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PERFECT SQUARES IN THE SEQUENCE 3, 5, 7, 11, ...

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Abstract: We prove that the only square terms in the sequence $\{u_n\}$, defined by $u_0 = 1$, $u_1 = 3$, $u_{n+2} = u_{n+1} + u_n - 1$ are u_0 and $u_{12} = 289$.

1 – Introduction

We trust that the reader did not assume that the sequence of the title is the sequence of odd primes! The sequence under consideration here is defined recursively by $u_{n+2} = u_{n+1} + u_n - 1$, with initial terms (omitted above) $u_0 = 1$ and $u_1 = 3$. The recursive relationship is, of course, very close to that of the sequence $\{F_n\}$ of Fibonacci numbers ($F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$), and one can readily show, by induction, that $u_n = 2F_n + 1$. Our purpose, here, is to show that $\{u_n\}$ has only two terms which are perfect squares: $u_0 = 1$ and $u_{12} = 289$.

The character of the terms of $\{F_n\}$ has been the subject of a number of investigations. The values of n have been found for which F_n is a square [1], for which F_n has the form $m(m{+}1)/2$ (i.e., is a triangular number) [5] or $m(3m{-}1)/2$ (a pentagonal number) [6], for which F_n is the product of consecutive integers [7] and [8], and for which $F_n = m(m{+}2)$ [9]. Among other results are the values of n for which F_n is of the form $m^2{+}1$, m^3 and $m^3 \pm 1$ [2], [3], [4], [9]. It is remarkable that F_n has none of the above forms if $n > 12$. Our result in this paper adds to this list the values of n such that F_n is of the form $2m(m{+}1)$ (twice the product of consecutive integers). Our approach involves using the periodicity of the sequence modulo any integer to show that, for each integer $n \neq 0$ or 12, there exists an integer $w(n)$ such that the Jacobi symbol $(u_n \,|\, w(n)) = (2F_n + 1 \,|\, w(n)) = -1$.

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Main Theorem. *The sequence* $\{u_n\}$ *contains exactly two terms which are perfect squares:* $u_0 = 1$ *and* $u_{12} = 289$.

Corollary. *The only terms of* $\{F_n\}$ *of the form* $2m(m+1)$ *are* $F_0 = 0$ *and* $F_{12} = 2 \cdot 8 \cdot 9$.

2 – Identities and preliminary lemmas

We will require the sequence of Lucas numbers $\{L_n\}$ which satisfies the same recursive relation as $\{F_n\}$, but with initial terms $L_0 = 2$, $L_1 = 1$. Let k, m and n be integers. Properties (1) through (6) are well-known.

(1) $$F_{-n} = (-1)^{n+1} F_n \; .$$

(2) $$F_{2n} = F_n L_n \quad \text{and} \quad L_{2n} = L_n^2 - 2(-1)^n \; ,$$

(3) $$L_n^2 - 5F_n^2 = (-1)^n \cdot 4 \; ,$$

(4) $$F_{m+n} = F_m L_n - (-1)^n F_{m-n} \; ,$$

(5) $$L_{4m} \equiv \begin{cases} -1 \pmod 8 & \text{if } 3 \nmid m, \\ 2 \pmod 8 & \text{if } 3 \mid m \; , \end{cases}$$

(6) $$F_n \text{ and } L_n \text{ are even iff } 3 \mid n$$
$$(\text{hence } u_n = 2F_n + 1 \text{ is a square only if } 3 \mid n) \; ,$$

(7) $$F_{2kt} \equiv \pm F_{2k} \pmod{L_{2k}}, \quad \text{if} \quad t \text{ is odd} \; .$$

Luo [5] has used (7). The proof readily follows from (4) — just notice that

$$F_{2kt} = F_{2k(t-1)} L_{2k} - (-1)^{2k} F_{2k(t-2)} \equiv -F_{2k(t-2)} \equiv ... \equiv (-1)^{\frac{t-1}{2}} F_{2k} \pmod{L_{2k}} \; .$$

Lemma 1. *If* $k = 2^u$, $u \geq 3$ *and* t *is odd, then* $(u_{2kt} \,|\, L_{2k}) = (u_{2k} \,|\, L_{2k})$.

Proof: From (7), $(u_{2kt} \,|\, L_{2k}) = (2F_{2kt} + 1 \,|\, L_{2k}) = (2F_{2k} + 1 \,|\, L_{2k})$ or $(-2F_{2k} + 1 \,|\, L_{2k})$. We prove the lemma by showing that the product of the two Jacobi symbols on the right is +1:

$$(2F_{2k} + 1 \,|\, L_{2k}) \cdot (-2F_{2k} + 1 \,|\, L_{2k}) = (1 - 4F_{2k}^2 \,|\, L_{2k}) = (25 - 20 \cdot 5F_{2k}^2 \,|\, L_{2k})$$

which, by (3),

$$= \Big(25 - 20\,(L_{2k}^2 - 4) \,|\, L_{2k}\Big) = (105 \,|\, L_{2k}) = (L_{2k} \,|\, 105) \; .$$

Now, $L_{2^4} = 2207 \equiv 2 \pmod{105}$, and by induction (using (2)), we have $L_{2\cdot 2^u} \equiv 2^2 - 2 \equiv 2 \pmod{105}$ for $u \geq 3$. Hence $(L_{2k} \,|\, 105) = (2 \,|\, 105) = +1$. ∎

Lemma 2. *If $k = 2^u$, $u \geq 4$, then* $(2F_{2k}+1 \,|\, L_{2k}) = (4F_k + L_k \,|\, 21)$.

Proof:

$$(2F_{2k}+1 \,|\, L_{2k}) = (2 \,|\, L_{2k})\,(4F_{2k}+2 \,|\, L_{2k})$$

which, by (2),

$$\begin{aligned} &= (4F_{2k} + L_k^2 - L_{2k} \,|\, L_{2k}) = (4F_{2k} + L_k^2 \,|\, L_{2k}) = (L_{2k} \,|\, 4F_{2k} + L_k^2) \\ &= (L_k^2 - 2 \,|\, L_k)\,(L_k^2 - 2 \,|\, 4F_k + L_k) \\ &= (-2 \,|\, L_k)\,(2 \,|\, 4F_k + L_k)\,(2L_k^2 - 4 \,|\, 4F_k + L_k)\ ; \end{aligned}$$

using (3), this

$$\begin{aligned} &= \Big(2L_k^2 - (L_k^2 - 5F_k^2) \,|\, 4F_k + L_k\Big) = (L_k^2 + 5F_k^2 \,|\, 4F_k + L_k) \\ &= (21F_k^2 \,|\, 4F_k + L_k) = (4F_k + L_k \,|\, 21)\ . \end{aligned}$$

The proof of the main theorem requires the following known congruence:

$$(8) \qquad F_{2kt+m} \equiv (-1)^t\, F_m \pmod{L_k}, \qquad \text{for all integers } k,\ t \text{ and } m\ .$$

3 – The Proof

Proof of the main theorem: It is readily seen that the sequence $\{u_n\} = \{2F_n + 1\}$ is periodic with period 8 modulo 3 and period 16 modulo 7. We find that $2F_n + 1$ is a quadratic residue modulo 3 only if $n \equiv 0$, 1, 2, 4 or 7 (mod 8) and a quadratic residue modulo 7 only if $n \equiv 0$, 4, 5, 8, 11, or 12 (mod 16). It follows that $2F_n + 1$ is a square only if $n \equiv 0$, 4, 8 or 12 (mod 16). Assume that $n \neq 0$, -4 or 12 and that u_n is a square.

Case 1. $n \not\equiv 0 \pmod{16}$. Then, $n \equiv \pm 4$, ± 8 or $\pm 12 \pmod{32}$.

We write $n = 2kt + m$ and use (8) to obtain a contradiction in each subcase.

1) $m = -4$. We take $k = 2^u$, $u \geq 4$, t odd. Then (using (1)),

$$2F_n + 1 \equiv -2F_{-4} + 1 \equiv 2F_4 + 1 \equiv 7 \pmod{L_{2^u}}\ .$$

Since $L_8 \equiv -2 \pmod 7$, it is easy to see, using (2) and induction, that $L_{2^u} \equiv 2 \pmod 7$; hence, $(7 \,|\, L_{2^u}) = -(L_{2^u} \,|\, 7) = -(2 \,|\, 7) = -1$, a contradiction.

2) $m = 4$. Then $n \equiv 4$ or 36 (mod 64). Taking $m = 4$, $k = 2^4$, and t even, we have

$$2F_n + 1 \equiv 2F_4 + 1 \equiv 7 \pmod{L_{2^4}} ,$$

so $n \equiv 4$ (mod 64) is eliminated as in 1). If $n \equiv 36$ (mod 64), then, since $3 \mid n$ by (6), $n \equiv 36$ (mod $3 \cdot 64$). Taking $m = 36$, $k = 3 \cdot 2^4$, and t even, we have

$$2F_n + 1 \equiv 2F_{36} + 1 \pmod{L_{3 \cdot 2^4}} ;$$

the congruence holds modulo 769, a divisor of L_{48}, and we find that

$$(2F_{36} + 1 \,|\, 769) = (435 \,|\, 769) = -1 .$$

3) $m = -8$. Again, $3 \mid n$ implies that $n \equiv 24$ (mod $3 \cdot 32$). Take $k = 3 \cdot 2^3$ and t even; using the factor 1103 of L_{24} yields $(2F_n + 1 \,|\, 1103) = (85 \,|\, 1103) = -1$.

4) $m = 8$. Taking $k = 2^3$ and t even eliminates this subcase, as in 2).

5) $m = -12$. In this subcase, $n \equiv -12$ (mod 64), or $n \equiv 20$ or 84 (mod 128). Taking $m = -12$, $k = 2^4$ and t even, we have $2F_n + 1 \equiv -2F_{12} + 1 \equiv -287$ (mod L_{16}), but $(-287 \,|\, L_{16}) = -1$. Upon taking $m = 20$ or 84, $k = 2^5$, t even, and q a divisor of $L_{32} = 1087 \cdot 4481$, $2F_n + 1 \equiv 2F_m + 1$ (mod q), and we find that $(2F_{20} + 1 \,|\, 4481) = -1$, and $(2F_{84} + 1 \,|\, 1087) = -1$.

6) $m = 12$. Take $k = 2^u$, $u \geq 4$ and t odd. Then

$$2F_n + 1 \equiv -2F_{12} + 1 \equiv -287 \pmod{L_{2^u}} ,$$

and

$$(-287 \,|\, L_{2^u}) = (L_{2^u} \,|\, 287) = (L_{2^u} \,|\, 7)\,(L_{2^u} \,|\, 41) = (L_{2^u} \,|\, 41) .$$

Now, using (2), it is easy to show that

$$L_{2^u} \equiv \begin{cases} 6 \pmod{41}, & \text{if } u \text{ is odd,} \\ -7 \pmod{41}, & \text{if } u \text{ is even ,} \end{cases}$$

and each of $(6 \,|\, 41)$ and $(-7 \,|\, 41)$ equals -1.

Case 2. $n \equiv 0$ (mod 16). Let $n = 2 \cdot 2^u t$, $u \geq 3$, t odd. If $u = 3$, then, since $3 \mid n$, $n = 48(t/3)$; by (6), $2F_n + 1 \equiv \pm 2F_{48} + 1$ (mod L_{48}). Since $769 \cdot 3167 \mid L_{48}$, we have

$$(2F_n + 1 \,|\, 769) = (2F_{48} + 1 \,|\, 769) \equiv (104 \,|\, 769) = -1 ,$$

or

$$(2F_n + 1 \,|\, 3167) = (-2F_{48} + 1 \,|\, 3167) \equiv (-780 \,|\, 3167) = -1 .$$

Assume $u \geq 4$. By Lemmas 1 and 2, $(2F_n + 1 \,|\, L_{2\cdot 2^u}) = (4F_k + L_k \,|\, 21)$. Now, $F_8 = 21$ divides F_k, and from the proof of Lemma 1, $L_k \equiv 2 \pmod{21}$; hence $(4F_k + L_k \,|\, 21) = (2 \,|\, 21) = -1$.

Finally, since t may be 0 in each of the above cases except for 1), 6) and Case 2, u_n is not a square except possibly when $n = 0$, -4 or 12. Clearly, only $u_0 = 1$ and $u_{12} = 289$ are squares; this completes the proof. ∎

Proof of the corollary: The proof is immediate, since, if $u_n = 2F_n + 1 = (2m+1)^2$, then $F_n = 2m\,(m+1)$. ∎

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