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ON THE SPECTRAL SET OF A SOLVABLE LIE ALGEBRA OF OPERATORS *

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Abstract: If L is a complex solvable finite dimensional Lie Algebra of operators acting on a Banach space E, and $\{x_i\}_{1\leq i\leq n}$ is a Jordan–Hölder basis of L, we study the relation between $Sp(L,E)$ and $\prod Sp(x_i)$, when L is a nilpotent or a solvable Lie algebra.

1 – Introduction

J.L. Taylor developed in [4] a notion of joint spectrum for a n-tuple $a=(a_1,...,a_n)$ of mutually commuting operators acting on a Banach space E, i.e., $a_i\in\mathcal{L}(E)$, the algebra of all bounded linear operators on E, and $[a_i,a_j]=0$, $1\leq i,j\leq n$. This interesting notion depends on the action of the a_i on E and extends in a natural way the classical definition of spectrum of a single operator. Taylor's joint spectrum, which we denote by $Sp(a,E)$, has many remarkable properties, among then the projection property and the fact that $Sp(a,E)$ is a compact non empty subset of $\mathbb{C}^n$. Another property, in which we are specially interested, is a well known fact about Taylor's joint spectrum, the relation between $Sp(a,E)$ and $Sp(a_i)$, $1\leq i\leq n$:

$$Sp(a,E)\subseteq\prod_{i=1}^{n} Sp(a_i)\ ,$$

where $Sp(a_i)$ denotes the spectral set of a_i.

In [1] we developed a spectral theory for complex solvable finite dimensional Lie algebras acting on a Banach space E. If L is such an algebra and $Sp(L,E)$

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denotes its spectrum, $Sp(L,E)$ is a compact non empty subset of L^* which also satisfies the projection property for ideals, see [1]. Besides, when L is a commutative algebra, $Sp(L,E)$ reduces to Taylor joint spectrum in the following sense. If $\dim L = n$ and if $\{a_i\}_{(1\le i\le n)}$ is a basis of L, we consider the n-tuple $a = (a_1, ..., a_n)$, then $\{(f(a_1), ..., f(a_n));\ f \in Sp(L,E)\} = Sp(a,E)$; i.e., $Sp(L,E)$, in terms of the basis of L^* dual of $\{a_i\}_{(1\le i\le n)}$, coincides with the Taylor joint spectrum of the n-tuple a. Then, the following question arises naturally: if $\{x_i\}_{(1\le i\le n)}$ is a basis of L, and if we consider, as above, $Sp(L,E)$ in terms of the basis of L^* dual of $\{x_i\}_{(1\le i\le n)}$, i.e., if we identify $Sp(L,E)$ with its coordinate expression $\{(f(x_1), ..., f(x_n));\ f \in Sp(L,E)\}$, does $Sp(L,E)$ satisfy the relation:

$$\Big\{(f(x_1), ..., f(x_n));\ f \in Sp(L,E)\Big\} \subseteq \prod_{i=1}^{n} Sp(x_i)\ .$$

The answer, even if $\{x_i\}_{(1\le i\le n)}$ is a Jordan–Hölder basis of L, see Section 2, in general is no.

In this paper we study this problem, i.e., the relation between $Sp(L,E)$ and $\prod_{i=1}^{n} Sp(x_i)$. Refining an idea of [1], we describe this relation by means of the structure of L, in a way which generalizes the well known result of the commutative case. Furthermore, when L is a nilpotent Lie algebra, in particular when L is a commutative algebra, we reobtain the previous inclusion and, when L is a solvable non nilpotent Lie algebra, we give an example in order to show that our characterization can not be improved.

The paper is organized as follows. In Section 2 we review several definitions and results of [1], and topics related to the theory of Lie algebras which we need for our work. In Section 3 we prove our main theorems for solvable and nilpotent Lie algebras. Finally, in Section 4 we give an example in order to show that our characterization can not be improved.

2 – Preliminaries

We briefly recall several definitions and results related to the spectrum of solvable Lie algebras of operators, see [1]. From now on, L denotes a complex solvable finite dimensional Lie algebra, and E a Banach space on which L acts as right continuous linear operators, i.e., L is a Lie subalgebra of $\mathcal{L}(E)$ with the opposite product. If $\dim(L) = n$ and f is a character of L, i.e., f belongs to L^* and $f(L^2) = 0$, where $L^2 = \{[x,y];\ x, y \in L\}$, let us consider the following chain complex, $(E\otimes\wedge L, d(f))$, where $\wedge L$ denotes the exterior algebra of L and $d_{p-1}(f)$ is such that:

$$d_{p-1}(f)\colon\ E\otimes\wedge^p L\to E\otimes\wedge^{p-1}L\ ,$$

$$d_{p-1}(f)\, e(x_1\wedge ...\wedge x_p)=\sum_{k=1}^{p}(-1)^{k+1}\, e(x_k-f(x_k))\,(x_1\wedge ...\wedge \hat{x_k}\wedge ...\wedge x_p)+ \\ +\sum_{1\le k<l\le p}(-1)^{k+l}\, e\Big([x_k,x_l]\wedge x_1\wedge ...\wedge \hat{x_k}\wedge ...\wedge \hat{x_l}\wedge ...\wedge x_p\Big)\ ,$$

where $\hat{}$ means deletion, and $e(x_1\wedge ...\wedge x_p)$ denotes an element of $E\otimes\wedge^p L$. If $p<0$ or $p\ge n$, we also define $d_p(f)\equiv 0$.

Let $H_*(E\otimes\wedge L, d(f))$ denote the homology of the complex $(E\otimes\wedge L, d(f))$. We now state our first definition.

Definition 1. With L and f be as above, the set $\{f\in L^*,\ f(L^2)=0,\ H_*(E\otimes\wedge L,\ d(f))\neq 0\}$ is the joint spectrum of L acting on E, and it is denoted by $Sp(L,E)$.

As we have said, in [1] we proved that $Sp(L,E)$ is a compact non empty subset of L^*, which reduces to Taylor joint spectrum, in the sense of the Introduction, when L is a commutative algebra. Besides, if I is an ideal of L, and π denotes the projection map from L^* to I^*, then:

$$Sp(I,E)=\pi(Sp(L,E))\ ,$$

i.e., the projection property for ideals still holds. With regard to this property, I ought to mention the paper [3], of C. Ott, who pointed out a gap in the proof of this result, and gave another proof of it. In any case, the projection property remains true.

From now to the end of the paragraph, we recall several results which we need for our main theorem. First, as in [1], we consider an $n-1$ dimensional ideal, L', of L and we decompose $E\otimes\wedge^p L$ in the following way:

$$E\otimes\wedge^p L=(E\otimes\wedge^p L')\oplus(E\otimes\wedge^{p-1}L')\wedge\langle x\rangle\ ,$$

where $x\in L$ and is such that $L'\oplus\langle x\rangle=L$, and where $\langle x\rangle$ denotes the one dimensional subspace of L generated by the vector x. If $\widetilde{f}$ denotes the restriction of f to L', we may consider the complex $(E\otimes\wedge^p L', d(\widetilde{f}))$ and, as L' is an ideal of codimension 1 of L, we may decompose the operator $d_p(f)$ as follows:

$$d_{p-1}(f)\colon\ E\otimes\wedge^p L'\to E\otimes\wedge^{p-1}L'\ ,$$

$$d_{p-1}(f) = d_{p-1}(\widetilde{f}) \;, \tag{2}$$

$$d_{p-1}(f)\colon\, (E \otimes \wedge^{p-1} L') \wedge \langle x \rangle \;\to\; E \otimes \wedge^{p-1} L' \oplus (E \otimes \wedge^{p-2} L') \wedge \langle x \rangle \;,$$

$$d_{p-1}(f)(a \wedge (x)) = (-1)^{p+1}\, L_{p-1}(a) + (d_{p-2}(\widetilde{f})(a)) \wedge (x) \;, \tag{3}$$

where $a \in E \otimes \wedge^{p-1} L'$, and L_{p-1} is the bounded linear endomorphism defined on $E \otimes \wedge^{p-1} L'$ by:

$$\begin{aligned} L_{p-1}\, e(x_1 \wedge ... \wedge x_{p-1}) &= e(x - f(x))\, (x_1 \wedge ... \wedge x_{p-1}) + \\ &\quad + \sum_{1 \le k \le p-1} (-1)^k\, e\Big([x, x_k] \wedge x_1 \wedge ... \wedge \hat{x_k} \wedge ... \wedge x_{p-1}\Big) \;, \end{aligned} \tag{4}$$

where $\hat{}$ means deletion, and $\{x_i\}_{(1 \le i \le p-1)}$ belongs to L'.

We use the map θ defined in [2,2]. We recall the main facts which we need for our work. Let $ad(x)$, $x \in L$, be the derivation of L defined by

$$ad(x)(y) = [x, y]\,, \qquad (y \in L) \;,$$

then $\theta(x)$ is the derivation of $\wedge L$ which extends $ad(x)$, and is defined by:

$$\theta(x)(x_1 \wedge ... \wedge x_p) = \sum_{i=1}^{p} \Big(x_1 \wedge ... \wedge ad(x)(x_i) \wedge ... \wedge x_p\Big) \;. \tag{5}$$

Observe that if we consider the map $1_E \otimes \theta(x)$, which we still denote by $\theta(x)$, then

$$\begin{aligned} L_{p-1}\, e(x_1 \wedge ... \wedge x_{p-1}) &= e(x - f(x))\, (x_1 \wedge ... \wedge x_{p-1}) \\ &\quad - \theta(x)\, e(x_1 \wedge ... \wedge x_{p-1}) \;. \end{aligned} \tag{6}$$

Finally, as L is a complex solvable finite dimensional Lie algebra, it is well known that there is a Jordan–Hölder sequence of ideals such that:

i) $\{0\} = L_0 \subseteq L_i \subseteq L_n = L$,

ii) $\dim L_i = i$,

iii) There is a k, $0 \le k \le n$, such that $L_k = L^2 = [L, L]$.

As a consequence, if we consider a basis of L, $\{x_j\}_{(1 \le j \le n)}$, such that $\{x_j\}_{(1 \le j \le i)}$ is a basis of L_i, we have:

$$[x_j, x_i] = \sum_{h=1}^{i} c_{ij}^h\, x_h \,, \qquad (i < j) \;. \tag{7}$$

Such a basis is a Jordan–Hölder basis of L.

In addition, if L is a nilpotent Lie algebra, we may add the condition:

iv) $[L, L_i] \subseteq L_{i-1}$.

Then, in terms of the previous basis, we have:

$$[x_j, x_i] = \sum_{h=1}^{i-1} c_{ij}^h \, x_h \,, \qquad (i < j) \ . \tag{8}$$

3 – The spectral set

First we give a definition which we need for our main theorems. We consider for p such that $0 \leq p \leq n-1$, the set of p-tuples of $[\![1, n-1]\!]$, I_p, defined as follows. If $p = 0$,

$$I_0 = \{1\} \ ,$$

and for p such that $1 \leq p \leq n-1$,

$$I_p = \Big\{(i_1, ..., i_p),\ 1 \leq i_1 < ... < i_j < ... < i_p \leq n-1\Big\} \ .$$

We observe that I_p has a natural order.

If $\alpha = (i_1, ..., i_p)$ and $\beta = (j_1, ..., j_p)$ belong to I_p, let $k = \min\{l, i_l \neq j_l\}$, then

i) $i_j = j_l$, $1 \leq l \leq k-1$,

ii) $i_k \neq j_k$.

If $i_k < j_k$ (resp. $j_k < i_k$) we put $\alpha < \beta$ (resp. $\beta < \alpha$).

Now, if L, L', x, and E are as in Section 2, let us consider a sequence $\{x_i\}_{(1 \leq i \leq n-1)}$ of elements of L'. If $\alpha = (i_1, ..., i_p)$ belongs to I_p we denote $(x_{i_1} \wedge ... \wedge x_{i_p})$ by (x_α), then:

$$(x_\alpha) = (x_{i_1} \wedge ... \wedge x_{i_p}) \ ,$$

if $p = 0$ we denote (x_0) by (1), then:

$$(x_0) = (1) \ .$$

In addition, as L' is an ideal of L, $\theta(x)(\wedge L') \subseteq \wedge L'$; thus, we have a well defined map which we still denote by $\theta(x)$:

$$\theta(x)\colon\ E \otimes \wedge L' \ \to\ E \otimes \wedge L' \ .$$

Now, if $(L_i)_{(0\leq i\leq n)}$ is a Jordan–Hölder sequence of L, and $\{x_i\}_{(1\leq i\leq n)}$ is a Jordan–Hölder basis of L associated to $(L_i)_{(0\leq i\leq n)}$, we set $L'=L_{n-1}$ and $x=x_n$.

In order to prove the following proposition we need to associate to each $\alpha\in I_p$, $0\leq p\leq n-1$, a number r_α. If α belongs to I_p, $\alpha=(i_1,...,i_p)$, and $[x_n,x_{i_k}]=\sum_{h=1}^{i_k} c^h_{i_k n}\,x_h$, we define for p such that $1\leq p\leq n-1$, $r_\alpha=\sum_{k=1}^p c^{i_k}_{i_k n}$, and if $p=0$, we define $r_1=0$. Then a standard calculation shows that:

$$\begin{aligned}\theta(x)\,e(x_\alpha)&=X+\Big(\sum_{k=1}^p c^{i_k}_{i_k n}\Big)\,e(x_\alpha)\\&=X+r_\alpha\,e(x_\alpha)\ ,\end{aligned}$$

where X belongs to $\bigoplus_{\beta<\alpha}E(x_\beta)$.

Besides, as x_n acts on E, $\bar{x}_n=x_n\otimes 1-1\otimes\theta(x_n)$ acts on $E\otimes\wedge L_{n-1}$ in a natural way, where 1 denotes the identity map of the corresponding spaces. Let us compute $Sp(\bar{x}_n,E\otimes\wedge L_{n-1})$, i.e. the spectrum of $\bar{x}_n$ in $E\otimes\wedge L_{n-1}$. If we decompose $E\otimes\wedge L_{n-1}$ by means of $E(x_\alpha)$, $\alpha\in I_p$, $0\leq p\leq n-1$, we have that $E\otimes\wedge L_{n-1}=\bigoplus_{(\alpha\in I_p,\,0\leq p\leq n-1)}E(x_\alpha)$. Now, as $\theta(x_n)$, acting on $\wedge L_{n-1}$ has an upper triangular form with diagonal entries r_α, $\bar{x}_n$, in the above decomposition has an upper triangular form with diagonal entries x_n-r_α, thus, $Sp(\bar{x}_n,E\otimes\wedge L_{n-1})=Sp(x_n)-\{r_\alpha,\,x\in I_p,\,0\leq p\leq n-1\}$. Finally, we observe that the spectrum of $\bar{x}_n$ depends on the structure of L as Lie algebra, and that in the commutative case, $\bar{x}_n=x_n\otimes 1$, and $Sp(\bar{x}_n,E\otimes\wedge L_{n-1})=Sp(x_n)$.

The first step to our main theorem is Proposition 1.

Proposition 1. *Let L be a complex solvable finite dimensional Lie algebra, acting as right continuous linear operators on a Banach space E. Let $(L_i)_{(0\leq i\leq n)}$ be a Jordan–Hölder sequence of L, and $\{x_i\}_{(1\leq i\leq n)}$ be a basis associated to this sequence. Then, if f is a character of L such that*

$$f(x_n)\notin Sp(\bar{x}_n,E\otimes\wedge L_{n-1})\ ,$$

f does not belong to $Sp(L,E)$.

Proof: First we decompose $E\otimes\wedge^p L$ as in Section 2:

$$E\otimes\wedge^p L=(E\otimes\wedge^p L_{n-1})\oplus(E\otimes\wedge^{p-1}L_{n-1})\wedge\langle x_n\rangle\ .$$

As L_{n-1} is an ideal of L, $ad(x_n)(L_{n-1})\subseteq L_{n-1}$, and

$$\theta(x_n)\,(E\otimes\wedge^{p-1}L_{n-1})\subseteq E\otimes\wedge^{p-1}L_{n-1}\ .$$

Then, by (4) and (5)

$$L_{p-1} = (x_n - \theta(x_n)) - f(x_n) \ .$$

Moreover, if we decompose $E \otimes \wedge^{p-1} L_{n-1}$ by means of $E(x_\alpha)$, it is obvious that:

$$E \otimes \wedge^{p-1} L_{n-1} = \bigoplus_{\alpha \in I_{p-1}} E(\alpha) \ .$$

Then, by the previous considerations and the above formula, L_{p-1} is an upper triangular matrix with diagonal entries $(x_n - r_\alpha) - f(x_n)$ associated to $\alpha = (i_1, ..., i_{p-1}) \in I_{p-1}$. Thus, if f satisfies the hypothesis, L_p is an invertible operator for each p, $0 \leq p \leq n-1$.

We now construct a homotopy operator, $(S_p)_{p \in \mathbb{Z}}$, for the complex $(E \otimes \wedge L, d(f))$, in order to see that $H_*(E \otimes \wedge L, d(f)) = 0$, which is equivalent to $f \notin Sp(L, E)$.

S_p is a map from $E \otimes \wedge^p L$ to $E \otimes \wedge^{p+1} L$, and we define it as follows:

$$S_p \colon \ E \otimes \wedge^p L \ \rightarrow \ E \otimes \wedge^{p+1} L \ ,$$

if $p < 0$ or $p > n-1$, we define $S_p \equiv 0$, if p is such that $0 \leq p \leq n-1$, we consider the decomposition of $E \otimes \wedge^p L$ set at the beginning of the proof, and we pose:

$$S_p\Big(E \otimes \wedge^{p-1} L_{n-1} \wedge \langle x \rangle\Big) = 0 \ , \tag{9}$$

and S_p restricted to $E \otimes \wedge^p L_{n-1}$ satisfies:

$$\begin{aligned} &S_p(E \otimes \wedge^p L_{n-1}) \subseteq E \otimes \wedge^p L_{n-1} \wedge \langle x_n \rangle \ , \\ &S_p = (-1)^p \, L_p^{-1} \wedge (x_n) \ . \end{aligned} \tag{10}$$

In order to verify that S_p is a homotopy operator we prove the following formula:

$$S_p \, d_p \, L_{p+1} = (-1)^p \, d_p \wedge (x_n) \ . \tag{11}$$

By (2) and (3), we have

$$\begin{aligned} d_p \, L_{p+1} &= d_p\Big((d_{p+1} - d_p \wedge (x_n))\Big) \, (-1)^{p+3} \\ &= (-1)^p \, d_p(d_p \wedge (x_n)) \\ &= (-1)^p \, (-1)^{p+2} \, L_p \, d_p \\ &= L_p \, d_p \ . \end{aligned}$$

Then,

$$d_p\, L_{p+1} = L_p\, d_p\ .$$

Thus,

$$S_p\, d_p\, L_{p+1} = S_p\, L_p\, d_p = (-1)^p\, d_p \wedge (x_n)\ .$$

Now, by means of formulas (9), (10), (11), an easy calculation shows that

$$d_p\, S_p + S_{p-1}\, d_{p-1} = I\ ,$$

for $p \in \mathbb{Z}$, i.e., $(S_p)_{p\in\mathbb{Z}}$ is a homotopy operator. ∎

In order to state our main theorems we consider the basis $\{x_i\}_{(1\leq i\leq n)}$ of (7), and we apply the definition of the beginning of the paragraph to L_j, the ideal generated by $\{x_i\}_{(1\leq i\leq j)}$, $1 \leq j \leq n$. We denote by I_p^j, $0 \leq p \leq j-1$, $1 \leq j \leq n$, the set of p-tuples associated to L_j and the ideal L_{j-1}, and if α belongs to I_p^j we denote by r_α^j the complex number associated to α. In addition, we observe that in Theorem 1 and 2 below, we consider the set $Sp(L,E)$ in terms of the basis of L^* dual of $\{x_i\}_{(1\leq i\leq n)}$, i.e., we identify $Sp(L,E)$ with its coordinate expression in the mentioned basis: $\{f(x_1),...,f(x_n);\ f \in Sp(L,E)\}$.

Now we state our main theorem.

Theorem 1. *Let L be a complex solvable finite dimensional Lie algebra, acting as right continuous linear operators on a Banach space E. Let $(L_i)_{(0\leq i\leq n)}$ be a Jordan–Hölder sequence of L, and $\{x_i\}_{(1\leq i\leq n)}$ be a basis associated to this sequence. Then, in terms of the basis of L^* dual of $\{x_i\}_{(1\leq i\leq n)}$, we have:*

$$Sp(L,E) \subseteq \prod_{1\leq j\leq n} Sp(\bar{x}_j,\ E \otimes \wedge L_{j-1})\ .$$

Proof: By means of an induction argument, the proof is a consequence of Proposition 1 and Theorem 3 of [1]. ∎

In the case of a nilpotent Lie algebra, Theorem 2 extends directly the commutative case.

Theorem 2. *Let L be a complex nilpotent finite dimensional Lie algebra, acting as right continuous linear operators on a Banach space E. Let $(L_i)_{(0\leq i\leq n)}$ be a Jordan–Hölder sequence of L and $\{x_i\}_{(1\leq i\leq n)}$ be a basis associated to this sequence. Then in terms of the basis of L^* dual of $\{x_i\}_{(1\leq i\leq n)}$, we have:*

$$Sp(L,E) \subseteq \prod_{i=1}^{n} Sp(x_i)\ .$$

In particular,

$$Sp(L,E) \subseteq \Big\{ f \in L^*,\ f(L^2) = 0 /\ \|f(x)\| \le \|x\|_{\mathcal{L}(E)},\ \forall\, x \in L \Big\} \ .$$

Proof: As L is a nilpotent Lie algebra we may consider a Jordan–Hölder sequence of L, $(L_j)_{(0\le j\le n)}$, such that $[L, L_j] \subseteq L_{j-1}$. Then for each $\alpha \in I_p^j$, $1 \le j \le n$, $0 \le p \le j-1$, we have:

$$r_\alpha^j = 0 \ ,$$

which implies that $Sp(x_i) = Sp(\bar{x}_i,\, E \otimes \wedge L_{i-1})$. Thus, by means of Theorem 1 we conclude the proof. ∎

4 – An example

We give an example in order to see that our Theorem 1 can not be, in general improved. We consider the solvable Lie algebra G_2 on two generators,

$$G_2 = \langle y \rangle \oplus \langle x \rangle \ ,$$

with relations $[x,y] = y$. Then, by Theorem 1:

$$Sp(G_2, E) \subseteq Sp(y) \times \Big(Sp(x) \cup Sp(x) - 1 \Big) \ .$$

Now, if $E = \mathbb{C}^2$, and y and x are the following matrices

$$y = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \ , \qquad x = \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix} ,$$

then, $L = \langle y \rangle \oplus \langle x \rangle$ defines a Lie subalgebra of $\mathcal{L}(\mathbb{C}^2)$ isomorphic to G_2, and an easy calculation shows that:

$$Sp(G_2, \mathbb{C}^2) = \{0\} \times \{1/2, -3/2\} \ .$$

However, as $Sp(x) = \{1/2,\ -1/2\}$, and $Sp(y) = 0$, Theorem 1 cannot be, in general, improved.

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