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EXISTENCE FOR QUASILINEAR ELLIPTIC SYSTEMS WITH QUADRATIC GROWTH HAVING A PARTICULAR STRUCTURE

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Abstract: In this paper, we consider the quasilinear elliptic system:

$$\begin{cases} -\displaystyle\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}\left(A_{ij}(x,u)\,\frac{\partial u^\gamma}{\partial x_j}\right) = \\ \qquad\qquad = G^\gamma(x,u,\nabla u) + F(x,u,\nabla u)\,Du^\gamma \quad \text{in } \mathcal{D}'(\Omega), \quad 1 \le \gamma \le m, \\ u \in (H_0^1(\Omega)\cap L^\infty(\Omega))^m \; . \end{cases}$$

The right hand side of this system consists of two parts: the first one, $G^\gamma(x,u,\nabla u)$, can have a quadratic growth in Du^δ for $\delta \le \gamma$, and possibly a small quadratic growth in Du^δ for $\delta > \gamma$; the second part is a coupling term with the particular structure $F(x,u,\nabla u)\,Du^\gamma$, where the nonlinearity F is the same for all the equations and can have linear growth in ∇u. We approximate the problem and assume that an L^∞-estimate on the approximated solutions is known. Without assuming any smallness on this L^∞-estimate we then prove that the approximations converge strongly in $(H_0^1(\Omega))^m$ and that the system admits at least one solution.

Introduction and results

In this paper we prove the existence of at least one solution for a quasilinear elliptic system whose right hand side has a quadratic growth with respect to the gradient but has a particular structure. More precisely, we consider the system

$$(1.1) \qquad \begin{cases} -\operatorname{div}(A(x,u)\,Du^\gamma) = \\ \qquad\qquad = G^\gamma(x,u,\nabla u) + F(x,u,\nabla u)\,Du^\gamma \quad \text{in } \mathcal{D}'(\Omega), \quad 1 \le \gamma \le m, \\ u \in (H_0^1(\Omega)\cap L^\infty(\Omega))^m \;, \end{cases}$$

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where Ω is a bounded open subset of $\mathbb{R}^N$, with boundary $\partial\Omega$ (no smoothness is assumed on $\partial\Omega$), where $u^\gamma : \Omega \to \mathbb{R}$ $(1 \le \gamma \le m)$ are the components of the unknown vector $\boldsymbol{u} = (u^1, ..., u^m)$, where $\nabla u : \Omega \to \mathbb{R}^{m\times N}$ is its gradients, i.e. the matrix whose γ-th row is the vector $Du^\gamma : \Omega \to \mathbb{R}^N$, and where $-\operatorname{div}(A(x,u)Du^\gamma) = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}(A_{ij}(x,u)\frac{\partial u^\gamma}{\partial x_j})$, with $A_{ij} \colon \Omega \times \mathbb{R}^m \to \mathbb{R}$ Carathéodory functions which satisfy for $\alpha > 0$ and $\beta > 0$:

$$\begin{cases} \text{a.e. } \; x \in \Omega, \quad \forall\, s \in \mathbb{R}^m, \quad \forall\, \xi \in \mathbb{R}^N \\ \displaystyle\sum_{i,j=1}^N A_{ij}(x,s)\, \xi_i\, \xi_j \ge \alpha\, |\xi|^2 \\ |A_{ij}(x,s)| \le \beta\ . \end{cases} \tag{1.2}$$

The functions $G^\gamma \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \to \mathbb{R}$ and $F \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \to \mathbb{R}^N$ are Carathéodory functions which satisfy:

$$|G^\gamma(x,s,\Xi)| \le C_0 + C_1 \sum_{\delta=1}^m |\xi^\delta| + C_2 \sum_{\delta=1}^\gamma |\xi^\delta|^2 + \eta \sum_{\delta=\gamma+1}^m |\xi^\delta|^2 , \quad 1 \le \gamma \le m\ , \tag{1.3}$$

$$|F(x,s,\Xi)| \le C_3 + C_4 |\Xi|\ , \tag{1.4}$$

where $\Xi = (\xi^1, ..., \xi^m) \in \mathbb{R}^{m\times N}$ with $\xi^\gamma \in \mathbb{R}^N$, and where C_0, C_1, C_2, C_3, C_4 and η are positive constants, η being small enough as precised later in hypothesis (1.10).

Assuming an L^∞-estimate on the solutions of a system which approximates (1.1), but without assuming any smallness of this L^∞-estimate, we will prove that problem (1.1) admits at least one solution. In fact, we will approximate problem (1.1) and prove that, whenever they are bounded in $(L^\infty(\Omega))^m$, the solutions of the approximated systems remain bounded and even compact in $(H_0^1(\Omega))^m$. We will then pass to the limit and obtain a solution of problem (1.1).

Approximation

For $\varepsilon > 0$, let $G_\varepsilon^\gamma(x,s,\Xi) \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \to \mathbb{R}$ and $F_\varepsilon(x,s,\Xi) \colon \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \to \mathbb{R}^N$ be Carathéodory functions such that:

$$\text{a.e. } \; x \in \Omega, \quad \forall\, s \in \mathbb{R}^m, \quad \forall\, \Xi \in \mathbb{R}^{m\times N}, \quad 1 \le \gamma \le m\ ,$$

$$|G_\varepsilon^\gamma(x,s,\Xi)| \le \frac{1}{\varepsilon}, \qquad |F_\varepsilon(x,s,\Xi)\, \xi^\gamma| \le \frac{1}{\varepsilon}\ , \tag{1.5}$$

$$|G_\varepsilon^\gamma(x,s,\Xi)| \le |G^\gamma(x,s,\Xi)|\, , \qquad |F_\varepsilon(x,s,\Xi)| \le |F(x,s,\Xi)|\ , \tag{1.6}$$

$$\begin{cases} G_\varepsilon^\gamma(x,s_\varepsilon,\Xi_\varepsilon) \to G^\gamma(x,s,\Xi)\, , \qquad F_\varepsilon(x,s_\varepsilon,\Xi_\varepsilon) \to F(x,s,\Xi) \\ \text{when } s_\varepsilon \to s \;\text{ in } \mathbb{R}^m \quad \text{ and } \quad \Xi_\varepsilon \to \Xi \;\text{ in } \mathbb{R}^{m\times N}\ . \end{cases} \tag{1.7}$$

Note that hypotheses (1.5), (1.6), (1.7) are satisfied for example when G^γ_ε and F_ε are defined by:

$$G^\gamma_\varepsilon(x,x,\Xi) = \frac{G^\gamma(x,s,\Xi)}{1+\varepsilon|G^\gamma(x,s,\Xi)|}\,, \qquad F_\varepsilon(x,s,\Xi) = \frac{F(x,s,\Xi)}{1+\varepsilon|F(x,s,\Xi)|\,|\Xi|}\ .$$

Now we consider the approximated problem:

$$(1.8)\qquad \begin{cases} -\operatorname{div}(A(x,u_\varepsilon)\,Du^\gamma_\varepsilon) = \\ \qquad\qquad = G^\gamma_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) + F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,Du^\gamma_\varepsilon \quad \text{in } \mathcal{D}'(\Omega), \quad 1\le\gamma\le m, \\ u^\gamma_\varepsilon \in (H^1_0(\Omega))^m\ . \end{cases}$$

In view of (1.5), an application of Schauder's fixed point theorem implies that problem (1.8) has at least one solution for $\varepsilon > 0$ given. Since the right hand side of each equation in (1.8) is bounded by $\frac{2}{\varepsilon}$, this solution belongs to $(L^\infty(\Omega))^m$ and satisfies $\|u^\gamma_\varepsilon\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon}$ for some constant C. We will from now on assume that we have the following $L^\infty(\Omega)$-estimate:

$$(1.9)\qquad \|u^\gamma_\varepsilon\|_{L^\infty(\Omega)} \le M\,, \qquad 1\le\gamma\le m\ ,$$

where M is independent of ε. Such an estimate can be proved in particular cases (see e.g. Theorem II.2 in A. Mokrane [4]).

We are now able to specify the smallness of the constant η which appears in the growth condition (1.3): we will assume that

$$(1.10)\qquad 0\le\eta\le\frac{C_2}{4}\left(\frac{1}{2m\exp(\frac{8C_2}{\alpha}M)}\right)^m\ .$$

We have the following theorem:

Theorem. *Under hypotheses (1.2), (1.3), (1.4), (1.5), (1.6), (1.7), (1.9), (1.10), problem (1.1) has at least one solution.*

Remark I.1. In the case $m = 2$ this existence result has been be proved in A. Bensoussan and J. Frehse [1]. For $m \ge 3$, the result has been announced in J. Frehse [3]. We prove here the Theorem using a method inspired by L. Boccardo, F. Murat and J.P. Puel [2], where the system (1.1) is studied under the stronger hypothesis that $|G^\gamma(x,s,\Xi)| \le b(|s|)\,(1+|\Xi|)$ where $b\colon \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing function.

Remark I.2. The second order operator $u^\gamma \to -\operatorname{div}(A(x,u)Du^\gamma)$ is the same for all the equations. On the other hand the coupling between the equations takes place mainly through the term $F(x,u,\nabla u)\,Du^\gamma$ where the nonlinearity

F has a linear growth in $|\nabla u|$ (note that F is the same for all the equations), and secondarily through the principal part of the operator (the matrix A depends on u) and through the term $G^\gamma(x,u,\nabla u)$ (which has a quadratic growth in $Du^1, Du^2, ..., Du^\gamma$). Note that if we neglect the coupling term $F(x,u,\nabla u)\,Du^\gamma$ and if we assume $\eta = 0$, the right hand side of the first equation has a quadratic growth only in Du^1, the right hand side of the second equation has a quadratic growth in Du^1 and Du^2, etc., until the last equation which has a quadratic growth in the whole gradient ∇u.

Remark I.3. From hypotheses (1.3) and (1.4) we deduce that H^γ defined by

$$H^\gamma(x,s,\Xi) = G^\gamma(x,s,\Xi) + F(x,s,\Xi)\,\xi^\gamma \ , \tag{1.11}$$

where $\Xi = (\xi^1, ..., \xi^m) \in \mathbb{R}^{m\times N}$ satisfies

$$\begin{aligned} |H^\gamma(x,s,\Xi)| \le C_0 + C_1 \sum_{\delta=1}^{m} |\xi^\delta| + C_2 \sum_{\delta=1}^{\gamma} |\xi^\delta|^2 + \eta \sum_{\delta=\gamma+1}^{m} |\xi^\delta|^2 \\ + [C_3 + C_4\,|\Xi|]\,|\xi^\gamma| \ . \end{aligned} \tag{1.12}$$

In the case $m = 2$ (i.e. two equations, and two unknowns u_1 and u_2) (1.12) implies that

$$\begin{cases} |H^1(x,s,\Xi)| \le C_0' + C_1'[|\xi^1| + |\xi^2|] + C_2'|\xi^1|^2 + \eta|\xi^2|^2 + C_4'|\xi^2|\,|\xi^1| \\ |H^2(x,s,\Xi)| \le C_0'' + C_2''[|\xi^1|^2 + |\xi^2|^2] \ , \end{cases} \tag{1.13}$$

where the constants C_0', C_1', C_2', C_4', C_0'', C_2'' do not depend on η.

We will prove in the present Remark that in the special case $m = 2$, whenever the functions H^1 and H^2 satisfy (1.13), then they can be written under the form (1.11), where G^1, G^2 and F satisfy (1.3) and (1.4); this will not be the case in general when $m \ge 3$ (see Remark I.4 below).

Indeed define

$$K(x,s,\Xi) = C_0' + C_1'[|\xi^1| + |\xi^2|] + C_2'|\xi^1|^2 + \eta|\xi^2|^2 + C_4'|\xi^2|\,|\xi^1| \ ,$$

$$F(x,s,\Xi) = C_4' \,\frac{H^1(x,s,\Xi)}{K(x,s,\Xi)}\,|\xi^2|\,\frac{\psi(|\xi^1|)}{|\xi^1|}\,\xi^1 \ ,$$

where $\psi\colon \mathbb{R} \to \mathbb{R}$ is a smooth function such that $0 \le \psi(t) \le 1$ for all t, $\psi(t) = 0$ if $|t| \le \frac{1}{2}$ and $\psi(t) = 1$ if $|t| \ge 1$. Define also

$$G^1(x,s,\Xi) = H^1(x,s,\Xi) - F(x,s,\Xi)\,\xi^1 \ ,$$

$$G^2(x,s,\Xi) = H^2(x,s,\Xi) - F(x,s,\Xi)\,\xi^2 \ .$$

Then F, G^1 and G^2 are Carathéodory functions which satisfy (1.11); moreover we have

$$|F(x,s,\Xi)| \le C_4'|\xi^2| \le C_4'|\Xi|, \quad \text{i.e. } (1.4)\ .$$

On the other hand

$$G^1(x,s,\Xi) = \frac{H^1(x,s,\Xi)}{K(x,s,\Xi)}\Big[K(x,s,\Xi) - C_4'|\xi^2|\,|\xi^1|\,\psi(|\xi^1|)\Big] =$$
$$= \frac{H^1(x,s,\Xi)}{K(x,s,\Xi)}\Big[C_0' + C_1'[|\xi^1|+|\xi^2|] + C_2'|\xi^1|^2 + \eta|\xi^2|^2 + C_4'|\xi^2|\,|\xi^1|\,\{1-\psi(|\xi^1|)\}\Big]$$

so that, in view of the properties of ψ,

$$|G^1(x,s,\Xi)| \le C_0' + C_1'[|\xi^1|+|\xi^2|] + C_2'|\xi^1|^2 + \eta|\xi^2|^2 + C_4'|\xi^2|, \quad \text{i.e. } (1.3) \text{ for } G^1\ .$$

Finally

$$\begin{aligned}|G^2(x,s,\Xi)| &\le |H^2(x,s,\Xi)| + |F(x,s,\Xi)|\,|\xi^2| \\ &\le C_0'' + C_2''[|\xi^1|^2 + |\xi^2|^2] + C_4'|\xi^2|^2, \quad \text{i.e. } (1.3) \text{ for } G^2\ .\end{aligned}$$

Remark I.4. Let us now prove that if $m \ge 3$, and if H^γ satisfy (1.12), then it can not in general be written under the form (1.11) with G^γ and F satisfying (1.3) and (1.4).

Consider for that the special case where $m = 3$, $N = 1$ (the ξ^γ are therefore scalars) and where

$$H^\gamma(x,s,\Xi) = a^\gamma\,|\xi^3|\,\xi^\gamma\,, \quad \gamma = 1,2,3\ , \tag{1.14}$$

with $a^\gamma \ne 0$, $a^1 \ne a^2$. Then H^γ satisfies (1.12) with $\eta = 0$.

If H^1 could be written under the form (1.11), with G^1 satisfying (1.3), we would have

$$G^1(x,s,\Xi) = H^1(x,s,\Xi) - F(x,s,\Xi)\,\xi^1 = \Big[a^1|\xi^3| - F(x,s,\Xi)\Big]\,\xi^1\ .$$

Since the growth condition (1.3) on G^1 does not allow G^1 to have a term of the form $|\xi^3|\,|\xi^1|$ (indeed, use of Young's inequality would give $|\xi^1|\,|\xi^3| \le \frac{\eta}{2}|\xi^3|^2 + \frac{1}{2\eta}|\xi^1|^2$, but here $C_2 = \frac{1}{2\eta}$ would depend on η), this implies that

$$a_1|\xi^3| - |F(x,s,\Xi)| \le C \quad \text{when } |\Xi| \text{ is large}\ .$$

Similarly if H^2 can be written under the form (1.11), with G^2 satisfying (1.3), we will have

$$\begin{aligned} G^2(x,s,\Xi) &= H^2(x,s,\Xi) - F(x,s,\Xi)\,\xi^2 \\ &= \Big[a^2|\xi^3| - F(x,s,\Xi)\Big]\,\xi^2 \\ &= [a^2-a^1]\,|\xi^3|\,\xi^2 + \Big[a_1|\xi^3| - F(x,s,\Xi)\Big]\,\xi^2 \ . \end{aligned}$$

But again the growth condition (1.3) on G^2 does not allow G^2 to have a term of the form $|\xi^3|\,|\xi^2|$. If $m=3$ and if H^1 and H^2 are given by (1.14) it is thus impossible to write H^γ under the form (1.11).

II – Proof of the Theorem

The proof of the Theorem will be performed in three steps: we will first prove an $(H_0^1(\Omega))^m$-estimate for u_ε, then the strong convergence in $(H_0^1(\Omega))^m$ of u_ε, and finally we will pass to the limit in the approximated problem (1.8).

II.1. $(H_0^1(\Omega))^m$-estimate

We have the following:

Proposition II.1. *Assume that (1.2), (1.3), (1.4) and (1.6) hold true. If the solutions u^ε of the approximated problem (1.8) satisfy (1.9), and if η satisfies*

$$0 \le \eta \le \frac{C_2}{4}\left(\frac{1}{2m\exp(\frac{2C_2}{\alpha}M)}\right)^m , \tag{2.1}$$

then u_ε remains bounded in $(H_0^1(\Omega))^m$.

Note that $\varphi\colon \mathbb{R}\to\mathbb{R}$ satisfies (2.1) as soon as (1.10) is satisfied.

Proof of Proposition II.1: Consider the test function(*)

$$v_\varepsilon^\gamma = (a)^\gamma\,\varphi'(u_\varepsilon^\gamma)\exp[\mu\,\psi(u_\varepsilon)] \ ,$$

where $\varphi\colon \mathbb{R}\to\mathbb{R}$ and $\psi\colon \mathbb{R}\to\mathbb{R}$ are defined by

$$\varphi(t) = e^{\lambda t} + e^{-\lambda t} - 2, \quad \forall\, t\in\mathbb{R}\,, \qquad \psi(s) = \sum_{\gamma=1}^m (a)^\gamma\,\varphi(s^\gamma), \quad \forall\, s\in\mathbb{R}^m \ , \tag{2.2}$$

(*) In the notation $(a)^\gamma$, γ denotes a power and not a superscript as it does in a^γ.

and where λ, μ and a are positive constants that we choose as

$$(2.3)\qquad \begin{cases} \lambda = \dfrac{2C_2}{\alpha}, \quad a = \dfrac{1}{2\,m\,e^{\lambda M}}, \quad \mu = \dfrac{C_3^2}{2\theta\alpha} + \dfrac{C_4^2}{2\theta\alpha}, \\ \text{where } \theta \text{ is any fixed number such that } 0 < \theta \le (a)^m\,\lambda\,\dfrac{C_2}{4}\ . \end{cases}$$

Since u_ε belongs to $(H_0^1(\Omega)\cap L^\infty(\Omega))^m$, the test function v_ε^γ belongs to $H_0^1(\Omega)$ and defining ψ_ε by $\psi_\varepsilon = \psi(u_\varepsilon)$, we have

$$(2.4)\qquad Dv_\varepsilon^\gamma = Du_\varepsilon^\gamma(a^\gamma)\,\varphi''(u_\varepsilon^\gamma)\exp[\mu\,\psi_\varepsilon] + \mu\,D\psi_\varepsilon(a)^\gamma\,\varphi'(u_\varepsilon^\gamma)\exp[\mu\,\psi_\varepsilon]\ .$$

We use v_ε^γ as test function in the γ-th equation of system (1.8) and sum up from $\gamma = 1$ to $\gamma = m$. We obtain:

$$(2.5)\qquad \sum_{\gamma=1}^m \int_\Omega A(x,u_\varepsilon)\,Du_\varepsilon^\gamma\,Du_\varepsilon^\gamma(a)^\gamma\,\varphi''(u_\varepsilon^\gamma)\exp[\mu\,\psi_\varepsilon]\,dx + {}$$
$$+\,\mu\sum_{\gamma=1}^m \int_\Omega A(x,u_\varepsilon)\,Du_\varepsilon^\gamma\,D\psi_\varepsilon(a)^\gamma\,\varphi'(u_\varepsilon^\gamma)\exp[\mu\,\psi_\varepsilon]\,dx =$$
$$= \sum_{\gamma=1}^m \int_\Omega G_\varepsilon^\gamma(x,u_\varepsilon,\nabla u_\varepsilon)(a)^\gamma\,\varphi'(u_\varepsilon^\gamma)\exp[\mu\,\psi_\varepsilon]\,dx$$
$$+ \sum_{\gamma=1}^m \int_\Omega F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,Du_\varepsilon^\gamma(a)^\gamma\,\varphi'(u_\varepsilon^\gamma)\exp[\mu\,\psi_\varepsilon]\,dx\ .$$

Noting that:

$$(2.6)\qquad D\psi_\varepsilon = \sum_{\gamma=1}^m (a)^\gamma\,\varphi'(u_\varepsilon^\gamma)\,Du_\varepsilon^\gamma\ ,$$

and using the coercivity condition (1.2) and the growth conditions (1.6), (1.3) on G_ε^γ we obtain:

$$(2.7)\qquad \alpha\sum_{\gamma=1}^m \int_\Omega |Du_\varepsilon^\gamma|^2\,(a)^\gamma\,\varphi''(u_\varepsilon^\gamma)\exp[\mu\,\psi_\varepsilon]\,dx + \alpha\,\mu\int_\Omega |D\psi_\varepsilon|^2\exp[\mu\,\psi_\varepsilon]\,dx \le$$
$$\le \sum_{\gamma=1}^m \int_\Omega \Big[C_0 + C_1\sum_{\delta=1}^m |Du_\varepsilon^\delta| + C_2\sum_{\delta=1}^\gamma |Du_\varepsilon^\delta|^2 + \eta\sum_{\delta=\gamma+1}^m |Du_\varepsilon^\delta|^2\Big](a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)|\exp[\mu\,\psi_\varepsilon]\,dx$$
$$+ \int_\Omega F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,D\psi_\varepsilon\exp[\mu\,\psi_\varepsilon]\,dx\ .$$

We estimate the second integral of the right hand side of (2.7) by using the growth conditions (1.6), (1.4) on F_ε and Youngs inequality. We obtain:

$$\int_\Omega F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,D\psi_\varepsilon \exp[\mu\,\psi_\varepsilon]\,dx \le \int_\Omega \Big[C_3 + C_4|\nabla u_\varepsilon|\Big]\,|D\psi_\varepsilon| \exp[\mu\,\psi_\varepsilon]\,dx \le$$

$$\begin{aligned}
&\le \int_\Omega \Big[\frac{\theta}{2} + \frac{C_3^2}{2\,\theta}\,|D\psi_\varepsilon|^2 + \frac{\theta}{2}\,|\nabla u_\varepsilon|^2 + \frac{C_4^2}{2\,\theta}\,|D\psi_\varepsilon|^2\Big] \exp[\mu\,\psi_\varepsilon]\,dx \\
&= \int_\Omega \Big[\frac{\theta}{2} + \Big(\frac{C_3^2}{2\theta} + \frac{C_4^2}{2\theta}\Big)\,|D\psi_\varepsilon|^2 + \frac{\theta}{2}\,|\nabla u_\varepsilon|^2\Big] \exp[\mu\,\psi_\varepsilon]\,dx\ .
\end{aligned} \tag{2.8}$$

We now estimate various terms of the first integral of the right hand side of (2.7); for what concerns the third term, we have, splitting the sum into $\delta=\gamma$ and $\delta<\gamma$, then reversing the order of $\sum_\gamma$ and $\sum_\delta$:

$$C_2 \sum_{\gamma=1}^m \sum_{\delta=1}^{\gamma} |Du_\varepsilon^\delta|^2\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| =$$

$$\begin{aligned}
&= C_2 \sum_{\gamma=1}^m |Du_\varepsilon^\gamma|^2\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| + C_2 \sum_{\gamma=1}^m \sum_{\delta=1}^{\gamma-1} |Du_\varepsilon^\delta|^2\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| \\
&= C_2 \sum_{\gamma=1}^m |Du_\varepsilon^\gamma|^2\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| + C_2 \sum_{\delta=1}^m \sum_{\gamma=\delta+1} |Du_\varepsilon^\delta|^2\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| \\
&= C_2 \sum_{\gamma=1}^m |Du_\varepsilon^\gamma|^2\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| + C_2 \sum_{\gamma=1}^m \sum_{\delta=\gamma+1}^m |Du_\varepsilon^\gamma|^2\,(a)^\delta\,|\varphi'(u_\varepsilon^\delta)|\ ;
\end{aligned} \tag{2.9}$$

for the fourth term we write:

$$\begin{aligned}
\eta \sum_{\gamma=1}^m \sum_{\delta=\gamma+1}^m |Du_\varepsilon^\delta|^2\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| &\le \eta \sum_{\gamma=1}^m \sum_{\delta=1}^m |Du_\varepsilon^\delta|^2\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| \\
&= \eta \sum_{\gamma=1}^m \sum_{\delta=1}^m |Du_\varepsilon^\gamma|^2\,(a)^\delta\,|\varphi'(u_\varepsilon^\delta)|\ .
\end{aligned} \tag{2.10}$$

Using (2.8), (2.9) and (2.10), inequality (2.7) becomes:

$$\begin{aligned}
&\sum_{\gamma=1}^m \int_\Omega |Du_\varepsilon^\gamma|^2 \exp[\mu\,\psi_\varepsilon] \Big\{\alpha(a)^\gamma\,\varphi''(u_\varepsilon^\gamma) - C_2(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| - \\
&\quad - C_2 \sum_{\delta=\gamma+1}^m (a)^\delta\,|\varphi'(u_\varepsilon^\delta)| - \eta \sum_{\delta=1}^m (a)^\delta\,|\varphi'(u_\varepsilon^\delta)| - \frac{\theta}{2}\Big\}\,dx +
\end{aligned} \tag{2.11}$$

$$+\int_\Omega |D\psi_\varepsilon|^2 \exp[\mu\,\psi_\varepsilon]\left\{\alpha\,\mu - \frac{C_3^2}{2\theta} - \frac{C_4^2}{2\theta}\right\} dx \leq$$

$$\leq \int_\Omega \frac{\theta}{2}\exp[\mu\,\psi_\varepsilon]\,dx + \sum_{\gamma=1}^m \int_\Omega \Big[C_0 + C_1 \sum_{\delta=1}^m |Du_\varepsilon^\delta|\Big]\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)|\exp[\mu\,\psi_\varepsilon]\,dx\ .$$

In view of the choices made in (2.3), of the hypothesis (2.1) made on η, and of Lemma II.1 that we state and prove below, we deduce from (2.11) that we have, for α_0 given by (2.15):

$$\alpha_0 \sum_{\gamma=1}^m \int_\Omega |Du_\varepsilon^\gamma|^2 \exp[\mu\,\psi_\varepsilon]\,dx \leq \tag{2.12}$$

$$\leq \int_\Omega \frac{\theta}{2}\exp[\mu\,\psi_\varepsilon]\,dx + \sum_{\gamma=1}^m \int_\Omega \Big[C_0 + C_1 \sum_{\delta=1}^m |Du_\varepsilon^\delta|\Big]\,(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)|\exp[\mu\,\psi_\varepsilon]\,dx\ ,$$

which using Young's inequality and the facts that $\exp[\mu\psi_\varepsilon] \geq 1$ and that $\|u_\varepsilon^\gamma\|_{L^\infty(\Omega)} \leq M$ (which implies that ψ_ε is bounded in $L^\infty(\Omega)$), implies that u_ε is bounded in $(H_0^1(\Omega))^m$. Proposition II.1 is proved. ∎

Lemma II.1. *Let λ, a, θ and η be such that*

$$\lambda = \frac{2\,C_2}{\alpha}, \quad a = \frac{1}{2\,m\,e^{\lambda M}}, \quad 0 < \theta \leq (a)^m\,\lambda\,\frac{C_2}{4}, \quad 0 \leq \eta \leq (a)^m\,\frac{C_2}{4}\ . \tag{2.13}$$

Then for any γ, $1 \leq \gamma \leq m$, and for any u_ε such that $|u_\varepsilon^\delta| \leq M$ for any δ, we have

$$\alpha(a)^\gamma\,\varphi''(u_\varepsilon^\gamma) - C_2(a)^\gamma\,|\varphi'(u_\varepsilon^\gamma)| - C_2 \sum_{\delta=\gamma+1}^m (a)^\delta\,|\varphi'(u_\varepsilon^\delta)| - \\ - \eta\sum_{\delta=1}^m (a)^\delta\,|\varphi'(u_\varepsilon^\delta)| - \frac{\theta}{2} \geq \alpha_0 \tag{2.14}$$

where α_0 is defined by

$$\alpha_0 = (a)^m\,\lambda\,\frac{C_2}{4}\ . \tag{2.15}$$

Proof of Lemma II.1: Since we have

$$\begin{cases} \forall\, t, \quad |t| \leq M, \quad |\varphi'(t)| \leq \lambda|e^{\lambda t} - e^{-\lambda t}| \leq \lambda\,e^{\lambda|t|} \leq \lambda\,e^{\lambda M}, \\ \varphi''(t) = \lambda^2(e^{\lambda t} + e^{-\lambda t}) \geq \lambda^2\,e^{\lambda|t|}\ , \end{cases}$$

we obtain for $1 \le \gamma \le m$ and for any u_ε with $|u_\varepsilon^\delta| \le M$, $1 \le \delta \le m$,

$$\alpha(a)^\gamma \varphi''(u_\varepsilon^\gamma) - C_2(a)^\gamma |\varphi'(u_\varepsilon^\gamma)| - C_2 \sum_{\delta=\gamma+1}^{m} (a)^\delta |\varphi'(u_\varepsilon^\delta)| - \eta \sum_{\delta=1}^{m} (a)^\delta |\varphi'(u_\varepsilon^\delta)| - \frac{\theta}{2} \ge \tag{2.16}$$
$$\ge \alpha(a)^\gamma \lambda^2 e^{\lambda |u_\varepsilon^\gamma|} - C_2(a)^\gamma \lambda e^{\lambda |u_\varepsilon^\gamma|} - C_2 \sum_{\delta=\gamma+1}^{m} (a)^\delta \lambda e^{\lambda M} - \eta \sum_{\delta=1}^{m} (a)^\delta \lambda e^{\lambda M} - \frac{\theta}{2} \,.$$

Since $\alpha \lambda^2 > C_2 \lambda$, the infinimum of the right hand side of (2.16) is achieved for $|u_\varepsilon^\gamma| = 0$; using also the fact that

$$0 < (a)^m < (a)^{m-1} < ... < (a)^{\gamma+1} < (a)^\gamma < ... < (a)^1 < (a)^0 = 1$$

we estimate from below the right hand side of (2.16) by

$$\alpha(a)^\gamma \lambda^2 - C_2(a)^\gamma \lambda - C_2\, m(a)^{\gamma+1} \lambda e^{\lambda M} - \eta\, m\, a\, \lambda e^{\lambda M} - \frac{\theta}{2} = \tag{2.17}$$
$$= (a)^\gamma \lambda \Big[\alpha \lambda - C_2 \lambda - C_2\, m\, a\, e^{\lambda M}\Big] - \Big[\eta\, m\, a\, \lambda e^{\lambda M} + \frac{\theta}{2}\Big] \,.$$

In view of the values of λ, a, θ and η given by (2.13), the right hand side of (2.17) is greater than

$$(a)^\gamma \lambda \frac{C^2}{2} - \Big[\eta\, m\, a\, \lambda e^{\lambda M} + \frac{\theta}{2}\Big] \ge \tag{2.18}$$
$$\ge (a)^m \lambda \frac{C_2}{2} - \Big[(a)^m \frac{C_2}{4} \frac{\lambda}{2} + \frac{1}{2} (a)^m \lambda \frac{C_2}{4}\Big] = (a)^m \lambda \frac{C_2}{4} \,,$$

which is α_0 by definition (2.15). Lemma II.1 is proved.

Remark II.1. In the proofs of Proposition II.1 and of Lemma II.1, θ is a fixed number such that $0 < \theta \le (a)^m \lambda \frac{C_2}{4}$. Actually, in these two proofs, we could have chosen $\theta = (a)^m \lambda \frac{C_2}{4}$. But it will be important in the proof of Proposition II.2 to have the possibility of choosing θ as small as we want.

II.2. Strong convergence in $(H_0^1(\Omega))^m$

Since by Proposition II.1 u_ε remains bounded in $(H_0^1(\Omega))^m$, we can extract a subsequence, still denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \quad \text{in} \quad (H_0^1(\Omega))^m \text{ weak} \,. \tag{2.19}$$

Proposition II.2. *Assume that (1.2), (1.3), (1.4) and (1.6) hold true. If the solutions u_ε of the approximated problem (1.8) satisfy (1.9) and (2.19), and if η satisfies*

$$0 \le \eta \le \frac{C_2}{4}\left(\frac{1}{2\,m\exp(\frac{8C_2}{\alpha}M)}\right)^m, \qquad \textit{(i.e. (1.10))}$$

then u_ε converges strongly to u in $(H_0^1(\Omega))^m$.

Proof of Proposition II.2: Let us set $\overline{u}_\varepsilon^\gamma = u_\varepsilon^\gamma - u^\gamma$, and write the system (1.8) under the form:

$$\begin{aligned}(2.20)\qquad & -\operatorname{div}\Big(A(x,u_\varepsilon)\,D\overline{u}_\varepsilon^\gamma\Big) - \operatorname{div}\Big(A(x,u_\varepsilon)\,Du^\gamma\Big) = \\ &= G_\varepsilon^\gamma(x,u_\varepsilon,\nabla u_\varepsilon) + F(x,u_\varepsilon,\nabla u_\varepsilon)\,D\overline{u}_\varepsilon^\gamma + F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,Du^\gamma, \quad 1\le\gamma\le m\ .\end{aligned}$$

We use in he γ-th equation of system (2.20) the test function:

$$\overline{v}_\varepsilon^\gamma = (\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon], \qquad \text{with } \ \overline{\psi}_\varepsilon = \overline{\psi}(\overline{u}_\varepsilon)\ ,$$

where $\overline{\varphi}\colon \mathbb{R}\to\mathbb{R}$ and $\overline{\psi}\colon \mathbb{R}\to\mathbb{R}$ are defined by

$$(2.21)\quad \overline{\varphi}(\overline{t}) = e^{\overline{\lambda}\,\overline{t}} + e^{-\overline{\lambda}\,\overline{t}} - 2, \quad \forall\,\overline{t}\in\mathbb{R}\,, \qquad \overline{\psi}(\overline{s}) = \sum_{\gamma=1}^m (\overline{a})^\gamma\,\overline{\varphi}(\overline{s}^\gamma), \quad \forall\,\overline{s}\in\mathbb{R}^m\ ,$$

and where $\overline{\lambda}$, $\overline{\mu}$ and $\overline{a}$ are positive constants that we choose as

$$(2.22)\qquad \begin{cases} \overline{\lambda} = \dfrac{4C_2}{\alpha}, \quad \overline{a} = \dfrac{1}{2\,m\,e^{2\overline{\lambda}M}}, \quad \overline{\mu} = \dfrac{C_3^2}{2\theta\alpha} + \dfrac{C_4^2}{2\theta\alpha}, \\ \text{where } \overline{\theta} \text{ is any fixed number such that } 0<\overline{\theta}\le(\overline{a})^m\,\overline{\lambda}\,\dfrac{C_2}{2}\ . \end{cases}$$

Summing up from $\gamma=1$ to $\gamma=m$, we obtain:

$$\begin{aligned}(2.23)\quad & \sum_{\gamma=1}^m \int_\Omega A(x,u_\varepsilon)\,D\overline{u}_\varepsilon^\gamma\,D\overline{u}_\varepsilon^\gamma(\overline{a})^\gamma\,\overline{\varphi}''(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\,+ \\ &\qquad + \overline{\mu}\sum_{\gamma=1}^m \int_\Omega A(x,u_\varepsilon)\,D\overline{u}_\varepsilon^\gamma\,D\overline{\psi}_\varepsilon(\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx \\ &\qquad + \sum_{\gamma=1}^m \int_\Omega A(x,u_\varepsilon)\,Du^\gamma\,D\overline{u}_\varepsilon^\gamma(\overline{a})^\gamma\,\overline{\varphi}''(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx \\ &\qquad + \overline{\mu}\sum_{\gamma=1}^m \int_\Omega A(x,u_\varepsilon)\,Du^\gamma\,D\overline{\psi}_\varepsilon(\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx =\end{aligned}$$

$$= \sum_{\gamma=1}^{m} \int_\Omega G_\varepsilon^\gamma(x,u_\varepsilon,\nabla u_\varepsilon)(\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx$$

$$+ \int_\Omega F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,D\overline{u}_\varepsilon^\gamma(\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx$$

$$+ \sum_{\gamma=1}^{m} \int_\Omega F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,Du^\gamma(\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\ .$$

Using the coercivity condition (1.2) and the growth condition (1.6), (1.3), and the fact that

$$D\overline{\psi}_\varepsilon = \sum_{\gamma=1}^{m} (\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\,D\overline{u}_\varepsilon^\gamma\ ,$$

we have:

$$\alpha\sum_{\gamma=1}^{m}\int_\Omega |D\overline{u}_\varepsilon^\gamma|^2(\overline{a})^\gamma\,\overline{\varphi}''(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx + \alpha\,\overline{\mu}\int_\Omega |D\overline{\psi}_\varepsilon|^2\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx \le \tag{2.24}$$

$$\le -\sum_{\gamma=1}^{m}\int_\Omega A(x,u_\varepsilon)\,Du^\gamma\,D\overline{u}_\varepsilon^\gamma(\overline{a})^\gamma\,\overline{\varphi}''(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx$$

$$-\,\overline{\mu}\sum_{\gamma=1}^{m}\int_\Omega A(x,u_\varepsilon)\,Du^\gamma\,D\overline{\psi}_\varepsilon(\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx$$

$$+\sum_{\gamma=1}^{m}\int_\Omega\Big[C_0 + C_1\sum_{\delta=1}^{m}|Du_\varepsilon^\delta| + C_2\sum_{\delta=1}^{\gamma}|Du_\varepsilon^\delta|^2 +$$

$$+\,\eta\sum_{\delta=\gamma+1}^{m}|Du_\varepsilon^\delta|^2\Big](\overline{a})^\gamma\,|\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)|\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx$$

$$+\int_\Omega F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,D\overline{\psi}_\varepsilon\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx$$

$$+\sum_{\gamma=1}^{m}\int_\Omega F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,Du^\gamma(\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\ .$$

We estimate the fourth integral of the right hand sided of (2.24) by using the growth conditions (1.6), (1.4) on F_ε and Young's inequality, as well as $(a+b)^2 \le 2a^2+2b^2$. We obtain:

$$\int_\Omega F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,D\overline{\psi}_\varepsilon\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx \le \tag{2.25}$$

$$\leq \int_\Omega \Big(C_3 + C_4|\nabla u_\varepsilon|\Big)\, |D\overline{\psi}_\varepsilon| \, \exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\, dx$$

$$\leq \int_\Omega \left[\frac{\overline{\theta}}{2} + \frac{C_3^2}{2\overline{\theta}}\, |D\overline{\psi}_\varepsilon|^2 + \frac{\overline{\theta}}{2}\, |\nabla u_\varepsilon|^2 + \frac{C_4^2}{2\overline{\theta}}\, |D\overline{\psi}_\varepsilon|^2\right] \exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\, dx$$

$$\leq \int_\Omega \left[\overline{\theta}\Big(\frac{1}{2} + |\nabla u|^2\Big) + \Big(\frac{C_3^2}{2\overline{\theta}} + \frac{C_4^2}{2\overline{\theta}}\Big)\, |D\overline{\psi}_\varepsilon|^2 + \overline{\theta}|\nabla \overline{u}_\varepsilon|^2\right] \exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\, dx \ .$$

We now estimate various terms of the third integral of the right hand side of (2.24); for what concerns the third term, we have, as in (2.9), (splitting the sum into $\delta = \gamma$ and $\delta < \gamma$, then reversing the order of and $\sum_\gamma$ and $\sum_\delta$), and then using $(a+b)^2 \leq 2a^2 + 2b^2$:

$$C_2 \sum_{\gamma=1}^{m} \sum_{\delta=1}^{\gamma} |Du_\varepsilon^\delta|^2 (\overline{a})^\gamma\, |\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)| = \tag{2.26}$$

$$= C_2 \sum_{\gamma=1}^{m} |Du_\varepsilon^\gamma|^2 (\overline{a})^\gamma\, |\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)| + C_2 \sum_{\gamma=1}^{m} \sum_{\delta=\gamma+1}^{m} |Du_\varepsilon^\gamma|^2 (\overline{a})^\delta\, |\overline{\varphi}'(\overline{u}_\varepsilon^\delta)|$$

$$\leq 2C_2 \sum_{\gamma=1}^{m} |D\overline{u}_\varepsilon^\gamma|^2 (\overline{a})^\gamma |\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)| + 2C_2 \sum_{\gamma=1}^{m} |Du^\gamma|^2 (\overline{a})^\gamma\, |\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)|$$

$$+ 2C_2 \sum_{\gamma=1}^{m} \sum_{\delta=\gamma+1}^{m} |D\overline{u}_\varepsilon^\gamma|^2 (\overline{a})^\delta\, |\overline{\varphi}'(\overline{u}_\varepsilon^\delta)| + 2C_2 \sum_{\gamma=1}^{m} \sum_{\delta=\gamma+1}^{m} |Du^\gamma|^2 (\overline{a})^\delta\, |\overline{\varphi}'(\overline{u}_\varepsilon^\delta)| \ ;$$

for the fourth term we write:

$$\eta \sum_{\gamma=1}^{m} \sum_{\delta=\gamma+1}^{m} |Du_\varepsilon^\delta|^2 (\overline{a})^\gamma\, |\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)| \leq \tag{2.27}$$

$$\leq \eta \sum_{\gamma=1}^{m} \sum_{\delta=1}^{m} |Du_\varepsilon^\delta|^2 (\overline{a})^\gamma\, |\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)| = \eta \sum_{\gamma=1}^{m} \sum_{\delta=1}^{m} |Du_\varepsilon^\gamma|^2 (\overline{a})^\delta\, |\overline{\varphi}'(\overline{u}_\varepsilon^\delta)|$$

$$\leq 2\eta \sum_{\gamma=1}^{m} \sum_{\delta=1}^{m} |D\overline{u}_\varepsilon^\gamma|^2 (\overline{a})^\delta\, |\overline{\varphi}'(\overline{u}_\varepsilon^\delta)| + 2\eta \sum_{\gamma=1}^{m} \sum_{\delta=1}^{m} |Du^\gamma|^2 (\overline{a})^\delta\, |\overline{\varphi}'(\overline{u}_\varepsilon^\delta)| \ .$$

Using (2.25), (2.26) and (2.27), inequality (2.24) becomes:

$$\sum_{\gamma=1}^{m} \int |D\overline{u}_\varepsilon^\gamma|^2 \exp[\overline{\mu}\,\overline{\psi}_\varepsilon] \bigg\{ \alpha(\overline{a})^\gamma\, \overline{\varphi}''(\overline{u}_\varepsilon^\gamma) - 2C_2(\overline{a})^\gamma |\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)| - \tag{2.28}$$

$$- 2C_2 \sum_{\delta=\gamma+1}^{m} (\overline{a})^\delta\, |\overline{\varphi}'(\overline{u}_\varepsilon^\delta)| - 2\eta \sum_{\delta=1}^{m} (\overline{a})^\delta\, |\overline{\varphi}'(\overline{u}_\varepsilon^\delta)| - \overline{\theta} \bigg\}\, dx +$$

$$+\int_\Omega |D\overline{\psi}_\varepsilon|^2 \exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\Big(\alpha\overline{\mu} - \frac{C_3^2}{2\overline{\theta}} - \frac{C_4^2}{2\overline{\theta}}\Big)\,dx \le$$

$$\le \overline{\theta}\int_\Omega \Big(\frac{1}{2} + |\nabla u|^2\Big)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx + R_\varepsilon\ ,$$

where R_ε is defined by:

$$\begin{aligned}
R_\varepsilon = &-\sum_{\gamma=1}^m \int_\Omega A(x,u_\varepsilon)\,Du^\gamma\,D\overline{u}_\varepsilon^\gamma(\overline{a})^\gamma\,\overline{\varphi}''(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\\
&-\overline{\mu}\sum_{\gamma=1}^m \int_\Omega A(x,u_\varepsilon)\,Du^\gamma\,D\overline{\psi}_\varepsilon(\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\\
&+\sum_{\gamma=1}^m \int_\Omega \Big[C_0 + C_1\sum_{\delta=1}^m |Du_\varepsilon^\delta|\Big](\overline{a})^\gamma\,|\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)|\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\\
&+\sum_{\gamma=1}^m \int_\Omega F_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\,Du^\gamma(\overline{a})^\gamma\,\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\\
&+2C_2\sum_{\gamma=1}^m \int_\Omega |Du^\gamma|^2(\overline{a})^\gamma\,|\overline{\varphi}'(\overline{u}_\varepsilon^\gamma)|\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\\
&+2C_2\sum_{\gamma=1}^m\sum_{\delta=\gamma+1}^m \int_\Omega |Du^\gamma|^2(\overline{a})^\delta\,|\overline{\varphi}'(\overline{u}_\varepsilon^\delta)|\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\\
&+2\eta\sum_{\gamma=1}^m\sum_{\delta=1}^m \int_\Omega |Du^\gamma|^2(\overline{a})^\delta\,|\overline{\varphi}'(\overline{u}_\varepsilon^\delta)|\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx\ .
\end{aligned}\tag{2.29}$$

We now apply to the first integral of (2.28) Lemma 11.1, where φ, λ and a are replaced by $\overline{\varphi}$, $\overline{\lambda}$ and $\overline{a}$, and where C_2, η, θ, and M are replaced by $2C_2$, 2η, 2θ and $2M$ (note indeed that we now have $|\overline{u}_\delta^\varepsilon| < 2M$); in view of the choices made in (2.22), of the hypothesis (1.10) made on η, and on Lemma II.1, we deduce from (2.28) that for $\overline{\alpha}_0$ given by:

$$\overline{\alpha}_0 = (\overline{a})^m\,\overline{\lambda}\,\frac{C_2}{2} = \bigg(\frac{1}{2\,m\exp(\frac{8C_2}{\alpha}M)}\bigg)^m\frac{2C_2^2}{\alpha}\ ,\tag{2.30}$$

we have:

$$\overline{\alpha}_0\sum_{\gamma=1}^m\int_\Omega |D\overline{u}_\varepsilon^\gamma|^2\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx \le \overline{\theta}\int_\Omega\Big(\frac{1}{2}+|\nabla u|^2\Big)\exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx + R_\varepsilon\ .\tag{2.31}$$

Since

$$\begin{cases}\overline{u}_\varepsilon^\gamma \rightharpoonup 0 & \text{in } H_0^1(\Omega) \text{ weak*},\ \ L^\infty(\Omega) \text{ weak and a.e.} x\in\Omega\\ \overline{\psi}_\varepsilon \rightharpoonup 0 & \text{in } H_0^1(\Omega) \text{ weak*},\ \ L^\infty(\Omega) \text{ weak and a.e.} x\in\Omega\end{cases}$$

and since $\overline{\varphi}'(0) = 0$ while F_ε satisfies (1.6), (1.4), it is easy to prove that:

$$R_\varepsilon \to 0 \ .$$

Similarly we have:

$$\int_\Omega \Big(\frac{1}{2} + |\nabla u|^2\Big) \exp[\overline{\mu}\,\overline{\psi}_\varepsilon]\,dx \ \to\ \int_\Omega \Big(\frac{1}{2} + |\nabla u|^2\Big)\,dx \ .$$

Since $\exp[\overline{\mu}\,\overline{\psi}_\varepsilon] \geq 1$, we deduce from (2.31) that

$$\limsup_{\varepsilon\to 0} \overline{\alpha}_0 \sum_{\gamma=1}^{m} \int_\Omega |D\overline{u}_\varepsilon^\gamma|^2\,dx \leq \overline{\theta} \int_\Omega \Big(\frac{1}{2} + |\nabla u|^2\Big)\,dx \ .$$

Since $\overline{\theta} > 0$ and since $\overline{\alpha}_0$ does not depend on $\overline{\theta}$, this implies that $\overline{u}_\varepsilon = u_\varepsilon - u$ tends to zero strongly in $H_0^1(\Omega)$. Proposition II.2 is proved. ∎

II.3. Passing to the limit

Because of the strong convergence in $(H_0^1(\Omega))^m$ of u_ε to u, and because of the hypotheses (1.6), (1.7), (1.3) and (1.4) on F_ε and G_ε^γ, passing to the limit in each term of equation (1.8) is easy. We thus have proved the existence of at least one solution of problem (1.1). This completes the proof of the Theorem. ∎

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