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UNIFORM STABILIZATION FOR ELASTIC WAVES SYSTEM WITH HIGHLY NONLINEAR LOCALIZED DISSIPATION *

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Recommended by E. Zuazua

Abstract: We show that the solutions of a system in elasticity theory with a nonlinear localized dissipation decay in an algebraic rate to zero, that is, denoting by $E(t)$ the total energy associated to the system, there exist positive constants C and γ satisfying:

$$E(t) \le C\,E(0)\,(1+t)^{-\gamma} .$$

1 – Introduction

In this work we study decay properties of the solutions for the following initial-boundary value problem related with the system of elastic waves with a localized nonlinear dissipative term:

$$u_{tt} - b^2\Delta u - (a^2 - b^2)\,\nabla div\, u + \alpha\, u + \rho(x, u_t) = 0, \quad \text{in } \Omega\times\mathbb{R} , \tag{1.1}$$

$$u(x,0) = u_o(x), \quad u_t(x,0) = u_1(x) \quad \text{in } \Omega , \tag{1.2}$$

$$u(x,t) = 0 \quad \text{in } \Gamma\times\mathbb{R} , \tag{1.3}$$

where the medium Ω is a bounded domain in $\mathbb{R}^3$ with C^2 boundary Γ. The function $u(x,t) = (u^1(x,t), u^2(x,t), u^3(x,t))$ is the vector displacement, $\Delta\, u = (\Delta\, u^1(x,t), \Delta\, u^2(x,t), \Delta\, u^3(x,t))$ is the Laplacian operator, $div\, u$ is the usual divergent of u and ∇ is the gradient operator. The coefficients a and b are related with Lamé coefficients of Elasticity Theory, $a^2 > b^2 > 0$, (see [1]), and $\alpha \ge 0$ is a constant. We can found applications for this system in geophysics and seismic waves propagation.

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In the case that $a^2 = b^2$ we have a vector wave equation. Many results related to wave equation can be generalized for the system (1.1). We observe that the solutions of the free system of elastic waves are a superposition of two waves which propagate with different fase velocities a and b (see [3]).

Our goal in this work is to show the uniform stabilization of the total energy for system (1.1)–(1.3). We observe that the vector function ρ, which appear in (1.1), represents a dissipative term which is localized in a neighborhood of part of the boundary of Ω. To prove this result we use some energy identities associated with localized multipliers in order to construct special difference inequalities for the energy of the system (1.1). These ideas come from Control Theory (see J.-L. Lions [16], V. Komornik [12], A. Haraux [8] and V. Komornik – E. Zuazua [15]). The main estimates in this work are obtained using Holmgren's Uniqueness Theorem and Nakao's Lemma.

About the stabilization of the local energy for the nondissipative system of linear elasticity in unbounded domains we refer B. Kapitonov [11] and R.C. Charão [2]. B. Kapitonov works in an exterior domain with geometrical condition on the boundary. There are some results related to stabilization of the total energy associated with systems of elasticity in bounded domains with localized dissipations. For the system (1.1) when the dissipative term $\rho(x, u_t)$ is linear and the localizing function $a(x)$ is unbounded, the stabilization of the energy is proved by M.A. Astaburuaga and R.C. Charão [4]. In the case of wave equation, stabilization results can be found in M. Nakao [19], [20], E. Zuazua [22], P. Martinez [17], [18] and L.R.T. Tébou [21] and the references therein.

The system (1.1) with $\rho \equiv 0$ damped by a linear boundary feedback is studied by F. Alabau and V. Komornik [13] and uniform stabilization is obtained. In [14] F. Alabau and Komornik have considered an anisotropic system of elasticity and have established uniform decay rates when feedback control is acting via natural and physically implementable boundary conditions. Their results require even more stringent geometric conditions. In fact, they must assume that the domain is a sphere.

A. Guesmia in [5] studied the stabilization of the energy for the system of elasticity in anisotropic domains in $\mathbb{R}^n$ with localized non linear dissipation given by an additional term $\rho(x, u_t) = (b_1(x)\, g_1(u_t^1), ..., b_n(x)\, g_n(u_t^n))$ where $u = (u^1, ..., u^n)$ is the solution of the system. The system considered by Guesmia is not coupled in the dissipative term. The dissipation is localized by functions $b_i(x)$ and they are effective only in a neighborhood of part of the boundary. For the non degenerate case $b_i(x) > b_0 > 0,\ i = 1, ..., n$ in such neighborhood, Guesmia assumes the following conditions on the functions $g_i(s)$

$$C_1|s|^r \le g_i(s) \le C_2|s|^{1/r}, \quad |s| \le 1, \quad s \in \mathbb{R},$$
$$C_1|s| \le g_i(s) \le C_2|s|, \quad |s| > 1, \quad s \in \mathbb{R},$$

for positive constants C_1, C_2 and r, with $r \ge 1$. Thus, the class of functions $g_i(s) = |s|^p s$ for positive constant p are not included in Guesmia's results.

In this work we consider the system of elasticity in a homogeneous isotropic medium in $\mathbb{R}^3$ with a dissipative localized term given by $\rho(x, u_t) = f(x, |u_t|)\, u_t$, where f is a positive function. Our hypotheses include the case $\rho(x, s) = a(x)\, |s|^p s$ for $p \in (-1, 2]$. The function $a(x)$ is also effective only on a neighborhood of part of the boundary of the domain.The system (1.1) can or not be coupled in the dissipative term. We obtain precise algebraic decay rates for the energy depending only on the growing rates r and p of the function f near zero and the infinity, respectively. So, in this sense our result generalizes the Guesmia's result for the system of elasticity in homogeneous isotropic domais. We note that our results can be extended for domains in $\mathbb{R}^n$, $n \ge 1$.

In [6] A. Guesmia consider the stabilization for the anisotropic system of elasticity with a nonlinear boundary feedback and the dissipation is effective only on a part of the boundary given by the same geometrical condition which we have used. The framework is based on integral inequalities and Nakao's Lemma. The conditions on the dissipative functions $g_i(s)$ are more restrictive then ours. For instance, to obtain the following estimate for the energy

$$E(t) \le M_1(1+t)^{-\frac{r}{1-r}}, \quad t > 0,$$

where M_1 is a constant independent of the solution u, A. Guesmia used the following hypothesis on the functions $g_i(s)$

$$C_3|s| \le |g_i(s)| \le C_4|s|, \quad |s| > 1, \quad s \in \mathbb{R}.$$

With the following condition on the functions $g_i(s)$

$$|g_i(s)| \le C_5|s|^\lambda, \quad |s| > 1,$$

with a very special λ, A. Guesmia obtain the same rate, but M_1 depends on the function u. We also observe that our results in this work extend for isotropic elasticity the result obtained by A. Guesmia in [7].

Now, we want mention two works of Mary A. Horn. In [9] M. Horn works with the uniform stabilization for the system of isotropic linear elasticity (1.1) (with

$\rho(x,s) \equiv 0$) using a nonlinear boundary feedback in the same sense of A. Guesmia [6]. The proof uses multipliers estimates and is based on trace regularity results and uniqueness (Holmgren) argument. The damping is effective only on a part of the boundary and is given by a continuously differentiable function $g(s),\ s \in \mathbb{R}^n$, such that

$$g(0) = 0\,, \quad g(s)\cdot s > 0, \quad s \in \mathbb{R}^n - \{0\}\ ,$$
$$m|s| \le |g(s)| \le M|s|\,, \quad |s| > 1\ ,$$

with m and M positive constants. The decay rate is given by a function $S(t)$ which is the solution of a differential equation of type

$$\frac{dS(t)}{dt} + q(S(t)) = 0\,, \quad S(0) = 0\ ,$$

where $q(x),\ x > 0$, is a special nonlinear function which is constructed using the function $g(s)$. M. Horn shows that $S(t) \to 0,\ t \to \infty$.

Finally, in [10] M. Horn studied the same problem for the non isotropic system of elasticity but the system is considered containing an additional light internal damping given by a term $\rho(x,u_t) = b(x)u_t$ with $b(x)$ a function such that $b(x) > 0$ for all $x \in \Omega$. The other dissipative function $g(s)$ on part of the boundary has the same properties assumed in M. Horn [9]. The coefficients a_{ijkl} of the elasticity tensor are assumed by M. Horn to be independent of both time and space. The decay rate for the stabilization of the energy is the same that in [9]. The framework also is the same but of course using Korn's Inequality. If the control function $g(s) = s$ then M. Horn observe that the decay rate is exponential.

To conclude, we observe that the dissipation function $\rho(x,s)$ which we have used in this paper, is more strongly non linear then the functions $g(s)$ or $g_i(s)$ which appear in the references.

2 – Hypotheses and results

Through this work the dot $\cdot$ will represent the usual inner product between two vectors in $\mathbb{R}^3$. Let us consider the following hypotheses on the function $\rho(x,s)$.

Let $x_o \in \mathbb{R}^3$ be a fixed vector and

$$\Gamma(x_0) = \Big\{x \in \Gamma;\ (x - x_o)\cdot \eta(x) \ge 0\Big\}$$

where $\eta = \eta(x)$ is the outward unit normal at $x \in \Gamma = \partial\Omega$. Let $\omega \subset \overline{\Omega}$ be a neighborhood of $\Gamma(x_o)$ and

$$a\colon \overline{\Omega} \to \mathbb{R}^+$$

a bounded function satisfying

$$a(x) \ge 0 \ \text{ in } \overline{\Omega}\,, \quad a(x) \ge a_o > 0 \ \text{ in } \omega\ . \tag{2.1}$$

We suppose that the function $\rho\colon \overline{\Omega}\times\mathbb{R}^3 \to \mathbb{R}^3$ is given by $\rho(x,s) = f(x,|s|)s$, $s \in \mathbb{R}^3$, $x \in \overline{\Omega}$ where

$$f\colon \overline{\Omega}\times\mathbb{R}^+ \to \mathbb{R}^+$$

is a differentiable function which satisfies the following conditions:

$$a(x)|s|^r \le f(x,|s|) \le K_o\, a(x)\,(|s|^r+1), \quad \text{if } |s| \le 1\ , \tag{2.2}$$

$$a(x)|s|^p \le f(x,|s|) \le K_1 a(x)\,(|s|^p+1), \quad \text{if } |s| \ge 1\ , \tag{2.3}$$

with $r \in (-1,+\infty)$, $p \in [-1,2]$ and $K_0,\ K_1$ are positive constants.

Hence, the dissipative term $\rho(x,u_t)$ is effective only on a part of $\overline{\Omega}$ that includes $\Gamma(x_o)$.

We also suppose that

$$\frac{\partial f}{\partial s}(x,s) \,\ge\, 0$$

for all $s \in \mathbb{R}^+$ and $x \in \overline{\Omega}$.

About the existence and uniqueness of solution for the problem (1.1) we have the following result.

Theorem 2.1 (Existence and Uniqueness)**.** *We assume that Γ is C^2 class and the initial data $u_o \in (H^1_o(\Omega)\cap H^2(\Omega))^3$, $u_1 \in (H^1_o(\Omega))^3$. Then, under above hypotheses on function ρ, the initial boundary value problem (1.1) has a unique solution $u = u(x,t)$ in the class*

$$u \in C\Big([0,\infty[, (H^1_o(\Omega)\cap H^2(\Omega))^3\Big)\cap C^1\Big([0,\infty[, (H^1_o(\Omega))^3\Big)\cap C^2\Big([0,\infty[, (L^2(\Omega))^3\Big).$$

Proof: The prof of this theorem is standard, for example, using semigroups theory. ∎

We want in this work prove the uniform stabilization of the total energy $E(t)$,

$$E(t) \,=\, 1/2\int_\Omega \Big(|u_t|^2 + b^2|\nabla u|^2 + (a^2-b^2)\,(div\,u)^2 + \alpha\,|u|^2\Big)\,dx\ .$$

In fact, this is possible because the energy $E(t)$ satisfies the identity

$$E(t) - E(t+T) = \int_t^{t+T}\int_\Omega \rho(x,u_t)\cdot u_t\,dx\,ds\,, \qquad t\geq 0,\quad T>0\;. \tag{2.4}$$

Then, the energy is a nonincreasing function of t because $\rho(x,u_t)\cdot u_t \geq 0$ always.

The identity (2.4) is obtained taking inner product between equation (1.1) and u_t and integrating over $[t,t+T]\times\Omega$.

Our result is the following

Theorem 2.2 (Stabilization)**.** *Under the hypotheses of Theorem 2.1, the total energy for the solution $u=u(x,t)$ of the problem (1.1)–(1.3) has the following asymptotic behavior in time*

$$E(t) = E(u(x,t)) \leq C\,E(0)\,(1+t)^{-\gamma_i}\,, \qquad i=1,2,3,4\;. \tag{2.5}$$

where C is a positive constant and the decay rates γ_i are given according to the cases:

Case 1: *If $r\geq 0$ and $0\leq p\leq 2$ then*

$$\gamma_1 = \min\left\{\frac{2}{r}, \frac{4(p+1)}{p}\right\};$$

Case 2: *If $r\geq 0$ and $-1\leq p<0$ then*

$$\gamma_2 = \min\left\{\frac{2}{r}, \frac{-4}{p}\right\};$$

Case 3: *If $-1<r<0$ and $0\leq p\leq 2$ then*

$$\gamma_3 = \min\left\{\frac{-2(r+1)}{r}, \frac{4(p+1)}{p}\right\};$$

Case 4: *If $-1<r<0$ and $-1\leq p<0$ then*

$$\gamma_4 = \min\left\{\frac{-2(r+1)}{r}, \frac{-4}{p}\right\}.$$

In order to prove this theorem we need some Lemmas and special estimates about the solution $u(x,t)$.

3 – Technical lemmas

We are going to prove for the energy of system (1.1) an estimate like

$$E(t)^{\varepsilon_i} \le C_i \left[E(t) - E(t+T)\right], \qquad t \ge 0 , \tag{3.1}$$

where C_i is a positive constant, $T > 0$ is fixed and $\varepsilon_i > 0$ is related with γ_i (given in Theorem 2.2), $i = 1, 2, 3, 4$.

Then, to show the decay property (2.5) we will use the following Nakao's Lemma(See Nakao [19]).

Lemma 3.1 (Nakao)**.** *Let $\Phi(t)$ be a nonnegative function on $\mathbb{R}^+$ satisfying*

$$\sup_{t \le s \le t+T} \Phi(s)^{1+\delta} \le\, g(t) \left\{\Phi(t) - \Phi(t+T)\right\}$$

with $T > 0$, $\delta \ge 0$ and $g(t)$ a nondecreasing continuous function. If $\delta > 0$ then $\Phi(t)$ has the decay property

$$\Phi(t) \le \left\{\Phi(0)^{-\delta} + \int_T^t g(s)^{-1}\, ds\right\}^{-1/\delta}, \qquad t \ge T \ .$$

If $\delta = 0$, instead of the above inequality, $\Phi(t)$ is such that

$$\Phi(t) \le C\, \Phi(0)\, \mathbf{e}^{\lambda\, t}, \qquad t \ge 0 \ ,$$

for some $\lambda > 0$. ∎

Now, to prove (3.1) we will need the Gagliardo–Niremberg Lemma.

Lemma 3.2 (Gagliardo–Niremberg)**.** *Let $1 \le r < p < \infty$, $1 \le q \le p$ and $0 \le m$. Then we have the inequality*

$$\|v\|_{W^{k,q}} \,\le\, C\, \|v\|^{\theta}_{W^{m,p}}\, \|v\|^{1-\theta}_{L^r}$$

for $v \in W^{m,p}(\Omega) \cap L^r(\Omega)$, where C is a positive constant and

$$\theta = \left(\frac{k}{N} + \frac{1}{r} - \frac{1}{q}\right) \left(\frac{m}{N} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \le 1$. ∎

In order to show (3.1) we have also developed some identities for the elastic waves system (1.1).

We take $h\colon \overline{\Omega} \to \mathbb{R}^3$ a vector field C^1 class such that

$$h(x) = \eta(x) \quad \text{on } \Gamma(x_o) \;, \tag{3.2}$$

$$h(x)\cdot\eta(x) \geq 0 \quad \text{on } \Gamma \;, \tag{3.3}$$

$$h(x) = 0 \quad \text{in } \Omega\backslash\widehat{\omega} \;, \tag{3.4}$$

where $\widehat{\omega}$ is a open set in $\mathbb{R}^3$ with the property

$$\Gamma(x_o) \subset \widehat{\omega}\cap\overline{\Omega} \subset \omega$$

and we consider the multiplier

$$M(u) \;=\; h * \nabla u \;\equiv\colon (h\cdot\nabla u^1,\, h\cdot\nabla u^2,\, h\cdot\nabla u^3)$$

where $u(x,t) = (u^1,u^2,u^3)$ is the solution of the problem (1.1)–(1.3).

We also take a function $m \in W^{1,\infty}(\Omega)$ such that $\frac{|\nabla m|^2}{m}$ is bounded and

$$1 \geq m \geq 0 \quad \text{in } \Omega \;, \tag{3.5}$$

$$m = 1 \quad \text{in } \widetilde{\omega} \;, \tag{3.6}$$

$$m = 0 \quad \text{in } \overline{\Omega}\backslash\omega \;, \tag{3.7}$$

where $\widetilde{\omega} \subset \overline{\Omega}$ is an open set in $\overline{\Omega}$ with $\Gamma(x_o) \subset \widetilde{\omega} \subset \omega \subset \overline{\Omega}$.

Then we have

Lemma 3.3 (Energy Identities)**.** *Let u be the solution of (1.1)–(1.3) and $T>0$ fixed. Then the following identities holds for all $t\geq 0$.*

$$\begin{aligned} &\int_t^{t+T}\!\!\int_\Omega \Big[-|u_t|^2 + b^2|\nabla u|^2 + (a^2-b^2)(div\,u)^2 - \alpha\,|u|^2\Big]\,dx\,ds \;= \\ &\qquad = -\int_\Omega u_t\,u\,dx\Big|_t^{t+T} - \int_t^{t+T}\!\!\int_\Omega \rho(x,u_t)\cdot u\;dx\,ds \;, \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\int_t^{t+T}\!\!\int_\Omega m(x)\,\Big[b^2|\nabla u|^2 + (a^2-b^2)(div\,u)^2 + \alpha\,|u|^2\Big]\,dx\,ds \;= \\ &\quad = -\int_\Omega m(x)\,u\cdot u_t\;dx\Big|_t^{t+T} \\ &\qquad + \int_t^{t+T}\!\!\int_\Omega m(x)\,u\,|u_t|^2\;dx\,ds - \int_t^{t+T}\!\!\int_\Omega m(x)\,u\cdot\rho(x,u_t)\;dx\,ds \\ &\qquad - \int_t^{t+T}\!\!\int_\Omega \Big[b^2\sum_{i=1}^3 u^i\,\nabla u^i\cdot\nabla m + (a^2-b^2)\,\nabla m\cdot u\,div\,u\Big]\,dx\,ds \;, \end{aligned} \tag{3.9}$$

$$
\begin{aligned}
&\frac{1}{2}\int_t^{t+T}\!\!\int_\Gamma (h\cdot\eta)\left[b^2\left|\frac{\partial u}{\partial\eta}\right|^2+(a^2-b^2)(div\,u)^2\right]d\Gamma\,ds\;+\\
&\quad+\int_t^{t+T}\!\!\int_\Gamma\left[b^2(h(x)*\nabla u)\cdot\frac{\partial u}{\partial\eta}+(a^2-b^2)div\,u(h(x)*\nabla u)\cdot\eta\right]d\Gamma\,ds=\\
&\quad=-\int_\Omega(h(x)*\nabla u)\cdot u_t\,dx\Big|_t^{t+T}-\int_t^{t+T}\!\!\int_\Omega(h(x)*\nabla u)\cdot\rho(x,u_t)\,dx\,ds\\
&\quad+\frac{1}{2}\int_t^{t+T}\!\!\int_\Omega div\,h\left[-|u_t|^2+b^2|\nabla u|^2+(a^2-b^2)(div\,u)^2-\alpha|u|^2\right]dx\,ds\\
&\quad-\int_t^{t+T}\!\!\int_\Omega\sum_{i,j,k=1}^{3}\left[b^2\,\frac{\partial h^k}{\partial x_j}\frac{\partial u^i}{\partial x_j}\frac{\partial u^i}{\partial x_k}+(a^2-b^2)\,\frac{\partial h^k}{\partial x_i}\frac{\partial u^j}{\partial x_j}\frac{\partial u^i}{\partial x_k}\right]dx\,ds\;,
\end{aligned}
\tag{3.10}
$$

$$
\begin{aligned}
&\int_\Omega(x-x_o)*\nabla u\cdot u_t\,dx\Big|_t^{t+T}+\int_t^{t+T}\!\!\int_\Omega((x-x_o)*\nabla u)\cdot\rho(x,u_t)\,dx\,ds\;+\\
&\quad+\frac{3}{2}\int_t^{t+T}\!\!\int_\Omega\left[|u_t|^2-b^2|\nabla u|^2-(a^2-b^2)(div\,u)^2+\alpha|u|^2\right]dx\,ds\\
&\quad+\frac{1}{2}\int_t^{t+T}\!\!\int_\Gamma(x-x_o)\cdot\eta\left[b^2|\nabla u|^2+(a^2-b^2)(div\,u)^2\right]d\Gamma\,ds\\
&\quad-\int_t^{t+T}\!\!\int_\Gamma\left[b^2((x-x_o)*\nabla u)\cdot\frac{\partial u}{\partial\eta}+(a^2-b^2)div\,u((x-x_o)*\nabla u)\cdot\eta\right]d\Gamma\,ds\\
&\quad+\int_t^{t+T}\!\!\int_\Omega\left[b^2|\nabla u|^2+(a^2-b^2)(div\,u)^2\right]dx\,ds\;=\;0\;,
\end{aligned}
\tag{3.11}
$$

where we have used the notation $|\nabla u|^2=\sum_{i=1}^{3}|\nabla u^i|^2$, *for* $u=(u^1,u^2,u^3)$. *The vector* $\eta=\eta(x)$ *is the usual normal at* $x\in\Gamma$.

Proof: The identities (3.8), (3.9) and (3.10) are obtained in a standard way taking the inner product between the equation (1.1) and the multipliers $M(u)=u$, $M(u)=m(x)\,u$ and $M(u)=h*\nabla u$ respectively, integrating in $\Omega\times[t,t+T]$ and using the fact that $u=0$ on $\Gamma\times\mathbb{R}$. The identity (3.11) is a particular case of (3.10) when $h(x)=(x-x_o)$. In (3.10) we do not have used the properties (3.2)–(3.4) for the vector field h.

Because $u=0$ on $\Gamma\times[0,\infty[$ we observe that

$$\nabla u^i=\frac{\partial u^i}{\partial\eta}\,\eta\quad\text{ on }\;\Gamma\times[0,\infty[,\quad i=1,2,3\;.$$

Then we have

$$(h*\nabla u)\cdot\frac{\partial u}{\partial\eta}\;=\;\sum_{i=1}^{3}\left(h\cdot\frac{\partial u^i}{\partial\eta}\eta\right)\frac{\partial u^i}{\partial\eta}\;=\;(h\cdot\eta)\Big|\frac{\partial u}{\partial\eta}\Big|^2$$

and

$$(div\,u)(h*\nabla\,u)\cdot\eta\;=\;(h\cdot\eta)\,(divu)^2\;,$$

where
$$\frac{\partial u}{\partial \eta} = \left(\frac{\partial u^1}{\partial \eta}, \frac{\partial u^2}{\partial \eta}, \frac{\partial u^3}{\partial \eta}\right).$$

With this observation we obtain from identity (3.10) in Lemma 3.3 the following energy identity

$$\begin{aligned}
&\frac{1}{2}\int_t^{t+T}\!\!\int_\Gamma (h\cdot\eta)\left[b^2\left|\frac{\partial u}{\partial \eta}\right|^2 + (a^2-b^2)(div\, u)^2\right] d\Gamma\, ds \;= \\
&\quad= \int_\Omega (h(x)*\nabla u)\cdot u_t\, dx\Big|_t^{t+T} + \int_t^{t+T}\!\!\int_\Omega (h(x)*\nabla u)\cdot \rho(x,u_t)\, dx\, ds \\
&\qquad+ \frac{1}{2}\int_t^{t+T}\!\!\int_\Omega (div\, h)\left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2 + \alpha|u|^2\right] dx\, ds \\
&\qquad+ \int_t^{t+T}\!\!\int_\Omega \sum_{i,j,k=1}^{3}\left[b^2\frac{\partial h^k}{\partial x_j}\frac{\partial u^i}{\partial x_j}\frac{\partial u^i}{\partial x_k} + (a^2-b^2)\frac{\partial h^k}{\partial x_i}\frac{\partial u^j}{\partial x_j}\frac{\partial u^i}{\partial x_k}\right] dx\, ds\;.
\end{aligned} \tag{3.12}$$

In the same way we have from identity (3.11) in Lemma 3.3 that

$$\begin{aligned}
&\frac{3}{2}\int_t^{t+T}\!\!\int_\Omega \left[|u_t|^2 - b^2|\nabla u|^2 - (a^2-b^2)(div\, u)^2 + \alpha|u|^2\right] dx\, ds \\
&\qquad+ \int_t^{t+T}\!\!\int_\Omega \left[b^2|\nabla u|^2 + (a^2-b^2)(div\, u)^2\right] dx\, ds \;= \\
&\quad= -\int_\Omega ((x-x_o)*\nabla u)\cdot u_t\, dx\Big|_t^{t+T} \\
&\qquad+ \frac{1}{2}\int_t^{t+T}\!\!\int_\Gamma ((x-x_o)\cdot\eta)\left[b^2\left|\frac{\partial u}{\partial \eta}\right|^2 + (a^2-b^2)(div\, u)^2\right] d\Gamma\, ds \\
&\qquad- \int_t^{t+T}\!\!\int_\Omega ((x-x_o)*\nabla u)\cdot\rho(x,u_t)\, dx\, ds\;. \blacksquare
\end{aligned} \tag{3.13}$$

4 – Main estimates

Next, C will denote different positive constants.

Lemma 4.1. *Let $u = u(x,t)$ be the solution of the problem (1.1)–(1.3). Then, the energy $E(t) = E(u(x,t))$ satisfies the estimate*

$$\begin{aligned}
E(t) \;\le\; C\Big\{E(t) - E(t+T)\Big\} + C\int_t^{t+T}\!\!\int_\omega (|u_t|^2 + |u|^2)\, dx\, ds \\
+ C\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)|\,\Big[|u| + |\nabla u|\Big]\, dx\, ds\,, \qquad t\ge 0\;.
\end{aligned}$$

Proof: Adding identity (3.8) in Lemma 3.3 and identity (3.13) we obtain

$$\begin{aligned}
&\frac{1}{2}\int_t^{t+T}\!\!\int_\Omega \Big[|u_t|^2 + b^2|\nabla u|^2 + (a^2-b^2)(div\,u)^2 + \alpha|u|^2\Big]\,dx\,ds \;=\\
&\quad= -\int_\Omega \Big[(x-x_o)*\nabla u + u\Big]\cdot u_t\,dx\Big|_t^{t+T}\\
(4.1)\qquad &\quad + \frac{1}{2}\int_t^{t+T}\!\!\int_\Gamma ((x-x_o)\cdot\eta)\left[b^2\left|\frac{\partial u}{\partial\eta}\right|^2 + (a^2-b^2)(div\,u)^2\right] d\Gamma\,ds\\
&\quad - \int_t^{t+T}\!\!\int_\Omega \Big[((x-x_o)*\nabla u) + u\Big]\cdot\rho(x,u_t)\,dx\,ds\;.
\end{aligned}$$

Therefore

$$\begin{aligned}
\int_t^{t+T} E(s)\,ds \;\le\; &\int_\Omega \Big[|(x-x_o)*\nabla u| + |u|\Big]\,|u_t|\;dx\Big|_t^{t+T}\\
&+\int_t^{t+T}\!\!\int_\Omega \Big[|(x-x_o)*\nabla u| + |u|\Big]\,|\rho(x,u_t)|\,dx\,ds\\
&+\frac{1}{2}\int_t^{t+T}\!\!\int_{\Gamma(x_o)} ((x-x_o)\cdot\eta)\left[b^2\left|\frac{\partial u}{\partial\eta}\right|^2 + (a^2-b^2)(div\,u)^2\right] d\Gamma\,ds
\end{aligned}$$

because $(x-x_o)\cdot\eta \le 0$ on $\Gamma\backslash\Gamma(x_o)$.

Let $M_o = \sup_{x\in\overline{\Omega}} |x-x_o|$, then we have

$$\begin{aligned}
\int_t^{t+T} E(s)\,ds \;\le\; &\int_\Omega \Big[M_o|\nabla u| + |u|\Big]\,|u_t|\;dx\,\Big|_t^{t+T}\\
&+\int_t^{t+T}\!\!\int_\Omega \Big[M_o|\nabla u| + |u|\Big]\,|\rho(x,u_t)|\,dx\,ds\\
&+\frac{1}{2}\int_t^{t+T}\!\!\int_{\Gamma(x_o)} M_o\left[b^2\left|\frac{\partial u}{\partial\eta}\right|^2 + (a^2-b^2)(div\,u)^2\right] d\Gamma\,ds \;\le\\
\le\; C_1&\Big[\|u_t(t+T)\|\,\Big\{\|\nabla u(t+T)\| + \|u(t+T)\|\Big\} + \|u_t\|\,\Big\{\|\nabla u\| + \|u_t\|\Big\}\Big]\\
&+\int_t^{t+T}\!\!\int_\Omega \Big[M_o|\nabla u| + |u|\Big]\,|\rho(x,u_t)|\,dx\,ds\\
&+\frac{1}{2}\int_t^{t+T}\!\!\int_{\Gamma(x_o)} M_o\left[b^2\left|\frac{\partial u}{\partial\eta}\right|^2 + (a^2-b^2)(div\,u)^2\right] d\Gamma\,ds
\end{aligned}$$

where $\|u\|$ is the norm of u in $[L^2(\Omega)]^3$ and $\|\nabla u\|^2 = \sum_{i=1}^{3} \|\nabla u^i\|^2$.

Using Poincare's inequality it results

$$\int_t^{t+T} E(s)\,ds \;\le\; C\Big[\|u_t(t+T)\|\,\|\nabla u(t+T)\| + \|u_t\|\,\|\nabla u\|\Big] \;+$$

$$+ \int_t^{t+T}\!\!\int_\Omega \Big[|\nabla u| + |u|\Big]\, |\rho(x,u_t)|\, dx\, ds$$
$$+ \frac{1}{2}\int_t^{t+T}\!\!\int_{\Gamma(x_o)} M_o \left[b^2 \left|\frac{\partial u}{\partial \eta}\right|^2 + (a^2 - b^2)(div\, u)^2\right] d\Gamma\, ds\ .$$

Thus

$$\begin{aligned}\int_t^{t+T} E(s)\, ds \ \le\ & C\Big[E(t) + E(t+T)\Big] \\ & + C\int_t^{t+T}\!\!\int_\Omega \Big[|\nabla u| + |u|\Big]\, |\rho(x,u_t)|\, dx\, ds \\ & + C\int_t^{t+T}\!\!\int_{\Gamma(x_o)} \left[b^2 \left|\frac{\partial u}{\partial \eta}\right|^2 + (a^2 - b^2)(div\, u)^2\right] d\Gamma\, ds\ .\end{aligned} \tag{4.2}$$

We need estimate the last term in (4.2). We observe that

$$\int_t^{t+T}\!\!\int_\Omega \left[b^2 \sum_{i=1}^3 u^i\, \nabla u^i \cdot \nabla m + (a^2 - b^2)\, \nabla m \cdot u\, div\, u\right] dx\, ds \ \le$$
$$\le \int_t^{t+T}\!\!\int_\Omega \left[\frac{b^2}{2} \sum_{i=1}^3 \left(\left|u^i\right|^2 \frac{|\nabla m|^2}{m} + m\left|\nabla u^i\right|^2\right)\right.$$
$$\left. + \frac{(a^2 - b^2)}{2}\left\{\frac{|\nabla m|^2}{m}\, |u|^2 + m(div\, u)^2\right\}\right] dx\, ds$$

where m is the function which appear in Lemma 3.3.

We also observe that $m = 0$ on $\Omega\backslash\omega$ and $\dfrac{|\nabla m|^2}{m}$ is bounded, by construction of m.

Then we have

$$\begin{aligned}&\int_t^{t+T}\!\!\int_\Omega \Big[b^2 \sum_{i=1}^3 u^i\, \nabla u^i \cdot \nabla m + (a^2 - b^2)\, \nabla m \cdot u\ div\, u\Big]\, dx\, ds \ \le \\ &\le C\int_t^{t+T}\!\!\int_\omega |u|^2\, dx\, ds + \frac{1}{2}\int_t^{t+T}\!\!\int_\Omega m\Big[b^2\, |\nabla u|^2 + (a^2 - b^2)(div\, u)^2\Big]\, dx\, ds\,.\end{aligned} \tag{4.3}$$

Using (4.3) we obtain, from identity (3.9) in Lemma 3.3, the following estimate

$$\begin{aligned}\int_t^{t+T}\!\!\int_\Omega m\Big[b^2\, |\nabla u|^2 + (a^2 - b^2)(div\, u)^2 + \alpha\, |u|^2\Big]\, dx\, ds \ \le& \\ \le C\Big\{\int_t^{t+T}\!\!\int_\omega \Big(|u_t|^2 + |u|^2\Big)\, dx\, ds + E(t) + E(t+T)\Big\}& \\ + C\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)|\, |u|\, dx\, ds&\end{aligned} \tag{4.4}$$

where we have used that $0 \le m \le 1,\ m = 0$ (see (3.5)–(3.7)) in $\Omega\backslash\omega$ and the estimate

$$\left| - \int_\Omega m\, u \cdot u_t \, dx \Big|_t^{t+T} \right| \le C(\Omega) \Big[E(t) + E(t+T) \Big]$$

due to Poincare's inequality.

Because $h(x) = \eta(x),\ x \in \Gamma(x_o)$ and $h{\cdot}\eta \ge 0$ on Γ (see (3.2)–(3.4)) we have from (3.12) the following estimate

$$\begin{aligned}
&\int_t^{t+T}\!\!\int_{\Gamma(x_o)} \left[b^2 \left| \frac{\partial u}{\partial \eta} \right|^2 + (a^2 - b^2)(div\, u)^2 \right] d\Gamma\, ds = \\
&\quad = \int_t^{t+T}\!\!\int_{\Gamma(x_o)} (h\cdot\eta) \left[b^2 \left| \frac{\partial u}{\partial \eta} \right|^2 + (a^2 - b^2)(div\, u)^2 \right] d\Gamma\, ds \\
&\quad \le \int_t^{t+T}\!\!\int_{\Gamma} (h\cdot\eta) \left[b^2 \left| \frac{\partial u}{\partial \eta} \right|^2 + (a^2 - b^2)(div\, u)^2 \right] d\Gamma\, ds \\
&\quad = 2 \int_\Omega (h * \nabla u)\cdot u_t \, dx \Big|_t^{t+T} \\
&\qquad + \int_t^{t+T}\!\!\int_\Omega (div\, h)\Big[|u_t|^2 - b^2 |\nabla u|^2 - (a^2 - b^2)(div\, u)^2 - \alpha\, |u|^2 \Big]\, dx\, ds \\
&\qquad + \int_t^{t+T}\!\!\int_\Omega \rho(x, u_t)\cdot (h * \nabla u)\, dx\, ds \\
&\qquad + \int_t^{t+T}\!\!\int_\Omega \sum_{i,j,k=1}^{3} \left[b^2 \frac{\partial h^k}{\partial x_j} \frac{\partial u^i}{\partial x_j} \frac{\partial u^i}{\partial x_k} + (a^2 - b^2) \frac{\partial h^k}{\partial x_i} \frac{\partial u^j}{\partial x_j} \frac{\partial u^i}{\partial x_k} \right] dx\, ds\,.
\end{aligned} \tag{4.5}$$

Because $h \in C^1(\overline{\Omega})$ and $h = 0$ in $\Omega\backslash\widehat{\omega}$ it implies that

$$\int_\Omega |(h * \nabla u)\cdot u_t|\, dx \Big|_t^{t+T} \le C\Big[E(t) + E(t+T) \Big]\,, \tag{4.6}$$

$$\left| \int_t^{t+T}\!\!\int_\Omega \rho(x, u_t)\cdot (h * \nabla u)\, dx\, ds \right| \le C \int_t^{t+T}\!\!\int_\Omega |\rho(x, u_t)|\, |\nabla u|\, dx\, ds \tag{4.7}$$

and

$$\begin{aligned}
&\left| \int_t^{t+T}\!\!\int_\Omega (div\, h)\Big[|u_t|^2 - b^2 |\nabla u|^2 - (a^2 - b^2)(div\, u)^2 + \alpha\, |u|^2 \Big]\, dx\, ds \right| \le \\
&\quad \le C \int_t^{t+T}\!\!\int_{\widehat{\omega}\cap\overline{\Omega}} \Big[|u_t|^2 + b^2 |\nabla u|^2 + (a^2 - b^2)(div\, u)^2 + \alpha\, |u|^2 \Big]\, dx\, ds\,.
\end{aligned} \tag{4.8}$$

Now, note that

$$\left| \int_t^{t+T}\!\!\int_\Omega \sum_{i,j,k=1}^3 \left[b^2 \frac{\partial h^k}{\partial x_j} \frac{\partial u^i}{\partial x_j} \frac{\partial u^i}{\partial x_k} + (a^2 - b^2) \frac{\partial h^k}{\partial x_i} \frac{\partial u^j}{\partial x_j} \frac{\partial u^i}{\partial x_k} \right] dx\, ds \right| \le$$

$$\tag{4.9} \le C \int_t^{t+T}\!\!\int_{\widehat{\omega}\cap\overline{\Omega}} \sum_{i,j,k=1}^3 \left[b^2 \left| \frac{\partial u^i}{\partial x_j} \right| \left| \frac{\partial u^i}{\partial x_k} \right| + (a^2 - b^2) \left| \frac{\partial u^j}{\partial x_j} \right| \left| \frac{\partial u^i}{\partial x_k} \right| \right] dx\, ds$$

$$\le C \int_t^{t+T}\!\!\int_{\widehat{\omega}\cap\overline{\Omega}} b^2 \, |\nabla u|^2 \, dx\, ds \; .$$

Substituting (4.6)–(4.8) and (4.9) in (4.5) we have

$$\int_t^{t+T}\!\!\int_{\Gamma(x_o)} \left[b^2 \left| \frac{\partial u}{\partial \eta} \right|^2 + (a^2 - b^2)(div\, u)^2 \right] d\Gamma\, ds \le$$

$$\tag{4.10} \le C \Bigg\{ E(t) + E(t+T) + \int_t^{t+T}\!\!\int_\omega |u_t|^2 \, dx\, ds$$

$$+ \int_t^{t+T}\!\!\int_{\widehat{\omega}\cap\overline{\Omega}} \left[b^2 \, |\nabla u|^2 + (a^2 - b^2)(div\, u)^2 + \alpha \, |u|^2 \right] dx\, ds$$

$$+ \int_t^{t+T}\!\!\int_\Omega |\rho(x, u_t)| \, |\nabla u| \, dx\, ds \Bigg\} \; .$$

Using that $0 \le m \le 1$ and $m(x) = 1$ in $\widehat{\omega}$ it results

$$\int_t^{t+T}\!\!\int_{\widehat{\omega}\cap\overline{\Omega}} \left[b^2 \, |\nabla u|^2 + (a^2 - b^2)(div\, u)^2 + \alpha |u|^2 \right] dx\, ds =$$

$$\tag{4.11} = \int_t^{t+T}\!\!\int_{\widehat{\omega}\cap\overline{\Omega}} m(x) \left[b^2 \, |\nabla u|^2 + (a^2 - b^2)(div\, u)^2 + \alpha |u|^2 \right] dx\, ds$$

$$\le \int_t^{t+T}\!\!\int_\Omega m(x) \left[b^2 \, |\nabla u|^2 + (a^2 - b^2)(div\, u)^2 + \alpha |u|^2 \right] dx\, ds \; .$$

From estimates (4.10) in (4.11) we obtain

$$\int_t^{t+T}\!\!\int_{\Gamma(x_o)} \left[b^2 \left| \frac{\partial u}{\partial \eta} \right|^2 + (a^2 - b^2)(div\, u)^2 \right] d\Gamma\, ds \le$$

$$\tag{4.12} \le C \Bigg\{ E(t) + E(t+T) + \int_t^{t+T}\!\!\int_\omega |u_t|^2 \, dx\, ds$$

$$+ \int_t^{t+T}\!\!\int_\Omega m(x) \left[b^2 \, |\nabla u|^2 + (a^2 - b^2)(div\, u)^2 + \alpha \, |u|^2 \right] dx\, ds$$

$$+ \int_t^{t+T}\!\!\int_\Omega |\rho(x, u_t)| \, |\nabla u| \, dx\, ds \Bigg\} .$$

Substituting (4.4) in (4.12) we have

$$
\begin{aligned}
&\int_t^{t+T}\!\!\int_{\Gamma(x_o)} \left[b^2\left|\frac{\partial u}{\partial \eta}\right|^2 + (a^2-b^2)(div\, u)^2\right] d\Gamma\, ds \;\le \\
&\le\; C\left\{E(t)+E(t+T)\;+\int_t^{t+T}\!\!\int_\omega (|u_t|^2+|u|^2)\,dx\,ds \right.\\
&\left. \quad + \int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)|\;(|\nabla u|+|u|)\,dx\,ds\right\}.
\end{aligned} \tag{4.13}
$$

From (4.13) and (4.2) and observing that $E(t)$ is a nonincreasing function, there exist a positive constant C_1 such that

$$
\begin{aligned}
TE(t+T) \;\le \int_t^{t+T} E(s)\,ds \;\le\; C_1\Big\{E(t)+E(t+T)+\int_t^{t+T}\!\!\int_\omega \Big[|u_t|^2+|u|^2\Big]\,dx\,ds \\
+\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)|\Big[|\nabla u|+|u|\Big]\,dx\,ds\Big\}\;.
\end{aligned}
$$

Now, we fix a large $T>0$ such that $T\ge 2\,C_1+1$ to obtain the conclusion of Lemma 4.1. ∎

Lemma 4.2. *Let u be the solution of problem (1.1)–(1.3). Then*

$$
\begin{aligned}
\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)|\Big[|\nabla u|+|u|\Big]\,dx\,ds \;\le\; C\Big[E(t)-E(t+T)\Big]^{\frac{1}{r+2}}\sqrt{E(t)} \\
+\;C\Big[E(t)-E(t+T)\Big]^{\frac{p+1}{p+2}}E(t)^{\frac{4-p}{4(p+2)}}
\end{aligned} \tag{4.14}
$$

for $r\ge 0$ *and* $0\le p\le 2$.

$$
\begin{aligned}
\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)|\Big[|\nabla u|+|u|\Big]\,dx\,ds \;\le\; C\Big[E(t)-E(t+T)\Big]^{\frac{1}{r+2}}\sqrt{E(t)} \\
+\,C\Big[E(t)-E(t+T)\Big]^{\frac{2}{4-p}}\sqrt{E(t)}
\end{aligned} \tag{4.15}
$$

for $r\ge 0$ *and* $-1\le p<0$.

$$
\begin{aligned}
\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)|\Big[|\nabla u|+|u|\Big]\,dx\,ds \;\le\; C\Big[E(t)-E(t+T)\Big]^{\frac{r+1}{r+2}}\sqrt{E(t)} \\
+\,C\Big[E(t)-E(t+T)\Big]^{\frac{p+1}{p+2}}E(t)^{\frac{4-p}{4(p+2)}}
\end{aligned} \tag{4.16}
$$

for $-1<r<0$ *and* $0\le p\le 2$.

$$
\begin{aligned}
\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)|\Big[|\nabla u|+|u|\Big]\,dx\,ds \;\le\; C\Big[E(t)-E(t+T)\Big]^{\frac{r+1}{r+2}}\sqrt{E(t)} \\
+\,C\Big[E(t)-E(t+T)\Big]^{\frac{2}{4-p}}\sqrt{E(t)}
\end{aligned} \tag{4.17}
$$

for $-1<r<0$ and $-1\le p<0$.

Proof: Using the assumptions on function ρ we have:

$$\begin{aligned}
&\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)|\,\left(|\nabla u|+|u|\right)\,dx\,ds \;\leq \\
&\quad\leq\; C\left\{\int_t^{t+T}\!\!\int_{\Omega_1} K_o\,a(x)\left(|u_t|^{r+1}+|u_t|\right)\left(|\nabla u|+|u|\right)\,dx\,ds \right.\\
&\qquad\left. +\int_t^{t+T}\!\!\int_{\Omega_2} K_1\,a(x)\left(|u_t|^{p+1}+|u_t|\right)\left(|\nabla u|+|u|\right)\,dx\,ds\right\} \\
&\quad=\; C(I_1+I_2)\;.
\end{aligned} \tag{4.18}$$

where we have considered for each $t\geq 0$.

$$\Omega_1=\Omega_1(t)=\Big\{x\in\Omega,\;|u_t(x,t)|\leq 1\Big\}\,,\qquad \Omega_2=\Omega\backslash\Omega_1\;.$$

From the proof of Existence Theorem we obtain that $\|u_{tt}\|^2_{L^2(\Omega)}$, $\|\nabla u_t\|^2_{L^2(\Omega)}$ and $\|\Delta u\|^2_{L^2(\Omega)}$ are bounded. Then, to estimate I_1 and I_2 we consider the four cases related in the Theorem about Stabilization:

For the case $r\geq 0$ and $0\leq p\leq 2$, using Poincare's inequality we obtain

$$\begin{aligned}
I_1 \;&\leq\; \|\sqrt{a}\|_{L^\infty(\Omega)}\,2\,K_o\int_t^{t+T}\!\!\int_{\Omega_1}\sqrt{a(x)}\;|u_t|\;\left(|\nabla u|+|u|\right)\,dx\,ds \\
&\leq\; C\left(\int_t^{t+T}\!\!\int_{\Omega_1} a(x)\,|u_t|^2\;dx\,ds\right)^{1/2}\left(\int_t^{t+T}\!\!\int_{\Omega_1}\left(|\nabla u|+|u|\right)^2\,dx\,ds\right)^{1/2} \\
&\leq\; C\left(\int_t^{t+T}\!\!\int_{\Omega_1} a(x)\,|u_t|^2\;dx\,ds\right)^{1/2}\left(\int_t^{t+T} E(s)\,ds\right)^{1/2} \\
&\leq\; C\left(\int_t^{t+T}\!\!\int_{\Omega_1} a(x)\,|u_t|^{r+2}\;dx\,ds\right)^{\frac{1}{r+2}}\sqrt{E(t)}
\end{aligned}$$

because $\dfrac{2}{r+2}+\dfrac{r}{r+2}=1$, where C depends on $|\Omega|$, $\|\sqrt{a}\|_{L^\infty(\Omega)}$ and the fixed T. We have used the fact that $E(t)$ is a nonincreasing function of t.

From the hypotheses (2.2) on the function ρ and using (2.4), we get

$$I_1\;\leq\; C\left(\int_t^{t+T}\!\!\int_{\Omega_1}\rho(x,u_t)\cdot u_t\;dx\,ds\right)^{\frac{1}{r+2}}\sqrt{E(t)}\;\leq\; C\Big[E(t)-E(t+T)\Big]^{\frac{1}{r+2}}\sqrt{E(t)}\,.$$

Also, we have

$$\begin{aligned} I_2 \;&\le\; 2\,K_1 \int_t^{t+T}\!\!\int_{\Omega_2} a(x)\,|u_t|^{p+1}\,(|\nabla u| + |u|)\;dx\,ds \\ &\le\; C\left(\int_t^{t+T}\!\!\int_{\Omega_2} a(x)^{\frac{p+2}{p+1}}\,|u_t|^{p+2}\;dx\,ds\right)^{\frac{p+1}{p+2}} \left(\int_t^{t+T}\!\!\int_{\Omega_2} (|\nabla u| + |u|)^{p+2}\;dx\,ds\right)^{\frac{1}{p+2}} \\ &\le\; C\left(\int_t^{t+T}\!\!\int_{\Omega_2} a(x)\,|u_t|^{p+2}\;dx\,ds\right)^{\frac{p+1}{p+2}} \left(\int_t^{t+T}\!\!\int_{\Omega} |\nabla u|^{p+2}\;dx\,ds\right)^{\frac{1}{p+2}} \end{aligned}$$

where we have used the Poincare's inequality in $W_o^{1,p+2}(\Omega)$ and the fact that $a(x)$ is a bounded function.

Using Gagliardo–Niremberg Lemma and Poincare's inequality we have

$$\begin{aligned} \|\nabla u\|_{(L^{p+2}(\Omega))^3} &\le C\,\|\nabla u\|^{\theta}_{(H^1(\Omega))^3}\,\|\nabla u\|^{1-\theta}_{(L^2(\Omega))^3} \le C\,\|u\|^{\theta}_{(H^2(\Omega)\cap H^1_o(\Omega))^3}\,\|\nabla u\|^{1-\theta}_{(L^2(\Omega))^3} \\ &\le C\,\|\Delta u\|^{\theta}_{(L^2(\Omega))^3}\,\|\nabla u\|^{1-\theta}_{(L^2(\Omega))^3} \le C\,\|\nabla u\|^{1-\theta}_{(L^2(\Omega))^3} \le CE(t)^{\frac{1-\theta}{2}} \end{aligned}$$

with $\theta = \frac{3p}{2(p+2)}$.

From above estimate and the assumption (2.3) on $\rho(x,s)$ and using (2.4), we conclude that

$$\begin{aligned} I_2 \;&\le\; C\left(\int_t^{t+T}\!\!\int_{\Omega_2} \rho(x,u_t)\cdot u_t\;dx\,ds\right)^{\frac{p+1}{p+2}} E(t)^{\frac{1-\theta}{2}} \\ &\le\; C\Big[E(t) - E(t+T)\Big]^{\frac{p+1}{p+2}}\,E(t)^{\frac{4-p}{4(p+2)}}\;. \end{aligned}$$

Combining the estimates for I_1 and I_2 with inequality (4.18) we conclude the proof of (4.14).

Next, we consider the case $r \ge 0$ and $-1 \le p < 0$. Then

$$I_1 \;\le\; C\Big[E(t) - E(t+T)\Big]^{\frac{1}{r+2}}\,\sqrt{E(t)}\;.$$

Poincare's inequality implies that

$$\begin{aligned} I_2 \;&\le\; 2\,K_1 \int_t^{t+T}\!\!\int_{\Omega_2} a(x)\,|u_t|\,(|\nabla u| + |u|)\;dx\,ds \\ &\le\; C\left(\int_t^{t+T}\!\!\int_{\Omega_2} a(x)\,|u_t|^2\;dx\,ds\right)^{\frac{1}{2}}\sqrt{E(t)}\;\le \end{aligned}$$

$$\leq C \left[\left(\int_t^{t+T}\!\!\int_{\Omega_2} \left(a(x)\,|u_t|^{2-\alpha}\right)^{\frac{4-p}{4}} dx\,ds \right)^{\frac{4}{4-p}} \right]^{1/2}$$

$$\cdot \left[\left(\int_t^{t+T}\!\!\int_{\Omega_2} (|u_t|^{\alpha})^{\frac{4-p}{-p}}\; dx\,ds \right]^{\frac{-p}{4-p}} \right)^{1/2} \sqrt{E(t)}$$

$$= C \left(\int_t^{t+T}\!\!\int_{\Omega_2} a(x)\,|u_t|^{p+2}\; dx\,ds \right)^{\frac{2}{4-p}} \left(\int_t^{t+T}\!\!\int_{\Omega_2} |u_t|^6\; dx\,ds \right)^{\frac{-p}{8-2p}} \sqrt{E(t)}$$

with $\alpha = \frac{-6p}{4-p}$.

From Theorem of Existence we have that $u \in W^{1,\infty}(0,\infty;(H^1_o(\Omega))^3)$, then $u_t \in L^\infty(0,\infty;(H^1_o(\Omega))^3)$. Therefore,

$$\|u_t\|_{(L^6(\Omega))^3} \,\leq\, C\, \|\nabla u_t\|_{(L^2(\Omega))^3} \,\leq\, C\;.$$

Thus, we have

$$I_2 \,\leq\, C \left(\int_t^{t+T}\!\!\int_{\Omega_2} a(x)\,|u_t|^{p+2}\; dx\,ds \right)^{\frac{2}{4-p}} \sqrt{E(t)}$$

$$\leq\, C \left(\int_t^{t+T}\!\!\int_{\Omega_2} \rho(x,u_t)\cdot u_t\; dx\,ds \right)^{\frac{2}{4-p}} \sqrt{E(t)} \,\leq\, C\Big[E(t) - E(t{+}T)\Big]^{\frac{2}{4-p}} \sqrt{E(t)}\,.$$

Combining above estimates for I_1 and I_2 with inequality (4.18), the result of (4.15) holds.

The case $-1 < r < 0$ and $0 \leq p \leq 2$. In this case, using Poincare's inequality and (2.4), we have

$$I_1 \,\leq\, C \int_t^{t+T}\!\!\int_{\Omega_1} a(x)\,|u_t|^{r+1}\,|\nabla u|\; dx\,ds$$

$$\leq\, C \left(\int_t^{t+T}\!\!\int_{\Omega_1} a(x)\,|u_t|^{r+2}\; dx\,ds \right)^{\frac{r+1}{r+2}} \left(\int_t^{t+T}\!\!\int_{\Omega} |\nabla u|^{r+2}\; dx\,ds \right)^{\frac{1}{r+2}}$$

$$\leq\, C \left(\int_t^{t+T}\!\!\int_{\Omega_1} \rho(x,u_t)\cdot u_t\; dx\,ds \right)^{\frac{r+1}{r+2}} \left(\int_t^{t+T}\!\!\int_{\Omega} |\nabla u|^{r+2}\; dx\,ds \right)^{\frac{1}{r+2}}$$

$$\leq\, C\Big[E(t) - E(t+T)\Big]^{\frac{r+1}{r+2}} \sqrt{E(t)}\;.$$

The estimate for I_2 is the same of case (4.14): $r \geq 0$ and $0 \leq p \leq 2$. From estimates for I_1, I_2 and inequality (4.18) we conclude the proof of (4.16).

The case $-1 < r < 0$ and $-1 \le p < 0$. In the same way of above case, we obtain

$$I_1 \le C\Big[E(t) - E(t+T)\Big]^{\frac{r+1}{r+2}} \sqrt{E(t)}\ .$$

The integral I_2 is estimated like in the second case (4.15), that is

$$I_2 \le C\Big[E(t) - E(t+T)\Big]^{\frac{2}{4-p}} \sqrt{E(t)}\ .$$

Combining these estimates with inequality (4.18) we conclude the proof of (4.17). ∎

Proposition 4.3. *The energy for the solution of problem (1.1)–(1.3) satisfies*

$$E(t) \le C\Big\{D_i(t)^2 + \int_t^{t+T}\int_\omega (|u|^2 + |u_t|^2)\,dx\,ds\Big\}, \quad i = 1,2,3,4\ , \tag{4.19}$$

where

$$D_1(t)^2 = \Big[E(t) - E(t+T)\Big] + \Big[E(t) - E(t+T)\Big]^{\frac{2}{r+2}} + \Big[E(t) - E(t+T)\Big]^{\frac{4(p+1)}{5p+4}}$$

if $r \ge 0$ *and* $0 \le p \le 2$*;*

$$D_2(t)^2 = \Big[E(t) - E(t+T)\Big] + \Big[E(t) - E(t+T)\Big]^{\frac{2}{r+2}} + \Big[E(t) - E(t+T)\Big]^{\frac{4}{4-p}}$$

for the case $r \ge 0$ *and* $-1 \le p < 0$*;*

$$D_3(t)^2 = \Big[E(t) - E(t+T)\Big] + \Big[E(t) - E(t+T)\Big]^{\frac{2(r+1)}{r+2}} + \Big[E(t) - E(t+T)\Big]^{\frac{4(p+1)}{5p+4}}$$

for the case $-1 < r < 0$ *and* $0 \le p \le 2$*;*

$$D_4(t)^2 = \Big[E(t) - E(t+T)\Big] + \Big[E(t) - E(t+T)\Big]^{\frac{2(r+1)}{r+2}} + \Big[E(t) - E(t+T)\Big]^{\frac{4}{4-p}}$$

for the case $-1 < r < 0$ *and* $-1 \le p < 0$*.*

Proof: The proof follows using Lemma 4.1, Lemma 4.2 and Young's inequality. ∎

Now, in order to estimate the last two terms of (4.19) we need the following result.

Proposition 4.4. *There exists a constant* $C > 0$ *such that*

$$\int_t^{t+T}\int_\Omega |u|^2\,dx\,ds \le C\left\{D_i(t)^2 + \int_t^{t+T}\int_\omega |u_t|^2\,dx\,ds\right\} \tag{4.20}$$

where u *is the solution of problem (1.1)–(1.3) with initial data* u_o, u_1 *such that* $E(0) \le R$, $R > 0$ *fixed. The constant* C *depends on* R.

Proof: We prove by contradiction. If (4.20) is false, exist a sequence of solutions $\{u_n\}_{n\geq 1}$ with initial data u_o^n, u_1^n and a sequence $\{t_n\}_{n\geq 1}$ such that

$$\lim_{n\to\infty} \frac{\displaystyle\int_{t_n}^{t_n+T} \|u_n(s)\|^2_{(L^2(\Omega))^3}\, ds}{\displaystyle D_i(t_n)^2 + \int_{t_n}^{t_n+T}\!\!\int_\omega |u_{nt}|^2\, dx\, ds} \;=\; +\infty\;. \tag{4.21}$$

Let $\lambda_n^2 = \displaystyle\int_{t_n}^{t_n+T} \|u_n(s)\|^2_{(L^2(\Omega))^3}\, ds$ and $v_n(t) = \frac{u(t+t_n)}{\lambda_n}$, $0\leq t\leq T$. Then we have from (4.21)

$$\mathcal{Q}_n^2(t_n) = \frac{1}{\lambda_n^2}\left\{D_i(t_n)^2 + \int_{t_n}^{t_n+T}\!\!\int_\omega |u_{nt}|^2\, dx\, ds\right\} \to 0\,, \qquad n\to\infty \tag{4.22}$$

according each case $i=1,2,3,4$.

Also, we have

$$\int_0^T\!\!\int_\Omega |v_n|^2\, dx\, ds \;=\; 1\;. \tag{4.23}$$

Thus, we have from inequality (4.19),

$$\begin{aligned}
E(v_n(t)) \;&=\; E\left(\frac{u_n(t+t_n)}{\lambda_n}\right) \;=\; \frac{1}{\lambda_n^2}\, E(u_n(t+t_n)) \;\leq\; \frac{1}{\lambda_n^2}\, E(u_n(t_n)) \;\leq \\
&\leq\; \frac{1}{\lambda_n^2}\left\{D_i(t_n)^2 + \int_{t_n}^{t_n+T}\!\!\int_\omega (|u_{nt}|^2 + |u_n|^2)\, dx\, ds\right\} \\
&\leq\; C\left\{\mathcal{Q}_n(t_n)^2 + \frac{1}{\lambda_n^2}\int_{t_n}^{t_n+T}\!\!\int_\omega |u_n(x,s)|^2\, dx\, ds\right\} \\
&\leq\; C\left\{\mathcal{Q}_n(t_n)^2 + \frac{1}{\lambda_n^2}\int_0^T\int_\Omega |u_n(x,\tau+t_n)|^2 dx\, d\tau\right\} \\
&=\; C\left\{\mathcal{Q}_n(t_n)^2 + \int_0^T\int_\Omega |v_n(x,\tau)|^2 dx\, d\tau\right\} \\
&\leq\; C\Big\{\mathcal{Q}_n(t_n)^2 + 1\Big\} \;\leq\; 2\,C \;<\; +\infty\,, \qquad 0\leq t\leq T, \;\; n \;\text{ large}\;.
\end{aligned}$$

Therefore,

$$\|v_{nt}(t)\|_{(L^2(\Omega))^3}\,, \;\; \|\nabla v_n\|_{(L^2(\Omega))^3} \leq C\,, \qquad 0\leq t\leq T, \;\; \forall\, n\;. \tag{4.24}$$

On the other hand, by Poincare's inequality and estimate (4.24) it results

$$\begin{aligned}
&\int_\Omega |v_n(x,t)|^2\, dx \;=\; \int_\Omega \frac{1}{\lambda_n^2}\, |u_n(x,t+t_n)|^2\, dx \;\leq \\
&\leq\; C\int_\Omega \frac{1}{\lambda_n^2}\, |\nabla u_n(x,t+t_n)|^2\, dx \;\leq\; C\int_\Omega |\nabla v_n(t)|^2 dx \;<\; C\,, \quad \forall\, t\in[0,T],\; \forall\, n.
\end{aligned} \tag{4.25}$$

From (4.24) and (4.25) we conclude

$$v_n \in W^{1,\infty}(0,T;(L^2(\Omega))^3) \cap L^\infty(0,T;(H_0^1(\Omega))^3) \ . \tag{4.26}$$

To take the limit of $v_n(t)$, first, we need check that

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\rho(x,u_{nt}(t+t_n)) = 0 \quad \text{in} \ \ L^1([0,T]\times\Omega) \ .$$

For the case $r \geq 0$ and $0 \leq p \leq 2$ we see, from the estimates for I_1 and I_2, that

$$\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)| \, dx \, ds \ \leq \ C\left\{\Big[E(t)-E(t+T)\Big]^{\frac{1}{r+2}} + \Big[E(t)-E(t+T)\Big]^{\frac{p+1}{p+2}}\right\} \tag{4.27}$$

and using the definition of $D_1(t)$ in Proposition 4.3 we obtain

$$\Big[E(t)-E(t+T)\Big]^{\frac{1}{r+2}} \leq C\, D_1(t) \ ,$$
$$\Big[E(t)-E(t+T)\Big]^{\frac{p+1}{p+2}} \leq C\, D_1(t)^{\frac{4+5p}{2(p+2)}} \ .$$

Therefore

$$\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_t)| \, dx \, ds \ \leq \ C\Big\{D_1(t) + D_1(t)^{\frac{4+5p}{2(p+2)}}\Big\} \ .$$

Using the definition of $\mathcal{Q}_n$ we can write

$$\begin{aligned} \frac{1}{\lambda_n}\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_{nt})| \, dx \, ds \ &\leq \ C\Big\{\mathcal{Q}_n(t_n) + \lambda_n^{-1}\, D_1(t_n)^{\frac{4+5p}{2(p+2)}}\Big\} \\ &= \ C\Big\{\mathcal{Q}_n(t_n) + \lambda_n^{\frac{3p}{2(p+2)}}\, \mathcal{Q}_n(t_n)^{\frac{4+5p}{2(p+2)}}\Big\} \ . \end{aligned}$$

Now, we observe that $\{\lambda_n\}_{n\geq 1}$ is a bounded sequence

$$\lambda_n = \left(\int_{t_n}^{t_n+T} \|u_n(s)\|^2 \, ds\right)^{1/2} \leq C\left(\int_{t_n}^{t_n+T} \|\nabla u_n(s)\|^2 \, ds\right)^{1/2} \leq \ C\, E(u_n(0)) \ \leq \ C\, R$$

because the initial data are in a ball $B(0,R)$.

Hence, (4.22) implies that

$$\begin{aligned} &\frac{1}{\lambda_n}\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_{nt})| \, dx \, ds \ \leq \\ &\qquad \leq \ C\Big\{\mathcal{Q}_n(t_n) + \lambda_n^{\frac{3p}{2(p+2)}}\, \mathcal{Q}_n(t_n)^{\frac{4+5p}{2(p+2)}}\Big\} \to 0\,, \qquad n\to\infty \ . \end{aligned} \tag{4.28}$$

For the case $r \geq 0$ and $-1 \leq p < 0$, from estimates for I_1 and I_2, we obtain

$$\begin{aligned}\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_{nt})|\,dx\,ds \;&\leq\; C\Big\{\Big[E(t)-E(t+T)\Big]^{\frac{1}{r+2}}+\Big[E(t)-E(t+T)\Big]^{\frac{2}{4-p}}\Big\}\\ &\leq\; C\,D_2(t)\ .\end{aligned}$$

Thus

$$\frac{1}{\lambda_n}\int_t^{t+T}\!\!\int_\Omega |\rho(x,u_{nt})|\,dx\,ds \;\leq\; C\,\mathcal{Q}_n(t_n) \;\to\; 0\,, \qquad n\to\infty\ . \tag{4.29}$$

For the others two cases we obtain the same conclusion as in (4.28) and (4.29) respectively. We have proved, in the four cases, that

$$\frac{1}{\lambda_n}\,\rho(x,u_{nt}(t+t_n))\to 0 \quad \text{ in } \ L^1([0,T]\times\Omega)\ . \tag{4.30}$$

Now we can take a limit of $\{v_n(t)\}_{n\geq 1}$. From (4.26) there exists a function $v(t)$ such that

$$v_n(t) \overset{*}{\rightharpoonup} v(t), \quad \text{ in } \ W^{1,\infty}(0,T;(L^2(\Omega))^3)\cap L^\infty(0,T;(H_o^1(\Omega))^3)$$

and strongly in $L^2([0,T]\times\Omega)$.

Then, the limit function satisfies:

$$v \,\in\, W^{1,\infty}(0,T;(L^2(\Omega))^3)\cap L^\infty(0,T;(H_o^1(\Omega))^3)\ ,$$

$$v_{tt}-b^2\Delta v-(a^2-b^2)\,\nabla\,div\,v+\alpha\,v \;=\; 0, \quad \text{ in } \ [0,T]\times\Omega\ , \tag{4.31}$$

$$\int_0^T\!\!\int_\omega |v_t(t)|^2\,dx\,ds \;=\; 0\ , \tag{4.32}$$

$$\int_0^T \|v(t)\|^2\,ds \;=\; 1\ . \tag{4.33}$$

In fact, semigroups theory says that

$$v \,\in\, C^1(0,T;(L^2(\Omega))^3)\cap C(0,T;(H_o^1(\Omega))^3)\ .$$

Because the function v satisfies (4.31) and (4.32), the Holmgren's Uniqueness Theorem implies that $v(x,t)=0$ in $\Omega\times[0,T]$. This result contradicts the condition (4.33) on the function v. Thus, the inequality (4.20) holds. ∎

5 – Proof of Theorem 2.2

Using the result of Proposition 4.3 and the estimate in the Proposition 4.4 we conclude

$$E(t) \le C\left\{D_i(t)^2 + \int_t^{t+T}\int_\omega |u_t|^2\,dx\,ds\right\} \tag{5.1}$$

where $D_i(t)$ $(i = 1, 2, 3, 4)$ are given in the Proposition 4.3.

Now, we shall estimate the last term in (5.1) and derive the decay estimates stated in the Theorem 2.2.

For the case $0 \le r$ and $0 \le p \le 2$, we see from the hypothesis (2.1) on function $a(x)$

$$\begin{aligned}\int_t^{t+T}\int_\omega |u_t|^2\,dx\,ds &\le C\int_t^{t+T}\int_\Omega a(x)\,|u_t|^2\,dx\,ds\\ &\le C\int_t^{t+T}\int_{\Omega_1} a(x)\,|u_t|^2\,dx\,ds + \int_t^{t+T}\int_{\Omega_2} a(x)\,|u_t|^2\,dx\,ds\\ &\le C\left\{\left[\int_t^{t+T}\int_{\Omega_1} a(x)\,|u_t|^{r+2}\,dx\,ds\right]^{\frac{2}{r+2}} + \int_t^{t+T}\int_{\Omega_2} a(x)\,|u_t|^{p+2}\,dx\,ds\right\}\\ &\le C\left\{\left[\int_t^{t+T}\int_{\Omega_1} \rho(x,u_t)\cdot u_t\,dx\,ds\right]^{\frac{2}{r+2}} + \int_t^{t+T}\int_{\Omega_2} \rho(x,u_t)\cdot u_t\,dx\,ds\right\}\end{aligned}$$

where the last C depends on $|\Omega|$, T and $||a||_\infty$.

Then, due to (2.4), we have

$$\int_t^{t+T}\int_\omega |u_t|^2\,dx\,ds \le C\left\{\Big[E(t) - E(t+T)\Big] + \Big[E(t) - E(t+T)\Big]^{\frac{2}{r+2}}\right\}\ . \tag{5.2}$$

From (5.1), (5.2) and the expression for $D_1(t)$we obtain

$$E(t) \le C\left\{\Big[E(t)-E(t+T)\Big] + \Big[E(t)-E(t+T)\Big]^{\frac{2}{r+2}} + \Big[E(t)-E(t+T)\Big]^{\frac{4(p+1)}{4+5p}}\right\}.$$

Then, because $E(t)$ is bounded, we conclude that

$$E(t) \le C\Big[E(t) - E(t+T)\Big]^{K_1}$$

where $K_1 = \min\left\{\frac{2}{r+2}, \frac{4(p+1)}{4+5p}\right\}$ is such that $0 < K_1 < 1$.

We have obtained the following inequality

$$\sup_{t\le s\le t+T} E(s)^{\frac{1}{K_1}} \le E(t)^{\frac{1}{K_1}} \le C\Big[E(t) - E(t+T)\Big]\ . \tag{5.3}$$

If we set $1+\gamma = \frac{1}{K_1}$, then $\gamma = \frac{1-K_1}{K_1}$ and applying Lemma 3.1 to (5.3) we obtain that

$$E(t) \le C_1(1+t)^{-\gamma_1} \tag{5.4}$$

with $\gamma_1 = \min\left\{\frac{2}{r}, \frac{4(p+1)}{p}\right\}$.

For the case $0 \le r$ and $-1 \le p < 0$ we have

$$\int_t^{t+T}\int_{\Omega_2} a(x)\,|u_t|^2\,dx\,ds \;\le\; C_1\Big[E(t)-E(t+T)\Big]^{\frac{4}{4-p}}\;.$$

Then, in the same way of the first case, we get

$$\int_t^{t+T}\int_{\omega} |u_t|^2\,dx\,ds \;\le\; C_1\left\{\Big[E(t)-E(t+T)\Big]^{\frac{2}{r+2}}+\Big[E(t)-E(t+T)\Big]^{\frac{4}{4-p}}\right\}. \tag{5.5}$$

From (5.1) , (5.5) and the definition of $D_2(t)$ we have

$$E(t) \;\le\; C\left\{E(t)-E(t+T)+\Big[E(t)-E(t+T)\Big]^{\frac{2}{r+2}}+\Big[E(t)-E(t+T)\Big]^{\frac{4}{4-p}}\right\}$$

and because this

$$E(t)^{\frac{1}{K_2}} \;\le\; C\Big[E(t)-E(t+T)\Big] \tag{5.6}$$

with $K_2=\min\left\{\frac{2}{r+2},\frac{4}{4-p}\right\}$ such that $0<K_2<1$.

Applying Lemma 3.1 to (5.6) we have

$$E(t) \;\le\; C(1+t)^{-\gamma_2}$$

with $\gamma_2=\min\left\{\frac{2}{r},\frac{-4}{p}\right\}$.

For the case $-1<r<0$ and $0\le p\le 2$, we see that

$$\begin{aligned}\int_t^{t+T}\int_{\Omega_1} a(x)\,|u_t|^2\,dx\,ds \;&\le\; C\int_t^{t+T}\int_{\Omega_1} a(x)\,|u_t|^{r+2}\,dx\,ds\\ &\le\; C\int_t^{t+T}\int_{\Omega_1} \rho(x,u_t)\cdot u_t\,dx\,ds \;\le\; C\Big[E(t)-E(t+T)\Big]\end{aligned}$$

and it follows that

$$\int_t^{t+T}\int_{\omega} |u_t|^2\,dx\,ds \;\le\; C\Big[E(t)-E(t+T)\Big]\;. \tag{5.7}$$

Thus, from (5.1), (5.7) and the definition of $D_3(t)$, we get

$$E(t)\le C\left\{E(t)-E(t+T)+\Big[E(t)-E(t+T)\Big]^{\frac{2(r+1)}{r+2}}+\Big[E(t)-E(t+T)\Big]^{\frac{4(p+1)}{4+5p}}\right\}. \tag{5.8}$$

Therefore

$$E(t)^{\frac{1}{K_3}} \;\le\; C_1\Big[E(t)-E(t+T)\Big]$$

with $K_3=\min\left\{\frac{2(r+1)}{r+2},\frac{4(p+1)}{4+5p}\right\}$ such that $0<K_3<1$.

We conclude by Lemma 3.1,

$$E(t) \;\le\; C(1+t)^{-\gamma_3}$$

with $\gamma_3=\min\left\{\frac{-2(r+1)}{r},\frac{4(p+1)}{p}\right\}$.

Finally, for the case $-1 < r < 0$ and $-1 \le p < 0$, we have

$$\int_t^{t+T}\!\!\int_\omega |u_t|^2\,dx\,ds \;\le\; C\left\{\Big[E(t)-E(t+T)\Big] + \Big[E(t)-E(t+T)\Big]^{\frac{4}{4-p}}\right\}$$

and by (5.1) and the definition of $D_4(t)$ it follows that

$$E(t) \;\le\; C\left\{E(t)-E(t+T)+\Big[E(t)-E(t+T)\Big]^{\frac{2(r+1)}{r+2}}+\Big[E(t)-E(t+T)\Big]^{\frac{4}{4-p}}\right\}.$$

Thus

$$E(t)^{\frac{1}{K_4}} \;\le\; C\Big[E(t)-E(t+T)\Big]$$

with $K_4 = \min\left\{\frac{2(r+1)}{r+2}, \frac{4}{4-p}\right\}$.

From this inequality, using Lemma 3.1, we obtain

$$E(t) \;\le\; C(1+t)^{-\gamma_4}$$

with $\gamma_4 = \min\left\{\frac{-2(r+1)}{r}, \frac{-4)}{p}\right\}$.

The proof of Theorem 2.2 is complete. ∎

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