PORTUGALIAE MATHEMATICA
Vol. 54 Fasc. 4 – 1997

# EXISTENCE OF EIGENVALUES FOR (UAA) MARKOV MAPS

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**Abstract:** We give a simple proof of the existence and of the invariance of eigenvalues for (uaa) Markov maps.

## 1 – Introduction

The space of the (uaa) conjugacy classes of the (uaa) Markov maps on train tracks is the interesting completion of the $C^r$ conjugacy classes, $r > 1$ (see [4], [9] and [11]).

Locally, the definition of (uaa) functions is that they approach affine structures but not necessarily a unique affine structure. Therefore, the map does not need to have derivative at a point. But, we prove the existence of derivative at fixed points. Similar to the $C^r$ case, we show the existence of the eigenvalues at the periodic orbits and their invariance in the (uaa) conjugacy classes of Markov maps. The proof is simple and geometric, but we leave open a difficult problem which is if they are or not a complete invariant (see [11]).

## 2 – The uniformly asymptotically affine (uaa) Markov maps on train tracks

### 2.1. A (uaa) homeomorphism

We are going to explain what means to say that a local homeomorphism $\phi\colon I \subset \mathbb{R} \to J \subset \mathbb{R}$ is (uaa) uniformly asymptotically affine at a point $x$. The

*Received*: May 11, 1996; *Revised*: September 19, 1996.
*1991 AMS Mathematics Subject Classification*: 58F15.
*Keywords*: Markov maps, Eigenvalues, Differentiability.

(uaa) notion is an interesting generalisation of the usual $C^1$ smooth notion. A function is $C^1$ smooth if when we look at smaller and smaller scales near a point $x$, it approaches an affine structure. Roughly, a function is (uaa) at $x$, if when we look at smaller and smaller scales near $x$, it approaches a set of affine structures. We mean exactly the following.

The local homeomorphism $\phi\colon I \subset \mathbb{R} \to J \subset \mathbb{R}$ is (uaa), if, and only if, there is a constant $c > 1$ and a continuous function $\varepsilon_c\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\varepsilon_c(0) = 0$ and with the property that, for all $x-\delta_1$, $x$, $x+\delta_2 \in I$ so that $0 < \delta_1, \delta_2 < \delta$ and $c^{-1} < \delta_2/\delta_1 < c$, we have that

$$\left|\log \frac{\phi(x) - \phi(x-\delta_1)}{\phi(x+\delta_2) - \phi(x)} \frac{\delta_2}{\delta_1}\right| < \varepsilon_c(\delta) \ . \tag{1}$$

### 2.2. The train tracks

Consider the space which is the disjoint union of $k$ copies $R_i$, $i \in \mathbb{Z}_k$, of $\mathbb{R}_{\geq 0}$. Let $\rho_k$ be the space obtained from this by identifying $0 \in R_i$ with $0 \in R_{i+1}$, for all $i \in \mathbb{Z}_k$. Let $0_k \in \rho_k$ be the point determined by $0 \in R_i$.

By $\mathrm{Iso}(\rho_k)$ we denote the set of all maps $i\colon \mathbb{R} \to \rho_k$ such that

**i)** $i(0) = 0_k$ and

**ii)** $i$ is an isometry away from 0.

A map $f\colon \rho_k \to \rho_{k'}$ is $C^r$, if, and only if, for all $i \in \mathrm{Iso}(\rho_k)$ and $j \in \mathrm{Iso}(\rho_{k'})$, the induced map $f_{i,j}$ for which the following diagram commutes is $C^r$:

$$\begin{array}{ccc} \mathbb{R} & \xrightarrow{f_{i,j}} & \mathbb{R} \\ \big\downarrow{\scriptstyle i} & & \big\downarrow{\scriptstyle j} \\ \rho_k & \xrightarrow{f} & \rho_{k'} \end{array}$$

A map $f\colon V \to V'$, where $V \subset \rho_k$ and $V' \subset \rho_{k'}$ are open sets, is $C^r$, if it extends to a $C^r$ map on $\rho_k$.

A *train track* $T$ is a compact set locally homeomorphic to a compact set of $\mathbb{R}$ or of $\rho_k$.

A point $x \in T$ is a *singularity* of $T$, if, and only if, there exists a neighbourhood, $V$, of $x$ and an homeomorphism $h\colon V \to \rho_k$ such that $h(x) = 0_k$. We call $k = k(x)$ the *order* of the singularity $x$.

For any singularity $x$ correspondent to a point $0_k$, let $I_1, ..., I_k$ be $k$ disjoint open intervals with an extrem commum point $x$ such that $\bigcup_{i=1}^{k} I_i \cup \{x\}$ is a neighbourhood of $x$. A *Spine* $E$ of $x$ is the union $\bigcup_{i=1}^{m} J_{l_i} \cup \{x\}$, where $J_{l_i}$ are disjoint open intervals with $x$ as an extreme point and $J_{l_i} \cap I_{l_i} \neq \emptyset$, for all $i = 1, ..., m$.

For any singularity $x$, let $E(x)$ be a set of admissible spines on $x$ with elements of the form $E^j = \{l_1^j, ..., l_p^j\}$, where $l_i^j \in \{1, ..., k\}$,

$$E(x) = \{E^1, ..., E^r\} \ .$$

We say that a spine $U = J_{l_1} \cup \cdots \cup J_{l_s}$ on $x$ is *admissible*, if, and only if, $\{l_1, ..., l_s\} \in E(x)$.

A chart $(i, U)$ on $T$ is an homeomorphism $i\colon\, U \subset T \to W$, where if $U$ contains a singularity $x$, then $U = J_{l_1} \cup \cdots \cup J_{l_s} \cup \{x\}$ is an admissible spine and $W \subset \rho_k$, otherwise $U$ is homeomorphic to an interval $I \subset \mathbb{R}$ and $W \subset \mathbb{R}$. If $W \subset \rho_k$, then $i(x) = 0_k$, if, and only if, $x$ is a singularity.

Two charts $(i, U)$ and $(j, V)$ on $T$ are $C^r$ *compatible*, if the overlap map $i \circ j^{-1}\colon\, j(U \cap V) \to i(U \cap V)$ is $C^r$.

A $C^r$ *atlas*, $A$, on $T$ is a set of charts

**i**) which cover $T$,

**ii**) if $E \subset T$ is an admissible spine, there exists a chart $(i, E) \in A$ and

**iii**) the $C^r$ norm of the overlap maps is bounded, independent of the overlap map considered (just depends upon the atlas $A$).

### 2.3. The (uaa) Markov maps

Let $T$ be a subset of a train track $T'$. Let $M\colon\, T \to T$ be a local homeomorphism.

A *Markov partition* for the homeomorphism $M$ is a collection $\mathcal{C} = \{C_1, ..., C_n\}$ of intervals $C_i \subset T$ such that:

**i**) $\operatorname{int} C_i \bigcap \operatorname{int} C_j = \emptyset$, if $i \neq j$ and $T = \bigcup_{i=1}^{n} C_i$;

**ii**) If $x \in C_i$ and $M(x) \in C_j$, then $M(C_i)$ contains $C_j$;

**iii**) For all $C_j \in \mathcal{C}$, there exists $C_i \in \mathcal{C}$ such that $M(C_i)$ contains $C_j$;

**iv**) Let a *n-cylinder* be defined by

$$C_{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{n-1}} = \left\{x \in C_{\varepsilon_0}\colon\ M^{j+1}(x) \in C_{\varepsilon_{j+1}},\ j = 0, 1, \ldots, n-2\right\} .$$

For all sequence $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$, we have that the $\lim_{n\to\infty} C_{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{n-1}}$ is a point.

The spaces between $n$-cylinders which do not contain any cylinder are called *gaps*. A *n-gap* is a gap between $n$-cylinders which is contained in a $(n-1)$-cylinder. We call $\Lambda_M = \bigcap_{n=1}^{\infty} \bigcup_{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_n} C_{\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_n}$ the *invariant set* of $M$.

**Definition 1.** An homeomorphism $M\colon T \to T$ is a *Markov map*, if there is a Markov partition for $M$.

Let $I, J \subset T$ be two intervals such that $M_{IJ} = M^p\colon I \to J$ is an homeomorphism, for some $p \geq 1$. Let $M_{JI} = M_{IJ}^{-1}\colon J \to I$ be the inverse map of the map $M_{IJ}$.

The Markov map $M$ is *(uaa)*, if, and only if, there is a constant $c > 1$ and a continuous function $\varepsilon_c\colon \mathbb{R}_0^+ \to \mathbb{R}_0^+$ with $\varepsilon_c(0) = 0$ and with the property that for all maps $M_{IJ}$ as above, for all charts $(i, I' \supset I), (j, J' \supset J) \in A$ and and for all $x, y, z \in J$ so that $0 < j(y) - j(x), j(z) - j(y) < \delta$ and $c^{-1} < (j(z) - j(y))/(j(y) - j(x)) < c$, we have that

$$\left|\log \frac{(i \circ M_{JI})(y) - (i \circ M_{JI})(x)}{(i \circ M_{JI})(z) - (i \circ M_{JI})(y)}\, \frac{j(z) - j(y)}{j(y) - j(x)}\right| < \varepsilon_c(\delta) . \tag{2}$$

We note that the (uaa) definition of a Markov map is stonger than just to say that a map is (uaa). Roughly, a Markov map is (uaa), if every inverse composition is (uaa) with the some constant $c$ and the function $\varepsilon_c$.

The Markov maps $M\colon T \to T$ and $N\colon P \to P$ are *topologically conjugate*, if there exists an homeomorphism $h\colon T \to P$, such that

**i**) $h \circ M|_{\Lambda_M} = N \circ h|_{\Lambda_M}$ and

**ii**) the singularities $x$ and $h(x)$ have the same order.

If the homeomorphism $h$ is (uaa), then we say that the Markov maps $M$ and $N$ are (uaa) conjugate.

## 3 – The existence of the eigenvalues $e(p)$ at the periodic points

Let $M\colon T \to T$ be a (uaa) Markov map with respect to an atlas $A$ on $T$. Let $p \in T$ be a periodic point with period $q$. Consider a local chart $(i, I) \in A$ such that $p \in I$. The eigenvalue $e(p)$ of $p$ is well defined, if the following limit exists and it is independent of the chart considered:

$$e(p) = \lim_{z \to p} \frac{(i \circ M^q)(z) - (i \circ M^q)(p)}{i(z) - i(p)} \ .$$

**Theorem 1.** *Let $M\colon T \to T$ be a (uaa) Markov map with respect to an atlas $A$ on $T$ and $p$ a periodic point of the Markov map $M$. The eigenvalue $e(p)$ is well defined and the set of all the eigenvalues is an invariant of the (uaa) conjugacy class.*

**Proof of Theorem 1:** We are going to prove, in two lemmas, the existence of eigenvalues for (uaa) Markov maps and that they are invariants of the (uaa) conjugacy classes.

We prove in Lemma 1 and Lemma 2, for a fixed point $p$, that the eigenvalue $e(p)$ is well defined and it is an invariant of the (uaa) conjugacy class. The proof for a periodic point $p'$ of period $q$ follows in the same way as for the fixed point $p$ using the composition $M^q$ of the Markov map $M$. ∎

**Lemma 1.** *Let $p$ be a fixed point of the (uaa) Markov map $M\colon T \to T$ with respect to the atlas $A$. Let $i\colon I \to I'$ be a chart in $A$, such that $p \in I$. Then the eigenvalue $e(p)$ is well defined by*

$$e(p) = \lim_{z \to p} \frac{(i \circ M)(z) - (i \circ M)(p)}{i(z) - i(p)} \ .$$

**Proof of Lemma 1:** We consider two different cases: the first case when the Markov map $M$ is orientation reversing and the second case when the Markov map $M$ is orientation preserving. We prove both cases in two steps. First, we prove that given a sequence of points $q_n = M(q_{n+1})$ converging to $p$, they define a candidate $\underline{e}(p)$ for the eigenvalue $e(p)$. Secondly, we show that for any $z$ converging to $p$, we obtain that

$$\lim_{z \to p} \frac{(i \circ M)(z) - (i \circ M)(p)}{i(z) - i(p)} = \underline{e}(p) \ .$$

Let $q_n \in T$ be a sequence of points $q_n$ such that

**i**) the point $M(q_n)$ is equal to the point $q_{n-1}$ and

**ii**) the point $q_n$ is close of the point $p$, for all $n \geq 0$.

Let $r_n$ be equal to the ratio between the distances of $|i(q_{n-1}) - i(p)|$ and $|i(q_n) - i(p)|$ (see Figure 1).


**Fig. 1** – The ratio $r_n$.

Since the Markov map $M$ is (uaa), the limit $r = \lim r_n$ exists and

$$\left|\frac{r}{r_n} - 1\right| \leq \varepsilon_c\Big(|i(q_{n-1}) - i(q_n)|\Big) .$$

Therefore,

$$|\underline{e}(p)| = \lim\left|\frac{(i \circ M)(q_n) - (i \circ M)(p)}{i(q_n) - i(p)}\right| = r . \tag{3}$$

First case: The Markov map $M$ is orientation reversing, $\underline{e}(p) = -r$.

For all point $z$ converging to $p$, let the point $q_n \in [p, z]$ be such that the ratio between the distance $|i(z) - i(q_n)|$ and the distance $|i(q_n) - i(p)|$ is bounded away from zero and infinity. Let $s_n$ be equal to the ratio between the distances of $|i(z) - i(q_n)|$ and $|i(q_n) - i(p)|$. Let $s_{n-1}$ be equal to the ratio between the distances of $|(i \circ M)(z) - i(q_{n-1})|$ and $|i(q_{n-1}) - i(p)|$ (see Figure 2).

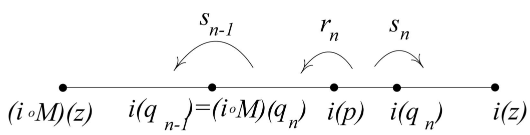


**Fig. 2** – The ratios $s_n$ and $s_{n-1}$.

By equality (3),

$$\begin{aligned}\frac{(i\circ M)(z)-(i\circ M)(p)}{i(z)-i(p)} &\in \underline{e}(p)\Big(1\pm\varepsilon_c\Big(|i(q_{n-1})-i(q_n)|\Big)\Big)\frac{1+s_{n-1}}{1+s_n}\\ &\subset \underline{e}(p)\Big(1\pm\varepsilon_c\Big(|i(z)-(i\circ M)(z)|\Big)\Big)\ .\end{aligned} \tag{4}$$

Therefore,

$$\lim_{z\to p}\frac{(i\circ M)(z)-(i\circ M)(p)}{i(z)-i(p)}=\underline{e}(p)\ .$$

Second case: The Markov map $M$ is orientation preserving, $\underline{e}(p)=r$.

For all point $z$ converging to $p$, either there is a point $q_n$ between $p$ and $z$ or there is not. In the case where there is a point $q_n$ between $p$ and $z$, we get a similar inequality to (4). Otherwise, we consider a point $q_n$ such that the ratio between $|i(q_n)-i(p)|$ and $|i(z)-i(p)|$ is bounded away from zero and infinity.

Let $s_n$ be equal to the ratio between the distances of $|i(q_n)-i(p)|$ and $|i(z)-i(p)|$. Let $s_{n-1}$ be equal to the ratio between the distance $|i(q_{n-1})-i(p)|$ and the distance $|(i\circ M)(z)-(i\circ M)(p)|$ (see Figure 3).

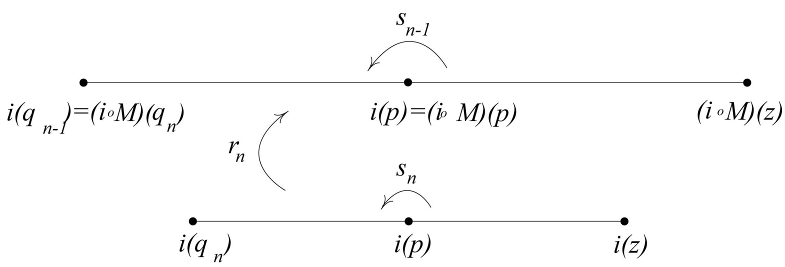


**Fig. 3** – The ratios $s_n$ and $s_{n-1}$.

By equality (3),

$$\begin{aligned}\frac{(i\circ M)(z)-(i\circ M)(p)}{i(z)-i(p)} &\in \underline{e}(p)\Big(1\pm\varepsilon_c\Big(|i(p)-i(q_{n-1})|\Big)\Big)\frac{s_n}{s_{n-1}}\\ &\subset \underline{e}(p)\Big(1\pm\varepsilon_c\Big(|(i\circ M)(z)-i(q_{n-1})|\Big)\Big)\ .\ \blacksquare\end{aligned} \tag{5}$$

**Lemma 2.** *The eigenvalue $e(p)$ is an invariant of the (uaa) conjugacy class of the Markov map $M\colon T\to T$ with respect to the atlas $A$.*

**Proof of Lemma 2:** Let $M$ and $N$ be (uaa) Markov maps with respect to the atlas $A$ and $B$, respectively. Let $h$ be the conjugate map between $M$ and $N$,

$$h \circ M = N \circ h \ .$$

Let $p$ and $p' = h(p)$ be fixed points of $M$ and $N$, respectively. Let $q_n$ and $q'_n = h(q_n)$ be two sequences of points converging to $p$ and $p'$, respectively, such that $M(q_n) = q_{n-1}$, for all $n \geq 0$. Let $t_n$ be equal to the ratio between the distances of $|i(q_{n-1}) - i(q_n)|$ and $|i(q_n) - i(p)|$ and $t'_n$ be equal to the ratio between the distances of $|j(q'_{n-1}) - j(q'_n)|$ and $|j(q'_n) - j(p')|$ (see Figure 4), where $i \in A$ and $j \in B$.

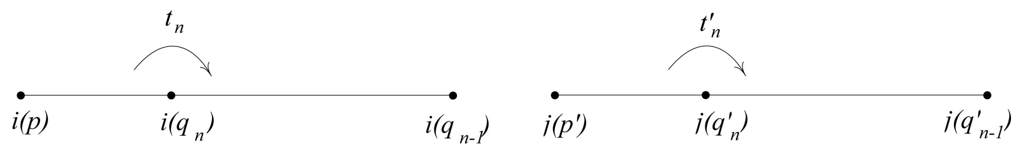


**Fig. 4** – The ratios $t_n$ and $t'_n$.

By Lemma 1,

$$e_M(p) = \pm \lim(1 + t_n) \quad \text{and} \quad e_N(p') = \pm \lim(1 + t'_n) \ .$$

Since the map $h$ is (uaa),

$$\frac{t_n}{t'_n} \ \in \ 1 \pm \varepsilon_c\Big(|i(q_{n-1}) - i(q_n)|\Big) \ .$$

Since the map $h$ preserves the order, we have

$$e_M(p) = e_N(p') \ . \ \blacksquare$$

*ACKNOWLEDGEMENT* – We are very grateful to Dennis Sullivan and David Rand for very useful discussions on this work. We also thank IHES, CUNY and University of Warwick for their hospitality.

We thank Calouste Gulbenkian Foundation, JNICT and PRAXIS for their financial support of A.A. Pinto. We thank JNICT for their financial support of F.F. Ferreira.

## REFERENCES

**[1]** Bedford, T. and Fisher, A.M. – *Ratio geometry, rigidity and the scenery process for hyperbolic cantor sets.*

**[2]** Douady, A. and Hubbard, J.H. – Etude dynamique des polynomes complexes I & II, *Publ. Math. Orsay* (1984–85).

**[3]** Faria, E. de – *Quasisymmetric distortion and rigidity of expanding endomorphisms of* $S^1$.

**[4]** Pinto, A.A. and Rand, D.A. – *Characterising rigidity and flexibility of pseudo-Anosov and other transversally laminated dynamical systems on surfaces*, Warwick preprint 95.

**[5]** Pinto, A.A. and Rand, D.A. – Classifying $C^{1+}$ structures on dynamical fractals: 1 The moduli space of solenoid functions for Markov maps on train tracks, *Ergodic Theory and Dyn. Syst.* (1995), 697–734.

**[6]** Pinto, A.A. and Rand, D.A. – Classifying $C^{1+}$ structures on dynamical fractals: 2 Embedded trees, *Ergodic Theory and Dyn. Syst.* (1995), 969–992.

**[7]** Pinto, A.A. and Rand, R.A. – Global phase space universality, smooth conjugacies and renormalisation: 2. The $C^{k+\alpha}$ case using rapid convergence of Markov families, *Nonlinearity,* 4 (1991), 1–31.

**[8]** Pinto, A.A. and Rand, D.A. – *The renormalization dynamics and Teichmüller spaces of less smooth conjugacy classes of golden rotations.*

**[9]** Pinto, A.A. and Sullivan, D. – *The solenoid and the circle*, Manuscript, 1994.

**[10]** Sullivan, D. – *Differentiable structures on fractal-like sets determined by intrinsic scaling functions on dual Cantor sets*, Proceedings of Symposia in Pure Mathematics, Vol.48 (1988), American Mathematical Society.

**[11]** Sullivan, D. – *Linking the universalities of Milnor–Thurston, Feigenbaum and Ahlfors–Bers*, in "Topological Methods in Modern Mathematics" (L. Goldberg and A. Phillips, eds.), Publish or Perish, (1993), 543–563.

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