\,\alpha(e^s)\,ds < \frac{1}{M}$. The relation $h(s_{n_0}) = 0$ implies $|h'(s_{n_0})| > 0$ (we have local uniqueness for the solutions of (4) since $w \in C^1(R,R)$ so that $h(s_{n_0}) = h'(s_{n_0}) = 0$ would imply $h(s) = 0$ for all $s \ge 1$) and since $\lim_{s\to\infty} h'(s) = 0$, there is a root s_{n_1} of $h(s)$ with $|h'(s)| < \frac{1}{2}\,|h'(s_{n_0})|$ for $s \ge s_{n_1}$. Let $T \in [s_{n_0}, s_{n_1}]$ be such that $|h'(s)|$ attains its maximal value on this interval at T.

Since $|h'(T)|$ is by construction equal to $\sup_{s_{n_0}\le s}\{|h'(s)|\}$, we have by the mean-value theorem that

$$|h(s)| = \Big|h(s) - h(s_{n_0})\Big| \le (s - s_{n_0})\,|h'(T)|\,, \qquad s_{n_0} \le s\ ,$$

and we obtain

$$\frac{|h(s)|}{s} \le |h'(T)|\,, \qquad s_{n_0} \le s\ .$$

Integrating (4) on $[T,s]$ $(T < s)$, we get

$$h'(s) - h'(T) + \int_T^s e^{2\tau}\,\alpha(e^\tau)\,w\Big(\frac{|h(\tau)|}{\tau}\Big)d\tau = 0\,, \qquad T \le s\ ,$$

thus

$$|h'(T)| \le |h'(s)| + \int_T^\infty e^{2\tau}\,\alpha(e^\tau)\,w\Big(\frac{|h(\tau)|}{\tau}\Big)d\tau\,, \qquad T \le s\ .$$

Letting $s \to \infty$ (remember that $\lim_{s\to\infty} h'(s) = 0$) we get, in view of the previous remarks,

$$\begin{aligned}|h'(T)| &\le \int_T^\infty e^{2\tau}\,\alpha(e^\tau)\,w\Big(\frac{|h(\tau)|}{\tau}\Big)d\tau \le M\int_T^\infty e^{2\tau}\,\alpha(e^\tau)\,\frac{|h(\tau)|}{\tau}\,d\tau \le \\ &\le M\,|h'(T)|\int_T^\infty e^{2\tau}\,\alpha(e^\tau)\,d\tau \le M\,|h'(T)|\int_{s_{n_0}}^\infty e^{2\tau}\,\alpha(e^\tau)\,d\tau < |h'(T)|\ ,\end{aligned}$$

a contradiction which shows that equation (4) has a solution $h(s)$ which is positive in $[c,\infty)$ for some $c \ge 1$.

To this solution there corresponds a solution $y(r)$ of (3), defined for $r \ge e$ and that is positive on $[e^c,\infty)$.

Let us define $u_1(x) = y(r)$, $r = |x| \ge b = \max\{a, e^c\}$. We have

$$\begin{aligned} rLu_1(x) &= \frac{d}{dr}\Big\{r\,\frac{dy}{dr}\Big\} + r\,f(x,u_1(x)) \\ &\le \frac{d}{dr}\Big\{r\,\frac{dy}{dr}\Big\} + r\,\alpha(r)\,w\Big(\frac{y(r)}{r}\Big) \\ &\le \frac{d}{dr}\Big\{r\,\frac{dy}{dr}\Big\} + r\,\alpha(r)\,w\Big(\frac{y(r)}{l\,n(r)}\Big) = 0\,, \qquad r \ge b\ ,\end{aligned}$$

so that $Lu_1(x) \le 0$ for all $x \in G_b$. Clearly $u_2(x) = 0$ satisfies $Lu_2(x) \ge 0$ in G_b. The Lemma shows that (1) has a solution $u(x)$ in G_b with $0 \le u(x) \le u_1(x) = y(r)$ for $|x| = r > b$ and $u(x) = u_1(x) > 0$ for $|x| = b$. Let now $d > b$. Since $u(x) \ge 0$ for $|x| = d > b$, by the maximum principle ($\Delta u(x) \le 0$ in $\{x \in R^2 \colon b < |x| < d\}$) we get that $u(x) > 0$ for $b < |x| < d$. This shows ($d > b$ was arbitrary) that $u(x)$ is a positive solution of (1) in G_b. ∎

3. To show the applicability of our result and its relation to other similar results from the literature ([2], [3], [4]) we consider the following

Example: The semilinear elliptic equation

$$\Delta u + \frac{u}{|x|^4}\, l\, n\left(\frac{u}{|x|} + 1\right) = 0\,, \qquad |x| > 1\ ,$$

has a positive solution in G_b for some $b \ge 1$.

Indeed, we can apply our theorem with $\alpha(r) = \frac{1}{r^3}$ for $r \ge 1$ and $w(s) = s\, l\, n(s+1)$, $s \ge 0$. We cannot apply the results of [2], [3] or [4] since it is impossible to find a function $g \in C^{\lambda}_{\rm loc}(R_+ \times R_+)$ with $g(r,t)$ nonincreasing of t in R_+ for each fixed $r > 0$, such that $f(t,x) \le t g(|x|,t)$, $|x| > 1$, $t \ge 0$.

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