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AN ALTERNATIVE FUNCTIONAL APPROACH TO EXACT CONTROLLABILITY OF REVERSIBLE SYSTEMS

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Abstract: A new functional approach is devised to establish an equivalence between the null-controllability of a given initial state and a certain individual observability property involving a momentum depending on the state. For instance if one considers the abstract second order control problem $y'' + Ay = Bh(t)$ in time T by means of a control function $h \in L^2(0,T,H)$ with $B \in \mathcal{L}(H)$, $B = B^* \geq 0$, a necessary and sufficient condition for null-controllability of a given state $[y^0, y^1] \in D(A^{1/2}) \times H$ is that the image of $[y^0, y^1]$ under the symplectic map lies in the dual space of the completion of the energy space with respect to a certain semi-norm. A similar property is derived for a general class of first order systems including the transport equation and Schrödinger equations. When A has compact resolvant the necessary and sufficient condition can be formulated by some conditions on the Fourier components of the initial state in a basis of "eigenstates" related to diagonalization of the quadratic form measuring the observability degree of the system under B.

The theory of exact controllability of infinite dimensional conservative systems has experienced an important breakthrough in 1986 with the introduction of the Hilbert uniqueness method by J.L. Lions [17, 18]. For instance if we consider the wave equation

$$u_{tt} - \Delta u = 0 \quad \text{in } \mathbb{R} \times \Omega\,, \qquad u = 0 \quad \text{on } \mathbb{R} \times \partial\Omega \tag{0.1}$$

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where Ω is a bounded smooth domain of $\mathbb{R}^N$ and the corresponding controlled problem

$$(0.2) \qquad y_{tt} - \Delta y = \chi_\omega \, h(t,x) \quad \text{in } (0,T)\times\Omega\,, \qquad y = 0 \quad \text{on } (0,T)\times\partial\Omega$$

in time T by means of an L^2 control confined in an open subset $\omega \subset \Omega$, the HUM method establishes an equivalence between the null-controllability of a given initial state $[y(0), y'(0)] := [y^0, y^1]$ under (0.2) and the observability property

$$(0.3) \qquad \forall[\phi^0,\phi^1] \in V\times H\,, \quad \Big|(y^0,\phi^1)_H - (\phi^0,y^1)_H\Big| \le C\Big\{\int_Q \phi^2(t,x)\; dx\, dt\Big\}^{\frac{1}{2}}$$

where $Q=(0,T)\times\Omega$, $H=L^2(\Omega)$, $V=H_0^1(\Omega)$, C is any finite positive constant and $\phi(t,x) \in C(\mathbb{R},V)\cap C^1(\mathbb{R},H)$ denotes the solution of (0.1) such that $\phi(0) = \phi^0$ and $\phi'(0) = \phi^1$. At least this result can be proved by the standard HUM method when the *uniqueness* property holds true, in the sense that solutions of (0.1) are *characterized* by their trace on $(0,T)\times\omega$. Indeed, in this case, (0.3) exactly means that the image of $[y^0,y^1]$ under the symplectic map lies in the dual space of the completion of the energy space with respect to the norm of that trace in $L^2((0,T)\times\omega)$. However when uniqueness fails, (0.3) still looks like a very reasonable characterization of null-controllable states, and this result was established in [11] by using a special eigenfunction expansion. This new result itself was still unsatisfactory since one feels that (0.3) could very well give the right conditions in a much more general context, independently of any boundedness of the domain and for quite arbitrary operators. The proof of this natural conjecture is the first object of this paper. Actually a similar property shall be first derived for a general class of first order systems including the transport equation and Schrödinger equations. Then we shall consider the general second order case. In addition to that, we shall establish a simple and general property enlighting the relationship between the first part of this paper and the results of [11]. This will lead us to the notion of "eigenstates", generally useful for second order problems and leading also to explicit formulas in some specific first-order problems.

The plan of this paper is as follows: in Sections 1 and 2 we characterize controllable states respectively for first and second order systems, in Sections 3 and 4 we develop the applications of eigenstates in both cases. Sections 5 and 6 are respectively devoted to point control of general second order problems and boundary control of the wave equation.

1 – The abstract Schrödinger equation

In this section we consider the first order evolution equation

$$\varphi' + C\varphi = 0\,, \qquad t \in \mathbb{R} \tag{1.1}$$

where C is a skew-adjoint operator on a real Hilbert space H and the corresponding controlled problem

$$y' + Cy = Bh(t) \quad \text{in} \ \ (0,T) \tag{1.2}$$

in time T by means of a control function $h \in L^2(0,T,H)$ with

$$B \in \mathcal{L}(H)\,, \qquad B = B^* \geq 0\ . \tag{1.3}$$

Theorem 1.1. *For any $y^0 \in H$, the two following conditions are equivalent:*

i) *There exists $h \in L^2(0,T;H)$ such that the mild solution y of (1.2) such that $y(0) = y^0$ satisfies $y(T) = 0$.*

ii) *There exists a finite positive constant C such that*

$$\forall \varphi^0 \in H\,, \qquad |(y^0,\varphi^0)_H| \,\leq\, C\Big\{\int_0^T |B\varphi(t)|_H^2\ dt\Big\}^{\frac{1}{2}} \tag{1.4}$$

where $\varphi(t) \in C(\mathbb{R},H)$ denotes the unique mild solution φ of (1.1) such that $\varphi(0) = \varphi^0$.

Proof: We proceed in 5 steps

Step 1. Let φ and y be a pair of strong solutions of (1.1) and (1.2), respectively. We have

$$\begin{aligned}\frac{d}{dt}\Big(y(t),\varphi(t)\Big) \;&=\; \Big(y'(t),\varphi(t)\Big) + \Big(y(t),\varphi'(t)\Big)\\ &=\; \Big(-Cy(t) + Bh(t),\ \varphi(t)\Big) + \Big(y(t), -C\varphi(t)\Big)\\ &=\; \Big(Bh(t),\varphi(t)\Big)\ .\end{aligned}$$

By integrating on $(0,T)$ we find

$$\Big(y(T),\varphi(T)\Big) - \Big(y(0),\varphi(0)\Big) \;=\; \int_0^T \Big(Bh(t),\varphi(t)\Big)\,dt\ . \tag{1.5}$$

By density, this identity is valid for mild solutions as well. Since B is bounded, self-adjoint and $B \geq 0$,

$$\int_0^T \Big(Bh(t), \varphi(t)\Big)\, dt \;=\; \int_0^T \Big(h(t), B\varphi(t)\Big)\, dt$$

finally if there exists $h \in L^2(0,T;H)$ such that the mild solution y of (1.2) with $y(0) = y^0$ satisfies $y(T) = 0$, we find as a consequence of (1.5)

$$-\Big(y(0), \varphi(0)\Big) \;=\; \int_0^T \Big(h(t), B\varphi(t)\Big)\, dt$$

and by the Cauchy–Schwartz inequality we obtain (1.4). Therefore i) implies ii).

Step 2. If $B \geq \alpha > 0$ we have for any mild solution φ of (1.1)

$$\int_0^T \Big(B\varphi(t), B\varphi(t)\Big)\, dt \;\geq\; \alpha^2 \int_0^T \Big(\varphi(t), \varphi(t)\Big)\, dt \;=\; \alpha^2\, T\, |\varphi(0)|^2$$

and in particular (1.4) is fulfilled. The proof of ii) $\Rightarrow$ i) in this special case is the object of

Lemma 1.2. *Assuming*

$$\exists\; \alpha > 0,\;\; B \geq \alpha \tag{1.6}$$

for each $y^0 \in H$, there exists $\varphi^0 \in H$ such that the mild solution y of (1.2) with $h = \varphi \in L^2(0,T;H)$ and $y(0) = y^0$ satisfies $y(T) = 0$.

Proof: We construct a bounded linear operator $\mathcal{A}$ on H in the following way: for any $z \in H$ we consider first the solution φ of (1.1) such that $\varphi(0) = z$. Then we consider the unique mild solution y of

$$y' + Cy = B\varphi(t) \quad \text{ in } \;(0,T), \quad y(T) = 0\;,$$

and finally we set

$$\mathcal{A}(z) = -y(0)\;.$$

By formula (1.5) we find

$$\Big(\mathcal{A}(z), z\Big) = -\Big(y(0), \varphi(0)\Big) = \int_0^T \Big(B\varphi(t), \varphi(t)\Big)\, dt \;\geq\; \alpha \int_0^T |\varphi(t)|^2\, dt \;=\; \alpha\, T |z|^2\;.$$

Hence $\mathcal{A}$ is coercive on H, and this implies $\mathcal{A}(H) = H$. Given any $y^0 \in H$, there exists $z \in H$ such that $\mathcal{A}(z) = -y^0$. This gives exactly the expected conclusion. ∎

Step 3. We now use a standard penalty method. For each $\varepsilon > 0$ we set

$$\beta_\varepsilon := B^2 + \varepsilon I \ .$$

As a consequence of Lemma 1.2 there exists a $\varphi^{0,\varepsilon} \in H$ such that the mild solution y_ε of (1.2) with Bh replaced by $\beta_\varepsilon\varphi_\varepsilon \in L^2(0,T;H)$ and $y_\varepsilon(0) = y^0$ satisfies $y(T) = 0$. By (1.5) we find

$$\begin{aligned} -\Big(y(0), \varphi_\varepsilon(0)\Big) &= \int_0^T \Big(\beta_\varepsilon\varphi_\varepsilon(t), \varphi_\varepsilon(t)\Big)\, dt \\ &\le C\left\{\int_0^T \Big(B^2\varphi_\varepsilon(t), \varphi_\varepsilon(t)\Big)\, dt\right\}^{\frac{1}{2}} \\ &\le C\left\{\int_0^T \Big(\beta_\varepsilon\varphi_\varepsilon(t), \varphi_\varepsilon(t)\Big)\, dt\right\}^{\frac{1}{2}} . \end{aligned}$$

In particular

$$\varepsilon \int_0^T |\varphi_\varepsilon(t)|^2\, dt + \int_0^T \Big(B\varphi_\varepsilon(t), B\varphi_\varepsilon(t)\Big)\, dt = \int_0^T \Big(\beta_\varepsilon\varphi_\varepsilon(t), \varphi_\varepsilon(t)\Big)\, dt \le C^2 . \tag{1.7}$$

Step 4. Convergence of $b_\varepsilon = \beta_\varepsilon\varphi_\varepsilon = \varepsilon\varphi_\varepsilon + B^2\varphi_\varepsilon$ along a subsequence. From (1.7) it is clear that

$$\sqrt{\varepsilon}\,\varphi_\varepsilon \quad \text{and} \quad B\varphi_\varepsilon \qquad \text{are bounded in} \quad L^2(0,T;H) . \tag{1.8}$$

Along a subsequence, we may assume

$$B\varphi_\varepsilon \rightharpoonup h \qquad \text{weakly in} \quad L^2(0,T;H) . \tag{1.9}$$

Then clearly

$$b_\varepsilon = \beta_\varepsilon\varphi_\varepsilon = \varepsilon\varphi_\varepsilon + B^2\varphi_\varepsilon \rightharpoonup Bh \qquad \text{weakly in} \quad L^2(0,T;H) . \tag{1.10}$$

Step 5. Conclusion. By passing to the limit, it is clear that the solution y of (1.2) with $y(0) = y^0$ and h as in step 4 satisfies $y(T) = 0$. The proof of Theorem 1.1 is now complete. ∎

2 – The abstract wave equation

In this section, we consider a real Hilbert space H and a positive self-adjoint operator A with dense domain $D(A) = W$. We also consider the space $V = D(A^{\frac{1}{2}})$ and its dual space V'. The equations (1.1) and (1.2) are replaced by the second order equation

$$\varphi'' + A\varphi = 0\,, \qquad t \in \mathbb{R} \tag{2.1}$$

and the corresponding controlled problem

$$y'' + Ay = Bh(t) \quad \text{in} \ \ (0,T) \tag{2.2}$$

in time T by means of a control function $h \in L^2(0,T,H)$ with

$$B \in \mathcal{L}(H)\,, \qquad B = B^* \geq 0\ . \tag{2.3}$$

In this section we shall represent a pair of functions by $[f,g]$ rather than (f,g) to avoid confusion with scalar products. On the other hand the symbol (f,g) will represent indifferently either the H-inner product of $f \in H$ and $g \in H$ or the duality product $(f,g)_{V,V'}$ when $f \in V$ and $g \in V'$, these two products being equal when $f \in V$ and $g \in H$.

Theorem 2.1. *For any* $[y^0,y^1] \in V{\times}H$, *the two following conditions are equivalent*

i) *There exists* $h \in L^2(0,T;H)$ *such that the mild solution* y *of (2.2) such that* $y(0) = y^0$ *and* $y'(0) = y^1$ *satisfies* $y(T) = y'(T) = 0$.

ii) *There exists a finite positive constant* C *such that*

$$\forall [\varphi^0,\varphi^1] \in V{\times}H\,, \qquad \Big|(y^0,\varphi^1) - (y^1,\varphi^0)\Big| \leq C\left\{\int_0^T |B\varphi(t)|^2\,dt\right\}^{\frac{1}{2}} \tag{2.4}$$

where $\varphi(t) \in C(\mathbb{R},V) \cap C^1(\mathbb{R},H)$ *denotes the unique mild solution of (2.1) such that* $\varphi(0) = \varphi^0$ *and* $\varphi'(0) = \varphi^1$.

Proof: It parallels exactly the proof of theorem 1.1.

Step 1. Let φ and y be a pair of strong solutions of (2.1) and (2.2), respectively. We have

$$\begin{aligned}\frac{d}{dt}\Big(y'(t),\varphi(t)\Big) &= \Big(y''(t),\varphi(t)\Big) + \Big(y'(t),\varphi'(t)\Big)\\ &= \Big(-Ay(t) + Bh(t),\, \varphi(t)\Big) + \Big(y'(t),\varphi'(t)\Big)\ .\end{aligned}$$

On the other hand

$$\begin{aligned}\frac{d}{dt}\Big(y(t),\varphi'(t)\Big) &= \Big(y(t),\varphi''(t)\Big)+\Big(y'(t),\varphi'(t)\Big)\\ &= \Big(y(t),-A\varphi(t)\Big)+\Big(y'(t),\varphi'(t)\Big)\ .\end{aligned}$$

By substracting these two identities we find

$$\frac{d}{dt}\Big[\Big(y'(t),\varphi(t)\Big)-\Big(y(t),\varphi'(t)\Big)\Big] = \Big(Bh(t),\varphi(t)\Big)\ .$$

By integrating on (0, T) we get

$$\Big[\Big(y'(t),\varphi(t)\Big)-\Big(y(t),\varphi'(t)\Big)\Big]_0^T = \int_0^T \Big(Bh(t),\varphi(t)\Big)\,dt\ . \tag{2.5}$$

By density, this identity is valid for mild solutions as well. Since B is bounded, self-adjoint and $B \ge 0$,

$$\int_0^T \Big(Bh(t),\varphi(t)\Big)\,dt = \int_0^T \Big(h(t),B\varphi(t)\Big)\,dt\ .$$

Finally if there exists $h \in L^2(0,T)$ such that the mild solution y of (2.2) with $[y(0),y'(0)] = [y^0,y^1]$ satisfies $y(T) = y'(T) = 0$, we find as a consequence of (2.5)

$$\Big(y^0,\varphi'(0)\Big)-\Big(y^1,\varphi(0)\Big) = \int_0^T \Big(h(t),B\varphi(t)\Big)\,dt$$

and by the Cauchy–Schwartz inequality we obtain (2.4). Therefore i) implies ii).

Step 2. Here the analog of Lemma 1.2, although slightly more difficult, is basically well-known. Indeed we have

Lemma 2.2. *Assuming*

$$\exists\ \alpha > 0,\ \ B \ge \alpha \tag{2.6}$$

for each $[y^0,y^1] \in V{\times}H$*, there exists* $[\varphi^0,\varphi^1] \in H{\times}V'$ *such that the mild solution* y *of (2.2) with* $h = \varphi \in L^2(0,T;H)$ *(the solution of (2.1) with initial data* $[\varphi^0,\varphi^1]$*) and* $[y(0),y'(0)] = [y^0,y^1]$ *satisfies* $y(T) = y'(T) = 0$.

Proof: We construct a bounded linear operator $\mathcal{A}$ on $H{\times}V'$ in the following way: for any $[\varphi^0,\varphi^1] \in H{\times}V'$ we consider first the solution φ of (2.1) initial data $[\varphi^0,\varphi^1]$. Then we consider the unique mild solution y of

$$y'' + Ay = B\varphi(t) \quad \text{in}\ \ (0,T), \quad y(T) = y'(T) = 0$$

and finally we set

$$\mathcal{A}\Big([\varphi^0,\varphi^1]\Big) = \Big[-y'(0), Ay(0)\Big] .$$

By formula (2.5) we find

$$\begin{aligned}\Big\langle \mathcal{A}([\varphi^0,\varphi^1]), [\varphi^0,\varphi^1]\Big\rangle_{H\times V'} &= (y(0),\varphi'(0)) - (y'(0),\varphi(0)) \\ &= \int_0^T (B\varphi(t),\varphi(t))\, dt \;\geq\; \alpha \int_0^T |\varphi(t)|^2\, dt .\end{aligned}$$

On the other hand it is known (cf. e.g. [5, 10]) that for any $T > 0$

$$\int_0^T |\varphi(t)|^2 dt \;\geq\; c(T)\left\{|\varphi(0)|^2 + |\varphi'(0)|^2_{V'}\right\} \;=\; c(T)\left\{|\varphi^0|^2 + |\varphi^1|^2_{V'}\right\}$$

with $c(T) > 0$. Hence $\mathcal{A}$ is coercive on $H\times V'$, and this implies $\mathcal{A}(H\times V') = H\times V'$. Then the conclusion is obvious. ∎

Step 3. We now use the penalty method. For each $\varepsilon > 0$ we set

$$\beta_\varepsilon := B^2 + \varepsilon I .$$

As a consequence of Lemma 2.2 there exists a pair $[\varphi^{0,\varepsilon},\varphi^{1,\varepsilon}] \in H\times V'$ such that the mild solution y_ε of (2.2) with Bh replaced by $\beta_\varepsilon\varphi_\varepsilon \in L^2(0,T;H)$ and $[y_\varepsilon(0), y'_\varepsilon(0)] = [y^0, y^1]$ satisfies $y(T) = y'(T) = 0$. By (2.5) we find

$$\begin{aligned}(y(0),\varphi'_\varepsilon(0)) - (y'(0),\varphi_\varepsilon(0)) &= \int_0^T (\beta_\varepsilon\varphi_\varepsilon(t),\varphi_\varepsilon(t))\, dt \\ &\leq C\left\{\int_0^T (B^2\varphi_\varepsilon(t),\varphi_\varepsilon(t))\, dt\right\}^{\frac12} \\ &\leq C\left\{\int_0^T (\beta_\varepsilon\varphi_\varepsilon(t),\varphi_\varepsilon(t))\, dt\right\}^{\frac12} .\end{aligned}$$

In particular

$$\varepsilon \int_0^T |\varphi_\varepsilon(t)|^2\, dt + \int_0^T (B\varphi_\varepsilon(t), B\varphi_\varepsilon(t))\, dt = \int_0^T (\beta_\varepsilon\varphi_\varepsilon(t),\varphi_\varepsilon(t))\, dt \;\leq\; C^2 . \tag{2.7}$$

Step 4. Convergence of $b_\varepsilon = \beta_\varepsilon\varphi_\varepsilon = \varepsilon\varphi_\varepsilon + B^2\varphi_\varepsilon$ along a subsequence. From (2.7) it is clear that

$$\sqrt{\varepsilon}\,\varphi_\varepsilon \;\text{ and }\; B\varphi_\varepsilon \quad \text{are bounded in } \; L^2(0,T;H) . \tag{2.8}$$

Along a subsequence, we may assume

$$B\varphi_\varepsilon \rightharpoonup h \quad \text{weakly in} \;\; L^2(0,T;H) \;. \tag{2.9}$$

Then clearly

$$b_\varepsilon = \beta_\varepsilon \varphi_\varepsilon = \varepsilon\varphi_\varepsilon + B^2\varphi_\varepsilon \;\rightharpoonup\; Bh \quad \text{weakly in} \;\; L^2(0,T;H) \;. \tag{2.10}$$

Step 5. Conclusion. By passing to the limit, it is clear that the solution y of (2.2) with $[y(0), y'(0)] = [y^0, y^1]$ and h as in step 4 satisfies $y(T) = y'(T) = 0$. The proof of Theorem 2.1 is now complete. ∎

3 – Eigenstates in the first order case. Examples

In our previous work [11] we noticed that in the case of the abstract equation (2.1) and if A^{-1} is compact, the quadratic form:

$$\Phi(\varphi^0, \varphi^1) \;=\; \int_0^T |B\varphi(t)|^2\, dt$$

where $\varphi(t) \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ denotes the unique mild solution of (2.2) such that $\varphi(0) = \varphi^0$ and $\varphi'(0) = \varphi^1$ is diagonalizable on $V \times H$ and if $[\varphi^0, \varphi^1]$ is an eigenvector of Φ, the state $J([\varphi^0, \varphi^1]) = [\varphi^1, -A\varphi^0]$ is null-controlable with control proportional to $B\varphi(t)$. A similar property holds for general first order systems, although generally there is no compactness. More precisely let (H, B, C) be as in theorem 1.1, and let us denote by $G(t)$ the isometry group generated by $(-C)$ (or equivalently, equation (1.1)). We have the following simple result

Theorem 3.1. *Let $\varphi \in H$ be such that for some $\lambda > 0$*

$$\int_0^T G(-t)\, B^2 G(t)\, \varphi\; dt \;=\; \lambda\varphi \;. \tag{3.1}$$

Then the solution y of

$$y' + Cy = -\frac{1}{\lambda}\, B^2(G(t)\,\varphi) \quad \text{in} \;\; (0,T), \quad y(0) = \varphi \tag{3.2}$$

satisfies $y(T) = 0$.

Proof: We have, by Duhamel's formula

$$\begin{aligned} y(T) &= G(T)\varphi - \frac{1}{\lambda}\int_0^T G(T-t)\,[B^2 G(t)\varphi]\;dt \\ &= G(T)\left[\varphi - \frac{1}{\lambda}\int_0^T G(-t)\,B^2 G(t)\varphi\;dt\right] \;=\; 0\;. \blacksquare \end{aligned}$$

Remark. In the first order case, the operator

$$\int_0^T G(-t)\,B^2 G(t)\varphi\;dt$$

is not compact except if B is compact, in which case controllabilty will only happen for data in a dense subset of H. Therefore eigenstates will only appear in special situations. We now consider two examples of application of the results of Sections 1 and 3. □

Example 3.2. *The periodic transport equation.* Let

$$\Omega = (0, 2\pi)\,, \qquad \omega = (\omega_1, \omega_2) \subset \Omega\;.$$

We consider the problem

$$y_t + y_x = \chi_\omega\, h\,, \qquad y(t,0) = y(t,2\pi)\;. \tag{3.3}$$

As a consequence of Theorem 1.1, a given state $y^0 \in L^2(\Omega) = H$ is null-controllable at $t = T$ if, and only if

(3.4)

$$\exists\, C \in \mathbb{R}^+, \quad \forall \varphi \in L^2(\Omega)\,, \qquad \left|\int_\Omega y^0(x)\,\varphi(x)\,dx\right| \le C\left\{\int_0^T\!\!\int_\omega \tilde\varphi^2(x-t)\;dx\,dt\right\}^{\frac12}$$

where $\tilde\varphi$ is the 2π-periodic extension of φ on $\mathbb{R}$.

1) First we notice that if $T + |\omega| < 2\pi$, the set of null-controllable states is not dense in H. More precisely if $y^0 \in L^2(\Omega) = H$ is null-controllable at $t = T$, we must have

$$\int_\Omega y^0(x)\,\varphi(x)\,dx \;=\; 0$$

for all $\varphi \in H$ such that $\tilde\varphi = 0$ a.e. on $(\omega_1 - T, \omega_2)$. To interpret this necessary condition we distinguish two cases

Case 1. $T < \omega_1$. In this case $J = (\omega_1 - T, \omega_2) \subset \Omega$ and the other $2m\pi$-translates of J do not intersect Ω. The necessary condition reduces to

$$y^0 = 0 \quad \text{ a.e. on } \; J^C = (0, \omega_1 - T] \cup [\omega_2, 2\pi)\;.$$

Case 2. $T \geq \omega_1$. In this case $J = (\omega_1 - T, \omega_2)$ and $J + 2\pi = (\omega_1 - T + 2\pi, \omega_2 + 2\pi)$ are the only $2m\pi$-translates of J which intersect Ω. The necessary condition becomes

$$y^0 = 0 \quad \text{a.e. on } [\omega_2, \omega_1 - T + 2\pi] \,.$$

Actually the set of null-controllable states is rather complicated when $T + |\omega| < 2\pi$. For instance if we consider the special case

$$T = \pi \,, \qquad \omega = \left(\pi, \frac{3\pi}{2}\right)$$

which is a subcase of case 2, the necessary condition is

$$\operatorname{supp}(y^0) \subset \left[0, \frac{3\pi}{2}\right] \,.$$

It is, however, easy to see that for instance $\chi_{(0,\frac{3\pi}{2})}$ is not controllable. In order to prove this, we first notice that by looking at the graphs

$$\int_0^T \!\! \int_\omega \tilde{\varphi}^2(x-t)\, dx\, dt \; = \int_0^{\frac{3\pi}{2}} \rho(u)\, \varphi^2(u)\, du$$

where

$$\rho(u) = u \quad \text{on} \quad \left(0, \frac{\pi}{2}\right), \qquad \rho(u) = \frac{\pi}{2} \quad \text{on} \quad \left(\frac{\pi}{2}, \pi\right), \qquad \rho(u) = \frac{3\pi}{2} - u \quad \text{on} \quad \left(\pi, \frac{3\pi}{2}\right).$$

Now we choose

$$\forall \varepsilon \in (0,1) \,, \qquad \varphi_\varepsilon(x) = \frac{\chi_{(\varepsilon,\pi)}(x)}{x} \,.$$

We obtain as $\varepsilon \to 0$

$$(\chi_{(0,\frac{3\pi}{2})}, \varphi_\varepsilon) \geq \int_\varepsilon^{\frac{\pi}{2}} \frac{du}{u} \sim \operatorname{Log} \frac{1}{\varepsilon}$$

while also

$$\int_0^{\frac{3\pi}{2}} \rho(u)\, {\varphi_\varepsilon}^2(u)\, du \leq C + \int_\varepsilon^{\frac{\pi}{2}} \frac{du}{u} \sim \operatorname{Log} \frac{1}{\varepsilon}$$

and therefore

$$\left\{ \int_0^{\frac{3\pi}{2}} \rho(u)\, {\varphi_\varepsilon}^2(u)\, du \right\}^{\frac{1}{2}} \leq \sqrt{C + \operatorname{Log} \frac{1}{\varepsilon}} \,.$$

In particular, letting $\varepsilon \to 0$ we can see that (3.4) is not fulfilled.

On the other hand, it is easy to see that the condition

$$\exists\, f \in L^2(0,2\pi)\,, \qquad y^0(x) = \chi_{(0,\frac{3\pi}{2})} \sqrt{x\Big(\frac{3\pi}{2} - x\Big)}\, f(x)$$

is sufficient in order for y^0 to be null-controllable in ω at $T = \pi$. In particular the condition

$$\exists\, \varepsilon > 0\,, \qquad |y^0(x)| \le C\, \chi_{(0,\frac{3\pi}{2})} \Big[x\Big(\frac{3\pi}{2} - x\Big)\Big]^{\varepsilon}$$

is sufficient.

2) If $T + |\omega| > 2\pi$, the set of null-controllable states is equal to H. Indeed in this case

$$\exists\, C \in \mathbb{R}^+\,, \quad \forall \varphi \in L^2(\Omega)\,, \qquad |\varphi|_H \;\le\; C \Big\{ \int_0^T \!\!\int_\omega \tilde{\varphi}^2(x-t)\; dx\, dt \Big\}^{\frac{1}{2}} .$$

Especially interesting is the case

$$T = 2\pi\ .$$

Indeed then by periodicity we have

$$\forall \varphi \in L^2(\Omega)\,, \qquad \int_0^{2\pi}\!\!\int_\omega \tilde{\varphi}^2(x-t)\; dx\, dt \;=\; \int_\omega \int_0^{2\pi} \tilde{\varphi}^2(x-t)\; dt\, dx \;=\; |\omega|\, |\varphi|_H^2$$

and this means that any $y^0 \in L^2(\Omega) = H$ is an eigenstate with eigenvalue $|\omega|$. Applying Theorem 3.1 we obtain that any $y^0 \in L^2(\Omega) = H$ is null-controllable in ω with control

$$-\frac{1}{|\omega|}\, \chi_\omega(x)\, \tilde{y}^0(x-t)\ . \tag{3.5}$$

Of course a direct calculation confirms this result. Indeed if y is the solution of

$$y_t + y_x = -\frac{1}{|\omega|}\, \chi_\omega(x)\, \tilde{y}^0(x-t)\,, \qquad y(t,0) = y(t,2\pi), \quad y(0,.) = y^0$$

we have by Duhamel's formula

$$y(2\pi,x) \;=\; \tilde{y}^0(x-2\pi) \;+ \int_0^{2\pi} -\frac{1}{|\omega|}\, \tilde{\chi}_\omega\Big(x - [2\pi - t]\Big)\, \tilde{y}^0\Big(x - t - [2\pi - t]\Big)\, dt\ ,$$

$$\tilde{y}^0(x) - \frac{1}{|\omega|} \int_0^{2\pi} \tilde{\chi}_\omega(x+t)\, \tilde{y}^0(x)\, dt \;=\; y^0(x) - \frac{1}{|\omega|}\, y^0(x) \int_0^{2\pi} \tilde{\chi}_\omega(x+t)\, dt \;=\; 0$$

since by periodicity

$$\forall x \in (0, 2\pi)\,, \qquad \int_0^{2\pi} \tilde{\chi}_\omega(x+t)\, dt \;=\; \int_0^{2\pi} \tilde{\chi}_\omega(t)\, dt \;=\; |\omega|\ . \ \square$$

Example 3.3. *A one dimensional Schrödinger equation.* Let

$$\Omega = (0, \pi)\,, \qquad \omega = (\omega_1, \omega_2) \subset \Omega\ .$$

We consider the problem

$$y_t + i\, y_{xx} = \chi_\omega\, h\,, \qquad y(t,0) = y(t,\pi) = 0\ . \tag{3.6}$$

As a consequence of Theorem 1.1, a given state $y^0 \in L^2(\Omega, \mathbb{C}) = H$ is null-controllable at $t = T$ if, and only if

$$\exists\, C \in \mathbb{R}^+\,, \quad \forall \varphi^0 \in L^2(\Omega, \mathbb{C})\,, \qquad \left| \int_\Omega y^0(x)\, \varphi^0(x)\, dx \right| \le C \left\{ \int_0^T\!\!\int_\omega |\varphi|^2(t,x)\, dx\, dt \right\}^{\frac{1}{2}} \tag{3.7}$$

where φ is the mild solution of

$$\varphi_t + i\varphi_{xx} = 0\,, \qquad \varphi(t,0) = \varphi(t,2\pi) = 0, \quad \varphi(0,.) = \varphi^0\ . \tag{3.8}$$

Here actually φ is given by

$$\varphi(t,x) \;=\; \sum_{m=1}^{\infty} c_m\, e^{-im^2 t} \sin mx \tag{3.9}$$

with

$$\varphi^0(x) \;=\; \sum_{m=1}^{\infty} c_m \sin mx$$

or in other terms

$$c_m \;=\; \frac{2}{\pi} \int_0^\pi \varphi^0(x) \sin mx\ dx\ .$$

Then a standard application of a variant to Ingham's Lemma (cf. e.g. [4, 6, 10]) shows that

$$\int_0^T\!\!\int_\omega |\varphi|^2(t,x)\ dx\, dt \;\ge\; c(T,\omega) \int_\Omega |\varphi|^2(0,x)\, dx$$

with $c(T,\omega) > 0$. In particular (3.7) is satisfied for any $y^0 \in L^2(\Omega) = H$, which means that here any state is null-controllable in arbitrarily small time.

Especially interesting is the case

$$T = 2\pi\ .$$

Indeed then by periodicity we have

$$\begin{aligned}\forall\varphi^0\in L^2(\Omega)\,,\qquad \int_0^{2\pi}\!\!\int_\omega |\varphi|^2(t,x)\;dx\,dt \;&=\; \int_\omega\int_0^{2\pi}|\varphi|^2(t,x)\;dt\,dx\\ &=\;\int_\omega\int_0^{2\pi}\Big|\sum_{m=1}^{\infty} c_m\, e^{-im^2t}\sin mx\Big|^2\,dt\,dx\\ &=\;2\pi\sum_{m=1}^{\infty}|c_m|^2\int_\omega\sin^2 mx\,dx\\ &=\;4\sum_{m=1}^{\infty}\delta_m|(\varphi^0,\psi_m)|^2\end{aligned}$$

with

$$\psi_m(x):=\sqrt{\frac{2}{\pi}}\,\sin mx\,,\qquad \delta_m=\int_\omega\sin^2 mx\,dx$$

and this implies that for any $m>0$, $\sin mx$ is an eigenstate with eigenvalue

$$\gamma_m\;=\;4\int_\omega\sin^2 mx\,dx\;.$$

Applying Theorem 3.1 we obtain that any $y^0\in L^2(\Omega)=H$ is null-controllable in ω at time $T=2\pi$ with control

$$-\chi_\omega(x)\sum_{m=1}^{\infty}\frac{c_m}{\gamma_m}\,e^{-im^2t}\sin mx\;. \tag{3.10}$$

Of course a direct calculation confirms this result. Indeed let us compute

$$\int_0^{2\pi}G(-t)\,[\chi_\omega\,G(t)\sin mx]\,dt$$

where $G(t)$ is the isometry group generated by (3.8). We have

$$G(t)\sin mx\;=\;e^{-im^2t}\sin mx\;.$$

Then we expand

$$\chi_\omega(x)\sin mx\;=\;a\sin mx\;+\sum_{p\neq m}c_p\sin px\;.$$

Multiplying by $\sin mx$ and integrating over Ω yields

$$a\int_\Omega\sin^2 mx\,dx\;=\int_\omega\sin^2 mx\,dx$$

hence

$$\frac{\pi}{2}\,a \;=\; \int_\omega \sin^2 mx\,dx\;.$$

On the other hand

$$\begin{aligned} G(-t)\,[\chi_\omega\,G(t)\sin mx] \;&=\; e^{-im^2t}\,G(-t)\,\chi_\omega\,\sin mx \\ &=\; a\sin mx \;+\; \sum_{p\neq m} c_p\,e^{i(p^2-m^2)t}\sin px \end{aligned}$$

and now by periodicity we find

$$\begin{aligned} \int_0^{2\pi} G(-t)\,[\chi_\omega\,G(t)\sin mx]\,dt \;&=\; 2\pi\,a\sin mx \\ &=\; 4\sin mx\int_\omega \sin^2 mx\,dx\;. \end{aligned}$$

Then the conclusion follows easily for eigenstates by Duhamel's formula and finally by linearity and continuity in the general case. ▫

4 – The second order case. Some examples

Let (H, A, V, B) be as in theorem 2.1. We have the following result

Theorem 4.1. *Let $[\varphi^0, \varphi^1] \in D(A)\times V$ be such that for some $\lambda > 0$*

$$\text{(4.1)}\qquad \forall[\psi^0,\psi^1]\in V\times H\,,\qquad \int_0^T (B\varphi(t), B\psi(t))\,dt \;=\; \lambda\Big[(A\varphi^0,\psi^0)+(\varphi^1,\psi^1)\Big]$$

where φ and ψ are the solutions of (2.1) with respective initial data $[\varphi^0, \varphi^1]$ and $[\psi^0, \psi^1]$. Then the solution y of

$$y'' + Ay = \frac{1}{\lambda}\,B^2\varphi(t) \quad \textit{in}\;\; (0,T),\;\; y(0)=\varphi^1,\;\; y'(0) = -A\varphi^0$$

satisfies $y(T) = y'(T) = 0$.

Proof: Let $[\psi^0, \psi^1]$ be any state in $V\times H$ and ψ the solution of (2.1) with initial data $[\psi^0, \psi^1]$. By formula (2.5) we find

$$\begin{aligned} \Big[(y'(t),\psi(t)) - (y(t),\psi'(t))\Big]_0^T \;&=\; \frac{1}{\lambda}\int_0^T (B^2\varphi(t),\psi(t))\,dt \\ &=\; \Big[(A\varphi^0,\psi^0)+(\varphi^1,\psi^1)\Big] \end{aligned}$$

hence

$$(y'(T),\psi(T)) - (y(T),\psi'(T)) \;=\; (y^1{+}A\varphi^0,\psi^0) - (y^0{-}\varphi^1,\psi^1) \;=\; 0\ .$$

Since the abstract wave equation generates an isometry group on $V{\times}H$, the pair $[\psi(T),\psi'(T)]$ is arbitrary in $V{\times}H$, hence $[\psi(T),-\psi'(T)]$ fills a dense subset of $H{\times}H$. We conclude that $y(T)=y'(T)=0$.

We now turn to the generalization of a result established in [11] in the special case $H=L^2(\Omega)$ and $B\varphi=\chi_\omega\varphi,\ \omega\subset\Omega$. We assume

$$A^{-1}\ \text{is compact}:\ H\longrightarrow H$$

or equivalently

$$\text{the inclusion map}:\ V\longrightarrow H\ \text{is compact}\ .$$

We set

$$\mathcal{H}:=V{\times}H$$

and we define $\mathcal{L}\in\mathcal{L}(\mathcal{H})$ by the formula:

$$\Big\langle\mathcal{L}[\varphi^0,\varphi^1],[\psi^0,\psi^1]\Big\rangle_{\mathcal{H}} = \int_0^T(B\varphi(t),B\psi(t))\,dt \tag{4.2}$$

$\forall[\varphi^0,\varphi^1]\in\mathcal{H},\ \forall[\psi^0,\psi^1]\in\mathcal{H}$, where φ andψ are the solutions of (2.1) with respective initial data $[\varphi^0,\varphi^1]$ and $[\psi^0,\psi^1]$. It is clear by definition that $\mathcal{L}$ is self-adjoint and ≥ 0 on $\mathcal{H}$. If we introduce the duality map $\mathcal{F}\colon\mathcal{H}\longrightarrow\mathcal{H}'=V'{\times}H$ we have

Proposition 4.2. *$\mathcal{L}\colon\mathcal{H}\longrightarrow\mathcal{H}$ is compact and more precisely we have*

$$\mathcal{L} \;=\; \mathcal{F}^{-1}\int_0^T S^*(t)\,B^2S(t)\;dt \tag{4.3}$$

where $S(t)\colon\mathcal{H}\longrightarrow H$ is the compact operator defined by

$$\forall[\varphi^0,\varphi^1]\in\mathcal{H}\,,\qquad S(t)[\varphi^0,\varphi^1]=\varphi(t)$$

and $S^*(t)\colon H\longrightarrow\mathcal{H}'$ is the adjoint of $S(t)$.

Proof: We have

$$\begin{aligned}\int_0^T(B\varphi(t),B\psi(t))\,dt \;&=\; \int_0^T\Big(B^2S(t)[\varphi^0,\varphi^1],\,S(t)[\psi^0,\psi^1]\Big)\,dt\\ &=\;\int_0^T\Big\langle S^*(t)B^2S(t)[\varphi^0,\varphi^1],\,[\psi^0,\psi^1]\Big\rangle_{\mathcal{H}',\mathcal{H}}\,dt\\ &=\;\int_0^T\Big\langle\mathcal{F}^{-1}S^*(t)B^2S(t)[\varphi^0,\varphi^1],\,[\psi^0,\psi^1]\Big\rangle_{\mathcal{H},\mathcal{H}}\,dt\ .\end{aligned}$$

Then (4.3) follows at once. Moreover since $S(t) \in \mathcal{L}(\mathcal{H}, V)$ it follows easily that $\int_0^T S^*(t)B^2S(t)\,dt$ is compact: $\mathcal{H} \longrightarrow \mathcal{H}'$. ∎

The following result is a natural generalization of Theorem 1.3 from [11].

Let us denote by $\mathcal{N}$ the kernel of $\mathcal{L}$ and let $\Phi_n = [\varphi_n^0, \varphi_n^1]$ be an orthonormal Hilbert basis of $\mathcal{N}^\perp$ in $\mathcal{H} := V \times H$ made of eigenvectors associated to the non-increasing sequence λ_n of eigenvalues of $\mathcal{L}$ repeated according to multiplicity. Then we have

Theorem 4.3. *In order for $[y^0, y^1] \in \mathcal{H}$ to be null-controllable under (2.2) at time T it is necessary and sufficient that the following set of two conditions is satisfied*

$$\forall [\phi^0, \phi^1] \in \mathcal{N}\,, \qquad (y^0, \phi^1) = (y^1, \phi^0) \tag{4.4}$$

$$\sum_{n=1}^{\infty} \frac{\left\{(y^0, \varphi_n^1) - (y^1, \varphi_n^0)\right\}^2}{\lambda_n} < \infty\,. \tag{4.5}$$

When these conditions are fulfilled, an exact control driving $[y^0, y^1]$ to $[0,0]$ is given by the explicit formula

$$B \sum_{n=1}^{\infty} \frac{(y^0, \varphi_n^1) - (y^1, \varphi_n^0)}{\lambda_n}\; B\,\varphi_n(t)\,. \tag{4.6}$$

Proof: We procced in 3 steps

Step 1. In order to show that controllabilty implies (4.4), we establish

$$\begin{aligned} \mathcal{N} &= \left\{[\phi^0, \phi^1] \in \mathcal{H}, \quad \int_0^T (B\phi(t), B\phi(t))\,dt = 0\right\} \\ &= \left\{[\phi^0, \phi^1] \in \mathcal{H}, \quad B\phi(t)) \equiv 0 \ \text{ on } (0,T)\right\}. \end{aligned}$$

Indeed if $[\phi^0, \phi^1] \in \mathcal{N}$,we have in particular

$$0 = \Big\langle \mathcal{L}[\phi^0, \phi^1], [\phi^0, \phi^1] \Big\rangle_{\mathcal{H}} = \int_0^T (B\phi(t), B\phi(t))\,dt$$

and this is equivalent to $B\phi(t)) \equiv 0$ on $(0,T)$. Conversely this last statement implies

$$\Big\langle \mathcal{L}[\phi^0, \phi^1], [\psi^0, \psi^1] \Big\rangle_{\mathcal{H}} = \int_0^T (B\phi(t), B\psi(t))\,dt = 0\,, \qquad \forall [\psi^0, \psi^1] \in \mathcal{H}$$

hence $\mathcal{L}[\phi^0, \phi^1] = 0$ and therefore $[\phi^0, \phi^1] \in \mathcal{N}$.

Step 2. We introduce

$$a_n = (y^0, \varphi_n^1) - (y^1, \varphi_n^0)\,, \qquad \psi_N = \sum_1^N a_n \frac{\varphi_n}{\lambda_n}\ ,$$
$$\psi_N^0 = \sum_1^N a_n \frac{\varphi_n^0}{\lambda_n}\,, \qquad \psi_N^1 = \sum_1^N a_n \frac{\varphi_n^1}{\lambda_n}\ .$$

We have

$$(y^0, \psi_N^1) - (y^1, \psi_N^0) \;=\; \sum_1^N a_n \frac{(y^0, \varphi_n^1) - (y^1, \varphi_n^0)}{\lambda_n} \;=\; \sum_1^N \frac{a_n^2}{\lambda_n}\ . \tag{4.7}$$

Also, by using the property of the eigenvectors $\Phi_n = [\varphi_n^0, \varphi_n^1]$ and introducing

$$\Psi_N \;=\; [\psi_N^0, \psi_N^1] \;=\; \sum_1^N a_n \frac{\Phi_n}{\lambda_n}$$

we obtain successively

$$\begin{aligned}
\int_0^T |B\psi_N(t)|^2\,dt \;&=\; \int_0^T \Big(B \sum_1^N a_n \frac{\varphi_n}{\lambda_n}(t),\ B\psi_N(t) \Big)\,dt \\
&=\; \sum_1^N \frac{a_n}{\lambda_n} \int_0^T (B\varphi_n(t), B\psi_N(t))\,dt \;=\; \sum_1^N \frac{a_n}{\lambda_n}\,\lambda_n \langle \Phi_n, \Psi_N \rangle_{\mathcal{H}} \\
&=\; \sum_1^N a_n \Big\langle \Phi_n, \sum_1^N a_n \frac{\Phi_n}{\lambda_n} \Big\rangle_{\mathcal{H}} \;=\; \sum_1^N \frac{a_n^2}{\lambda_n}
\end{aligned} \tag{4.8}$$

as a consequence of orthonormality. By Theorem 2.1 we have, assuming $[y^0, y^1] \in \mathcal{H}$ to be null-controllable under (2.2) at time T

$$(y^0, \psi_N^1) - (y^1, \psi_N^0) \;\le\; C \left\{ \int_0^T |B\psi_N(t)|^2\,dt \right\}^{\frac{1}{2}}$$

and by (4.7)–(4.8) this is equivalent to

$$\sum_1^N \frac{a_n^2}{\lambda_n} \;\le\; C \left\{ \sum_1^N \frac{a_n^2}{\lambda_n} \right\}^{\frac{1}{2}}$$

or finally

$$\forall N \ge 1\,, \qquad \sum_1^N \frac{a_n^2}{\lambda_n} \le C^2\ .$$

Step 3. We construct a sequence of approximated controls under condition (4.4). First of all we introduce the symplectic map J defined by

$$\forall[\varphi^0,\varphi^1]\in V\times H\,,\qquad J([\varphi^0,\varphi^1])=[\varphi^1,-A\varphi^0]\;. \tag{4.9}$$

Since the sequence $\Phi_n=[\varphi_n^0,\varphi_n^1]$ is an orthonormal Hilbert basis of $\mathcal{N}^\perp$ in $\mathcal{H}:=V\times H$, it follows that $J\Phi_n=[\varphi_n^1,-A\varphi_n^0]$ is an orthonormal Hilbert basis of the orthogonal of $J(\mathcal{N})$ in $J\mathcal{H}:=H\times V'$ for the corresponding inner product which is in fact the usual one. Now we have

$$\forall[y^0,y^1]\in\mathcal{H},\;\;\forall[\phi^0,\phi^1]\in\mathcal{H}\,,\qquad \Big\langle[y^0,y^1],J[\phi^0,\phi^1]\Big\rangle_{J\mathcal{H}}=(y^0,\phi^1)+\langle y^1,-A\phi^0\rangle_{V'}=(y^0,\phi^1)-(y^1,\phi^0)$$

and therefore (4.4) is equivalent to orthogonality of $[y^0,y^1]$ to $J(\mathcal{N})$ in $J\mathcal{H}$. Moreover if $[y^0,y^1]$ satisfies (4.4), the Fourier components of $[y^0,y^1]$ in the basis $J\Phi_n=[\varphi_n^1,-A\varphi_n^0]$ of the orthogonal of $J(\mathcal{N})$ in $J\mathcal{H}$ are precisely the coefficients

$$a_n=(y^0,\varphi_n^1)-(y^1,\varphi_n^0)\;.$$

Therefore the state

$$[y_N^0,y_N^1]=\sum_1^N a_n\,J\Phi_n$$

is an approximation of $[y^0,y^1]$ in $J(\mathcal{H})$. As a consequence of Theorem 4.1, for each N the solution y_N of

$$y_N''+Ay_N=B^2\,\psi_N(t)\,,\qquad y_N(0)=y_N^0,\;\;y_N'(0)=y_N^1$$

satisfies $y_N(T)=y_N'(T)=0$.

Step 4. Convergence of the approximated controls. Keeping the notation of steps 3 and 4, we have for $1\le P\le N$

$$\begin{aligned}\int_0^T|B\psi_N(t)-B\psi_P(t)|^2\,dt &= \int_0^T\Big(B\sum_P^N a_n\frac{\varphi_n}{\lambda_n}(t),\,B\psi_N(t)-B\psi_P(t)\Big)\,dt\\ &=\sum_P^N\frac{a_n}{\lambda_n}\int_0^T\Big(B\varphi_n(t),\,B\psi_N(t)-B\psi_P(t)\Big)\,dt\\ &=\sum_P^N\frac{a_n}{\lambda_n}\,\lambda_n\Big\langle\Phi_n,\Psi_N-\Psi_P\Big\rangle_{\mathcal{H}}\\ &=\sum_P^N a_n\Big\langle\Phi_n,\sum_P^N a_n\frac{\Phi_n}{\lambda_n}\Big\rangle_{\mathcal{H}}=\sum_P^N\frac{a_n^2}{\lambda_n}\end{aligned}$$

as a consequence of orthonormality. Therefore $\{B\psi_N\}_{N\geq 1}$ is a Cauchy sequence in $L^2(0,T;H)$. Setting

$$h := \lim_{N\to\infty} B\psi_N$$

since $y_N(T) = y'_N(T) = 0$ it follows immediately that

$$\lim_{N\to\infty} y_N = y$$

in $C([0,T],V)\cap C^1([0,T],H)\cap L^2([0,T],V')$. In particular $y(0)=y^0$, $y'(0)=y^1$ and

$$y'' + Ay = Bh(t)\,, \qquad y(T) = y'(T) = 0\ .$$

Formula (4.6) is satisfied in the sense

$$\sum_{n=1}^{\infty} \frac{(y^0,\varphi_n^1)-(y^1,\varphi_n^0)}{\lambda_n}\, B\varphi_n(t) \;=\; \lim_{N\to\infty}\sum_{n=1}^{N} \frac{(y^0,\varphi_n^1)-(y^1,\varphi_n^0)}{\lambda_n}\, B\varphi_n(t)$$

in the strong topology of $L^2(0,T;H)$.

Remark 4.4. In contrast with the first order case where diagonalization of the basic quadratic form was generally impossible due to non-compactness, in bounded domains Theorem 4.3 will be always applicable. □

We conclude this section by some typical examples borrowed from [11].

Example 4.5. Let

$$\Omega = (0,\pi)\,, \qquad \omega = (\omega_1,\omega_2) \subset \Omega\ .$$

We consider the problem

$$y_{tt} - y_{xx} = \chi_\omega h\,, \qquad y(t,0) = y(t,\pi) = 0\ . \tag{4.10}$$

As a consequence of Theorem 2.1, a given state $[y^0,y^1] \in H_0^1(\Omega)\times L^2(\Omega)$ is null-controllable at $t=T$ if, and only if there exists $C\in\mathbb{R}^+$ such that

$$\forall[\varphi^0,\varphi^1] \in H_0^1(\Omega)\times L^2(\Omega)\ ,$$

$$\left|\int_\Omega y^0(x)\,\varphi^1(x)\,dx \;-\int_\Omega y^1(x)\,\varphi^0(x)\,dx\right| \;\leq\; C\left\{\int_0^T\!\!\int_\omega |\varphi|^2(t,x)\;dx\,dt\right\}^{\frac12}$$

where φ is the mild solution of

$$\varphi_{tt} - \varphi_{xx} = 0\,, \qquad \varphi(t,0) = \varphi(t,\pi) = 0\,, \qquad \varphi(0,.) = \varphi^0\,, \qquad \varphi_t(0,.) = \varphi^1\ .$$

Here φ is given by

$$\varphi(t,x) \;=\; \sum_{m=1}^{\infty} \Big[c_m \cos mt + d_m \sin mt\Big] \sin mx$$

with

$$\varphi^0(x) = \sum_{m=1}^{\infty} c_m \sin mx\,, \qquad \varphi^1(x) = \sum_{m=1}^{\infty} d_m \sin mx$$

or in other terms

$$c_m = \frac{2}{\pi} \int_0^{\pi} \varphi^0(x) \sin mx \; dx\,, \qquad d_m = \frac{2}{\pi} \int_0^{\pi} \varphi^1(x) \sin mx \; dx\ .$$

If T is small, by the finite propagation property of the wave equation, there is in general an infinite-dimensional space of non-controllable states. For instance if

$$\omega_1 > 0\,, \qquad \omega_2 < \pi \qquad \text{and} \qquad T < \inf\{\omega_1, \pi - \omega_2\}\ ,$$

it is easily seen that

$$\left| \int_\Omega y^0(x)\, \varphi^1(x)\, dx \;-\; \int_\Omega y^1(x)\, \varphi^0(x)\, dx \right| \;=\; 0$$

for all $[\varphi^0, \varphi^1] \in H_0^1(\Omega) \times L^2(\Omega)$ with

$$\varphi^0 = \varphi^1 \equiv 0, \qquad \text{a.e. on } \ [\omega_1 - T, \omega_2 + T]\ .$$

In particular this implies

$$\operatorname{supp} y^0 \cup \operatorname{supp} y^1 \;\subset\; [\omega_1 - T, \omega_2 + T]\ .$$

Especially interesting is the case

$$T = 2\pi\ .$$

Indeed then by periodicity we have

$\forall [\varphi^0, \varphi^1] \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\begin{aligned}
\int_0^{2\pi}\!\!\int_\omega \varphi^2(t,x)\; dx\, dt \;&=\; \int_\omega \int_0^{2\pi} \varphi^2(t,x)\; dt\, dx \\
&=\; \int_\omega \int_0^{2\pi} \left\{ \sum_{m=1}^{\infty} \Big[c_m \cos mt + d_m \sin mt\Big] \sin mx \right\}^2 dt\, dx \\
&=\; \pi \sum_{m=1}^{\infty} (c_m^2 + d_m^2) \int_\omega \sin^2 mx\, dx
\end{aligned}$$

and this implies that for any $m > 0$, $[\sin mx, 0]$ and $[0, \sin mx]$ are two eigenstates with eigenvalue

$$\lambda_m = \frac{2}{m^2} \int_\omega \sin^2 mx\, dx\ .$$

Applying Theorem 4.3, after some calculations taking account of the normalization in $V \times H$ we obtain that any $[y^0, y^1] \in H_0^1(\Omega) \times L^2(\Omega)$ is null-controllable in ω at time $T = 2\pi$ with control

$$h(t,x) \;=\; \chi_\omega(x) \sum_{m=1}^{\infty} \frac{m y_m^0 \sin mt - y_m^1 \cos mt}{2 \int_\omega \sin^2 mx\, dx} \sin mx$$

with

$$y_m^0 = \frac{2}{\pi} \int_0^\pi y^0(x) \sin mx\, dx\,, \qquad y_m^1 = \frac{2}{\pi} \int_0^\pi y^1(x) \sin mx\, dx\ .\ \square$$

Example 4.6. Let

$$\Omega = (0, \pi)\,, \qquad \omega = (\omega_1, \omega_2) \subset \Omega\ .$$

We consider the problem

$$(4.11) \qquad y_{tt} + y_{xxxx} = \chi_\omega h\,, \qquad y(t,0) = y(t,\pi) = y_{xx}(t,0) = y_{xx}(t,\pi) = 0\ .$$

As a consequence of Theorem 2.1, a given state $[y^0, y^1] \in H^2 \cap H_0^1(\Omega) \times L^2(\Omega)$ is null-controllable at $t = T$ if, and only if there exists $C \in \mathbb{R}^+$ such that

$$\forall [\varphi^0, \varphi^1] \in H^2 \cap H_0^1(\Omega) \times L^2(\Omega)\ ,$$
$$\left| \int_\Omega y^0(x)\, \varphi^1(x)\, dx \;-\; \int_\Omega y^1(x)\, \varphi^0(x)\, dx \right| \;\le\; C \left\{ \int_0^T \!\! \int_\omega \varphi^2(t,x)\ dx\, dt \right\}^{\frac{1}{2}}$$

where φ is the mild solution of

$$\varphi_{tt} + \varphi_{xxxx} = 0\,, \qquad \varphi(t,0) = \varphi(t,\pi) = \varphi_{xx}(t,0) = \varphi_{xx}(t,\pi) = 0$$

such that

$$\varphi(0,.) = \varphi^0\,, \qquad \varphi_t(0,.) = \varphi^1\ .$$

Here φ is given by

$$\varphi(t,x) \;=\; \sum_{m=1}^{\infty} \Big[c_m \cos m^2 t + d_m \sin m^2 t \Big] \sin mx$$

with

$$\varphi^0(x) = \sum_{m=1}^{\infty} c_m \sin mx\,, \qquad \varphi^1(x) = \sum_{m=1}^{\infty} d_m \sin mx$$

or in other terms

$$c_m = \frac{2}{\pi} \int_0^\pi \varphi^0(x) \sin mx \, dx \,, \qquad d_m = \frac{2}{\pi} \int_0^\pi \varphi^1(x) \sin mx \, dx \ .$$

As in the Schrödinger case, a variant to Ingham's Lemma shows that any state is null-controllable in arbitrarily small time. Here Theorem 2.1 is useless.

Especially interesting is the case

$$T = 2\pi \ .$$

Indeed then by periodicity we have

$\forall [\varphi^0, \varphi^1] \in H^2 \cap H_0^1(\Omega) \times L^2(\Omega)$,

$$\begin{aligned} \int_0^{2\pi}\!\!\int_\omega \varphi^2(t,x)\; dx\, dt \; &= \int_\omega \int_0^{2\pi} \varphi^2(t,x)\; dt\, dx \\ &= \int_\omega \int_0^{2\pi} \Big\{ \sum_{m=1}^\infty \Big[c_m \cos m^2 t + d_m \sin m^2 t \Big] \sin mx \Big\}^2 dt\, dx \\ &= \pi \sum_{m=1}^\infty (c_m^2 + d_m^2) \int_\omega \sin^2 mx \, dx \end{aligned}$$

and this implies that for any $m > 0$, $[\sin mx, 0]$ and $[0, \sin mx]$ are two eigenstates with eigenvalue

$$\gamma_m = \frac{2}{m^4} \int_\omega \sin^2 mx \, dx \ .$$

Here we obtain that any $[y^0, y^1] \in H^2 \cap H_0^1(\Omega) \times L^2(\Omega)$ is null-controllable in ω at time $T = 2\pi$ with control

$$h(t,x) \;=\; \chi_\omega(x) \sum_{m=1}^\infty \frac{m^2\, y_m^0 \sin mt - y_m^1 \cos mt}{2 \int_\omega \sin^2 mx \, dx} \sin mx$$

with

$$y_m^0 = \frac{2}{\pi} \int_0^\pi y^0(x) \sin mx \, dx \,, \qquad y_m^1 = \frac{2}{\pi} \int_0^\pi y^1(x) \sin mx \, dx \ . \ \square$$

5 – A natural framework for pointwise control

In this section, we consider a real Hilbert space H and a positive self-adjoint operator A with dense domain $D(A) = W$. We also consider the space $V = D(A^{\frac{1}{2}})$ and its dual space V'. We consider the following control problem

$$y'' + Ay = h(t)\, \gamma \quad \text{in} \;\; (0,T) \tag{5.1}$$

in time T by means of a control function $h \in L^2(0,T)$ with

$$\gamma \in \mathcal{L}(V,\mathbb{R}) = V' \ . \tag{5.2}$$

In this section we shall represent a pair of functions by $[f,g]$ rather than (f,g) to avoid confusion with scalar products. On the other hand the symbol (f,g) will represent the H-inner product of $f \in H$ and $g \in H$ and the duality product $(f,g)_{V',V}$ when $f \in V'$ and $g \in V$ will be denoted by $\langle f,g\rangle$.

Theorem 5.1. *For any $[y^0,y^1] \in V\times H$, the two following conditions are equivalent*

- **i)** *There exists $h \in L^2(0,T)$ such that the mild solution y of (5.1) such that $y(0)=y^0$ and $y'(0)=y^1$ satisfies $y(T)=y'(T)=0$.*
- **ii)** *There exists a finite positive constant C such that*

$$\forall[\varphi^0,\varphi^1] \in V\times H\,, \qquad |(y^0,\varphi^1)-(y^1,\varphi^0)| \le C\left\{\int_0^T |\langle\gamma,\varphi(t)\rangle|^2\,dt\right\}^{\frac{1}{2}} \tag{5.3}$$

where $\varphi(t) \in C(\mathbb{R},V)\cap C^1(\mathbb{R},H)$ denotes the unique mild solution of (2.1) such that $\varphi(0)=\varphi^0$ and $\varphi'(0)=\varphi^1$.

Proof: It parallels exactly the proof of theorem 2.1.

Step 1. Considering first the case were $\gamma \in V$, let φ and y be a pair of strong solutions of (5.1) and (2.1), respectively, by a calculation similar to step 1 of Theorem 2.1 we get

$$\Big[(y'(t),\varphi(t)) - (y(t),\varphi'(t))\Big]_0^T = \int_0^T h(t)\,\langle\gamma,\varphi(t)\rangle\,dt \ .$$

By density, this identity is valid for mild solutions as well in the general case $\gamma \in V'$. Therefore if there exists $h \in L^2(0,T)$ such that the mild solution y of (5.1) with $[y(0),y'(0)] = [y^0,y^1]$ satisfies $y(T)=y'(T)=0$, we find

$$(y^0,\varphi'(0)) - (y^1,\varphi(0)) = \int_0^T h(t)\langle\gamma,\varphi(t)\rangle\,dt$$

and by the Cauchy–Schwartz inequality we obtain (5.3). Therefore i) implies ii).

Step 2. For each $\varepsilon > 0$ we construct a bounded linear operator

$$\mathcal{M}_\varepsilon \in \mathcal{L}(V\times H, V'\times H)$$

in the following way: for any $[\varphi^0, \varphi^1] \in V \times H := \mathcal{H}$ we consider first the solution φ of (2.1) with initial data $[\varphi^0, \varphi^1]$. Then we consider the unique mild solution y of

$$y'' + Ay = \langle \gamma, \varphi(t) \rangle \gamma + \varepsilon A \varphi(t) \quad \text{in } (0,T), \quad y(T) = y'(T) = 0 \tag{5.4}$$

and finally we set

$$\mathcal{M}_\varepsilon([\varphi^0, \varphi^1]) = [-y'(0), y(0)] \ .$$

We find

$$\begin{aligned} \Big\langle \mathcal{M}_\varepsilon([\varphi^0, \varphi^1]),\, [\varphi^0, \varphi^1]) \Big\rangle_{\mathcal{H}',\mathcal{H}} &= (y(0), \varphi'(0)) - \langle y'(0), \varphi(0) \rangle \\ &= \int_0^T \langle \gamma, \varphi(t) \rangle^2 \, dt + \int_0^T |A^{\frac{1}{2}} \varphi(t)|^2 \, dt \ . \end{aligned}$$

On the other hand it is known (cf. e.g. [10]) that for any $T > 0$

$$\int_0^T |A^{\frac{1}{2}} \varphi(t)|^2 \, dt \ \geq \ c(T) \left\{ |A^{\frac{1}{2}} \varphi(0)|^2 + |\varphi'(0)|^2 \right\} \ = \ c(T) \left\{ |\varphi^0|_V^2 + |\varphi^1|^2 \right\}$$

with $c(T) > 0$. Hence $\mathcal{M}_\varepsilon$ is coercive: $V \times H \to V' \times H$, and this implies $\mathcal{M}_\varepsilon(V \times H) = V' \times H$.

Step 3. For each $\varepsilon > 0$ we set

$$\beta_\varepsilon(z) := \langle \gamma, z \rangle \gamma + \varepsilon A z \ .$$

As a consequence of step 2 there exists a pair $[\varphi^{0,\varepsilon}, \varphi^{1,\varepsilon}] \in V \times H$ such that the mild solution y_ε of (5.1) with $h(t)\gamma$ replaced by $\beta_\varepsilon \varphi_\varepsilon \in L^2(0,T;V')$ and $[y_\varepsilon(0), y_\varepsilon'(0)] = [y^0, y^1]$ satisfies $y(T) = y'(T) = 0$. By (5.4) we find

$$\begin{aligned} (y(0), \varphi_\varepsilon'(0)) - (y'(0), \varphi_\varepsilon(0)) &= \int_0^T (\beta_\varepsilon \varphi_\varepsilon(t), \varphi_\varepsilon(t)) \, dt \\ &\leq C \left\{ \int_0^T \langle \gamma, \varphi_\varepsilon(t) \rangle^2 \, dt \right\}^{\frac{1}{2}} \\ &\leq C \left\{ \int_0^T (\beta_\varepsilon \varphi_\varepsilon(t), \varphi_\varepsilon(t)) \, dt \right\}^{\frac{1}{2}} \ . \end{aligned}$$

In particular

$$\varepsilon \int_0^T |A^{\frac{1}{2}} \varphi_\varepsilon(t)|^2 \, dt + \int_0^T \langle \gamma, \varphi_\varepsilon(t) \rangle^2 \, dt \ = \int_0^T (\beta_\varepsilon \varphi_\varepsilon(t), \varphi_\varepsilon(t)) \, dt \ \leq \ C^2 \ .$$

Step 4. Convergence of $b_\varepsilon = \beta_\varepsilon \varphi_\varepsilon$ along a subsequence. From step 3 it is clear that

$$\sqrt{\varepsilon}\,\varphi_\varepsilon \quad \text{is bounded in } \; L^2(0,T;V')$$

and

$$h_\varepsilon(t) = \langle \gamma, \varphi_\varepsilon(t)\rangle \quad \text{is bounded in } \; L^2(0,T)\ .$$

Along a subsequence, we may assume

$$h_\varepsilon \rightharpoonup h \quad \text{weakly in } \; L^2(0,T)\ .$$

Then clearly

$$b_\varepsilon = \beta_\varepsilon \varphi_\varepsilon \rightharpoonup h(t)\gamma \quad \text{weakly in } \; L^2(0,T;V')\ .$$

Step 5. Conclusion. By passing to the limit, it is clear that the solution y of (5.1) with $[y(0), y'(0)] = [y^0, y^1]$ and h as in step 4 satisfies $y(T) = y'(T) = 0$. The proof of Theorem 5.1 is now complete. ∎

In the sequel we use a generalization of Theorem 4.1. Let (H, A, V) be as in theorem 2.1 and let $\mathcal{B} \in \mathcal{L}(V, V')$ be such that $\mathcal{B} = \mathcal{B}^*$ and

$$\forall v \in V\,, \qquad \langle \mathcal{B}v, v\rangle \geq 0\ . \tag{5.5}$$

We have the following result

Theorem 5.2. *Let $[\varphi^0, \varphi^1] \in V \times H$ be such that for some $\lambda > 0$*

$$\forall [\psi^0, \psi^1] \in V \times H\,, \qquad \int_0^T \langle \mathcal{B}\varphi(t), \psi(t)\rangle\, dt \;=\; \lambda\Big[\langle A\varphi^0, \psi^0\rangle + (\varphi^1, \psi^1)\Big] \tag{5.6}$$

where φ and ψ are the solutions of (2.1) with respective initial data $[\varphi^0, \varphi^1]$ and $[\psi^0, \psi^1]$. Then the solution y of

$$y'' + Ay = \frac{1}{\lambda}\,\mathcal{B}\varphi(t) \quad \text{in } \; (0,T), \;\; y(0) = \varphi^1, \;\; y'(0) = -A\varphi^0$$

satisfies $y(T) = y'(T) = 0$.

Proof: Essentially identical to that of Theorem 4.1. ∎

We now turn to special case

$$\mathcal{B}(v) := \langle \gamma, v\rangle\, \gamma\ . \tag{5.7}$$

We set

$$\mathcal{H} := V \times H$$

and we define $\mathcal{L} \in \mathcal{L}(\mathcal{H})$ by the formula:

$$\Big\langle \mathcal{L}[\varphi^0,\varphi^1],[\psi^0,\psi^1]\Big\rangle_{\mathcal{H}} = \int_0^T \langle \mathcal{B}\varphi(t),\psi(t)\rangle\,dt \tag{5.8}$$

$\forall[\varphi^0,\varphi^1]\in\mathcal{H}$, $\forall[\psi^0,\psi^1]\in\mathcal{H}$, where φ and ψ are the solutions of (2.1) with respective initial data $[\varphi^0,\varphi^1]$ and $[\psi^0,\psi^1]$. It is clear by definition that $\mathcal{L}$ is self-adjoint and ≥ 0 on $\mathcal{H}$. If we introduce the duality map $\mathcal{F}\colon \mathcal{H}\longrightarrow\mathcal{H}' = V'\times H$ we have

Proposition 5.3. *$\mathcal{L}\colon\mathcal{H}\longrightarrow\mathcal{H}$ is compact and more precisely we have*

$$\mathcal{L} \;=\; \mathcal{F}^{-1}\int_0^T \mathcal{S}^*(t)\,\mathcal{B}\mathcal{S}(t)\,dt \tag{5.9}$$

where $\mathcal{S}(t)\colon\mathcal{H}\longrightarrow V$ is the bounded operator defined by

$$\forall[\varphi^0,\varphi^1]\in\mathcal{H}\,,\qquad \mathcal{S}(t)[\varphi^0,\varphi^1] = \varphi(t)$$

and $\mathcal{S}^*(t)\colon V'\longrightarrow\mathcal{H}'$ is the adjoint of $\mathcal{S}(t)$.

Proof: Formula (5.9) is immediate to check along the lines of proof of (4.3). However to prove that $\int_0^T \mathcal{S}^*(t)\,\mathcal{B}\mathcal{S}(t)\,dt$ is compact: $\mathcal{H}\longrightarrow\mathcal{H}'$ we need a specific argument. Here compactness does not follow from an hypothesis on the imbedding $V\longrightarrow H$ but is a consequence of the special structure of $\mathcal{B}$. As a preliminary step, we establish

Lemma 5.4. *For any $\gamma\in V'$ we have*

$$\mathcal{S}^*(t)\,\gamma \;\in\; C([0,T];\mathcal{H}')\ . \tag{5.10}$$

Proof: Since the mappings $\gamma\to\mathcal{S}^*(t)\,\gamma$ are uniformly equicontinuous: $V'\to\mathcal{H}'$, it is sufficient to prove (5.10) when for instance $\gamma\in V$. In this case setting

$$z = \gamma + A\gamma \;\in\; V'$$

we have

$$\forall t\in[0,T],\quad \forall\theta\in[0,T]\ ,$$

$$\begin{aligned}\Big\|\mathcal{S}^*(t)\gamma-\mathcal{S}^*(\theta)\gamma\Big\|_{\mathcal{H}'} &= \sup_{\|\Phi\|_{\mathcal{H}}\le 1}\Big|\Big\langle\gamma,\ \mathcal{S}(t)\Phi-\mathcal{S}(\theta)\Phi\Big\rangle_{V',V}\Big| \\ &= \sup_{\|\Phi\|_{\mathcal{H}}\le 1}\Big|\Big\langle z,\ \mathcal{S}(t)\mathcal{J}\Phi-\mathcal{S}(\theta)\mathcal{J}\Phi\Big\rangle_{V',V}\Big|\end{aligned}$$

where $\mathcal{J}: \mathcal{H} = V\times H \to D(A^{\frac{3}{2}})\times D(A) \subset D(A)\times V$ is defined by

$$\forall \Phi = [\varphi^0, \varphi^1] \in \mathcal{H}, \qquad \mathcal{J}\Phi = \Big[(I+A)^{-1}\varphi^0, \,(I+A)^{-1}\varphi^1\Big] \,.$$

In particular we have

$$\|\mathcal{S}(t)\mathcal{J}\Phi - \mathcal{S}(\theta)\mathcal{J}\Phi\|_V \;\le\; C\,|t-\theta|\,\|\Phi\|_{\mathcal{H}}$$

and therefore

$$\forall t \in [0,T], \quad \forall \theta \in [0,T], \qquad \|\mathcal{S}^*(t)\gamma - \mathcal{S}^*(\theta)\gamma\|_{\mathcal{H}'} \le C\,\|z\|_{V'}\,|t-\theta|$$

concluding the proof of Lemma 5.4. ∎

Proof of Proposition 5.3 (continued): We have for all $t \in [0,T]$,

$$\forall \Phi = [\varphi^0, \varphi^1] \in \mathcal{H}, \qquad \mathcal{S}^*(t)\,\mathcal{B}\mathcal{S}(t)\,\Phi = \langle \gamma, \mathcal{S}(t)\Phi\rangle\, \mathcal{S}^*(t)\,\gamma \,.$$

By Lemma 5.4, for $t \in [0,T]$, $\mathcal{S}^*(t)\gamma$ remains in a fixed compact subset of V'. On the other hand for $t \in [0,T]$ and $\Phi = [\varphi^0, \varphi^1] \in \mathcal{H}$ in the unit ball of $\mathcal{H}$, $\langle \gamma, \mathcal{S}(t)\Phi\rangle$ remains in a bounded interval of $\mathbb{R}$. Therefore $\mathcal{S}^*(t)\,\mathcal{B}\mathcal{S}(t)\Phi$ remains in a fixed compact subset of V' and so does the integral $\int_0^T \mathcal{S}^*(t)\,\mathcal{B}\mathcal{S}(t)\Phi\,dt$. The conclusion follows easily. ∎

The following result is a natural generalization of Theorem 3.3 from [11].

Let us denote by $\mathcal{N}$ the kernel of $\mathcal{L}$ and let $\Phi_n = [\varphi_n^0, \varphi_n^1]$ be an orthonormal Hilbert basis of $\mathcal{N}^\perp$ in $\mathcal{H} := V\times H$ made of eigenvectors associated to the non-increasing sequence λ_n of eigenvalues of $\mathcal{L}$ repeated according to multiplicity. Then we have

Theorem 5.5. *In order for $[y^0, y^1] \in \mathcal{H}$ to be null-controllable under (5.1) at time T it is necessary and sufficient that the following set of two conditions is satisfied*

$$\forall [\phi^0, \phi^1] \in \mathcal{N}, \qquad (y^0, \phi^1) = (y^1, \phi^0)\ , \tag{5.11}$$

$$\sum_{n=1}^{\infty} \frac{\Big\{(y^0, \varphi_n^1) - (y^1, \varphi_n^0)\Big\}^2}{\lambda_n} < \infty\ . \tag{5.12}$$

When these conditions are fulfilled, an exact control driving $[y^0, y^1]$ to $[0,0]$ is given by the explicit formula

$$\gamma \sum_{n=1}^{\infty} \frac{(y^0, \varphi_n^1) - (y^1, \varphi_n^0)}{\lambda_n}\, \langle \gamma, \varphi_n(t)\rangle\ . \ \blacksquare \tag{5.13}$$

In the special case

$$H = L^2(\Omega)\,, \qquad \gamma = \delta(x - x_0), \quad x_0 \in \Omega$$

we obtain the point control problem

$$y'' + Ay = h(t)\,\delta(x - x_0) \quad \text{in} \;\; (0,T) \tag{5.14}$$

in time T by means of a control function $h \in L^2(0,T)$. Assuming

$$D(A^{\frac{1}{2}}) \,\subset\, C(\overline{\Omega}) \tag{5.15}$$

with continuous imbedding, we obtain

Corollary 5.6. *In order for* $[y^0, y^1] \in \mathcal{H} = D(A^{\frac{1}{2}}) \times L^2(\Omega)$ *to be null-controllable at* x_0 *at time* T *under (5.14) it is necessary and sufficient that (5.11) and (5.12) be satisfied. When these conditions are fulfilled, an exact control driving* $[y^0, y^1]$ *to* $[0,0]$ *is given by the explicit formula*

$$h(t) = \sum_{n=1}^{\infty} \frac{(y^0, \varphi_n^1) - (y^1, \varphi_n^0)}{\lambda_n}\; \varphi_n(t, x_0) \;. \blacksquare \tag{5.16}$$

Example 5.7. Let

$$\Omega = (0,\pi)\,, \qquad \xi \in \Omega \;.$$

We consider the problem

$$y_{tt} - y_{xx} = h(t)\,\delta(x - \xi)\,, \qquad y(t,0) = y(t,\pi) = 0 \;. \tag{5.17}$$

As a consequence of Corollary 5.6, a given state $[y^0, y^1] \in H_0^1(\Omega) \times L^2(\Omega)$ is null-controllable at $t = T$ if, and only if there exists $C \in \mathbb{R}^+$ such that

$$\forall [\varphi^0, \varphi^1] \in H_0^1(\Omega) \times L^2(\Omega)\,,$$
$$\left| \int_\Omega y^0(x)\,\varphi^1(x)\,dx \;-\; \int_\Omega y^1(x)\,\varphi^0(x)\,dx \right| \;\le\; C \left\{ \int_0^T \varphi^2(t,\xi)\,dt \right\}^{\frac{1}{2}}$$

where φ is the mild solution of

$$\varphi_{tt} - \varphi_{xx} = 0\,, \qquad \varphi(t,0) = \varphi(t,\pi) = 0\,, \qquad \varphi(0,.) = \varphi^0\,, \qquad \varphi_t(0,.) = \varphi^1 \;.$$

Here φ is given by

$$\varphi(t,x) \;=\; \sum_{m=1}^{\infty} \Big[c_m \cos mt + d_m \sin mt \Big] \sin mx$$

with

$$\varphi^0(x) = \sum_{m=1}^{\infty} c_m \sin mx\,, \qquad \varphi^1(x) = \sum_{m=1}^{\infty} d_m \sin mx$$

or in other terms

$$c_m = \frac{2}{\pi} \int_0^{\pi} \varphi^0(x) \sin mx\, dx\,, \qquad d_m = \frac{2}{\pi} \int_0^{\pi} \varphi^1(x) \sin mx\, dx\ .$$

If T is small, by the finite propagation property of the wave equation,there is in general an infinite-dimensional space of non-controllable states.

Especially interesting is the case

$$T = 2\pi\ .$$

Indeed then by periodicity we have

$$\forall[\varphi^0, \varphi^1] \in H_0^1(\Omega)\times L^2(\Omega)\ ,$$

$$\int_0^{2\pi} \varphi^2(t,\xi)\, dt \ = \int_0^{2\pi} \left\{ \sum_{m=1}^{\infty} \Big[c_m \cos mt + d_m \sin mt\Big] \sin m\xi \right\}^2 dt$$

$$= \pi \sum_{m=1}^{\infty} (c_m^2 + d_m^2) \sin^2 m\xi$$

and this implies that for any $m > 0$, $[\sin mx, 0]$ and $[0, \sin mx]$ are two eigenstates with eigenvalue

$$\gamma_m = \frac{2}{m^2}\ \sin^2 m\xi\ .$$

Applying Theorem 5.6, after some calculations we obtain that any $[y^0, y^1] \in H_0^1(\Omega)\times L^2(\Omega)$ is null-controllable at ξ int time $T = 2\pi$ if and only if

$$\forall m \in \mathbb{N}^*\,, \qquad \sin m\xi = 0 \implies y_m^0 = y_m^1 = 0$$

and

$$\sum_{\sin m\xi \neq 0} \frac{1}{\sin^2 m\xi} \left\{ m^2 (y_m^0)^2 + (y_m^1)^2 \right\} \ < \ \infty$$

with

$$y_m^0 = \frac{2}{\pi} \int_0^{\pi} y^0(x) \sin mx\, dx\,, \qquad y_m^1 = \frac{2}{\pi} \int_0^{\pi} y^1(x) \sin mx\, dx\ .$$

In such a case a control is given explicitely by

$$h(t) \ = \sum_{m=1}^{\infty} \frac{1}{2 \sin m\xi}\,(m y_m^0 \sin mt - y_m^1 \cos mt)\ . \ \square$$

Example 5.8. Let
$$\Omega = (0,\pi)\,, \qquad \xi \in \Omega\ .$$
We consider the problem

(5.18) $$y_{tt}+y_{xxxx} = h(t)\,\delta(x-\xi)\,, \quad y(t,0) = y(t,\pi) = y_{xx}(t,0) = y_{xx}(t,\pi) = 0\ .$$

As a consequence of Corollary 5.6, a given state $[y^0,y^1] \in H_0^1(\Omega)\times L^2(\Omega)$ is null-controllable under (5.18) at $t=T$ if, and only if there exists $C \in \mathbb{R}^+$ such that

$$\forall[\varphi^0,\varphi^1] \in H^2\cap H_0^1(\Omega)\times L^2(\Omega)\ ,$$
$$\left|\int_\Omega y^0(x)\,\varphi^1(x)\,dx \;-\; \int_\Omega y^1(x)\,\varphi^0(x)\,dx\right| \;\le\; C\left\{\int_0^T \varphi^2(t,\xi)\,dt\right\}^{\frac12}$$

where φ is the mild solution of

$$\varphi_{tt}+\varphi_{xxxx} = 0\,, \qquad \varphi(t,0) = \varphi(t,\pi) = \varphi_{xx}(t,0) = \varphi_{xx}(t,\pi) = 0$$

such that
$$\varphi(0,.) = \varphi^0\,, \qquad \varphi_t(0,.) = \varphi^1\ .$$

Here φ is given by

$$\varphi(t,x) \;=\; \sum_{m=1}^\infty \Big[c_m\cos m^2t + d_m\sin m^2t\Big]\sin mx$$

with
$$\varphi^0(x) \;=\; \sum_{m=1}^\infty c_m\sin mx\,, \qquad \varphi^1(x) \;=\; \sum_{m=1}^\infty d_m\sin mx$$

or in other terms
$$c_m = \frac{2}{\pi}\int_0^\pi \varphi^0(x)\sin mx\,dx\,, \qquad d_m = \frac{2}{\pi}\int_0^\pi \varphi^1(x)\sin mx\,dx\ .$$

Applying Theorem 5.6, after some calculations we obtain that any $[y^0,y^1] \in H_0^1(\Omega)\times L^2(\Omega)$ is null-controllable at ξ in time $T=2\pi$ under (5.18) if and only if

$$\forall m\in\mathbb{N}^*\,, \qquad \sin m\xi = 0 \implies y_m^0 = y_m^1 = 0$$

and
$$\sum_{\sin m\xi\neq 0} \frac{1}{\sin^2 m\xi}\Big\{m^4(y_m^0)^2 + (y_m^1)^2\Big\} \;<\; \infty$$

with
$$y_m^0 = \frac{2}{\pi}\int_0^\pi y^0(x)\sin mx\,dx\,, \qquad y_m^1 = \frac{2}{\pi}\int_0^\pi y^1(x)\sin mx\,dx\ .$$

In such a case a control is given explicitely by

$$h(t) = \sum_{m=1}^{\infty} \frac{1}{2\sin m\xi} \,(m^2 y_m^0 \sin mt - y_m^1 \cos mt) \;. \;\square$$

We conclude this section with an example which is available in any domain. This case has been considered by Graham and Russell in [2]. In the case

$$H = L^2(\Omega)\,, \qquad \gamma = \chi_\omega$$

we obtain the point control problem

$$y'' + Ay = h(t)\,\chi_\omega(x) \quad \text{in} \;\; (0,T) \tag{5.19}$$

in time T by means of a control function $h \in L^2(0,T)$. We obtain

Corollary 5.9. *In order for $[y^0, y^1] \in \mathcal{H} = D(A^{\frac{1}{2}}) \times L^2(\Omega)$ to be null-controllable at time T under (5.19) it is necessary and sufficient that (5.11) and (5.12) be satisfied. When these conditions are fulfilled, an exact control driving $[y^0, y^1]$ to $[0,0]$ is given by the explicit formula*

$$h(t) = \sum_{n=1}^{\infty} \frac{(y^0, \varphi_n^1) - (y^1, \varphi_n^0)}{\lambda_n} \int_\omega \varphi_n(t,x)\,dx \;. \;\blacksquare \tag{5.20}$$

6 – Boundary control of the wave equation

In this section, we consider the real Hilbert space $H = L^2(\Omega)$ where Ω is a bounded domain of $\mathbb{R}^N$ and we set $V = H_0^1(\Omega)$, $V' = H^{-1}(\Omega)$. We consider the wave equation

$$\varphi_{tt} - \Delta\varphi = 0 \;\; \text{in} \;\; \mathbb{R}\times\Omega\,, \qquad \varphi = 0 \;\; \text{on} \;\; \mathbb{R}\times\partial\Omega \tag{6.1}$$

and the boundary control problem

$$y_{tt} - \Delta y = 0 \;\; \text{in} \;\; (0,T)\times\Omega\,, \qquad y = Bh(t,\sigma) \;\; \text{on} \;\; (0,T)\times\partial\Omega \tag{6.2}$$

in time T by means of a control function

$$h \in L^2(0,T,L^2(\Gamma))$$

with

$$B \in \mathcal{L}(L^2(\Gamma), L^2(\Gamma)) , \qquad B = B^* \geq 0 . \tag{6.3}$$

In this section we shall represent a pair of functions by $[f, g]$ rather than (f, g) to avoid confusion with scalar products. On the other hand the symbol (f, g) will represent indifferently either the H-inner product of $f \in H$ and $g \in H$ or the duality product $(f, g)_{V,V'}$ when $f \in V$ and $g \in V'$, these two products being equal when $f \in V$ and $g \in H$. The main result of this section is the following

Theorem 6.1. *For any* $[y^0, y^1] \in V \times H$, *the two following conditions are equivalent*

i) *There exists* $h \in L^2(0, T; L^2(\Gamma))$ *such that the mild solution* y *of (6.2) such that* $y(0) = y^0$ *and* $y'(0) = y^1$ *satisfies* $y(T) = y'(T) = 0$.

ii) *There exists a finite positive constant* C *such that*

$$\forall [\varphi^0, \varphi^1] \in V \times H , \quad \left|(y^0, \varphi^1) - (y^1, \varphi^0)\right| \leq C \left\{ \int_0^T \!\! \int_\Gamma \left| B \frac{\partial \varphi}{\partial \nu}(t, \sigma) \right|^2 dt \, d\sigma \right\}^{\frac{1}{2}} \tag{6.4}$$

where $\varphi(t) \in C(\mathbb{R}, V) \cap C^1(\mathbb{R}, H)$ *denotes the unique mild solution of (6.1) such that* $\varphi(0) = \varphi^0$ *and* $\varphi'(0) = \varphi^1$.

Proof: It parallels the proof of theorem 2.1.

Step 1. Let φ and y be a pair of strong solutions of (6.1) and (6.2), respectively. We have

$$\begin{aligned} \frac{d}{dt}(y'(t), \varphi(t)) &= (y''(t), \varphi(t)) + (y'(t), \varphi'(t)) \\ &= (\Delta y(t), \varphi(t)) + (y'(t), \varphi'(t)) . \end{aligned}$$

On the other hand

$$\begin{aligned} \frac{d}{dt}(y(t), \varphi'(t)) &= (y(t), \varphi''(t)) + (y'(t), \varphi'(t)) \\ &= (y(t), \Delta\varphi(t)) + (y'(t), \varphi'(t)) . \end{aligned}$$

By substracting these two identities we find

$$\frac{d}{dt}\Big[(y'(t), \varphi(t)) - (y(t), \varphi'(t))\Big] = \int_\Omega (\varphi \, \Delta y - y \, \Delta\varphi) \, dx = \int_\Gamma \left(\varphi \, \frac{\partial y}{\partial \nu} - y \, \frac{\partial \varphi}{\partial \nu} \right) d\sigma .$$

By integrating on (0, T) and using $\varphi = 0$ on $\mathbb{R}\times\partial\Omega$ we get

$$\text{(6.5)}\qquad \Big[(y'(t),\varphi(t)) - (y(t),\varphi'(t))\Big]_0^T \;=\; -\int_0^T\!\!\int_\Gamma Bh(t,\sigma)\,\frac{\partial\varphi}{\partial\nu}(t,\sigma)\;d\sigma\,dt\;.$$

By density and as a consequence of the so-called "hidden regularity property" (cf. e.g. [16, 19]), this identity is valid for mild solutions as well. Since B is bounded, self-adjoint and $B \geq 0$,

$$\int_0^T\!\!\int_\Gamma Bh(t,\sigma)\,\frac{\partial\varphi}{\partial\nu}(t,\sigma)\;d\sigma\,dt \;=\; \int_0^T\!\!\int_\Gamma h(t,\sigma)\,B\frac{\partial\varphi}{\partial\nu}(t,\sigma)\;d\sigma\,dt\;.$$

Finally if there exists $h \in L^2(0,T;L^2(\Gamma))$ such that the mild solution y of (6.2) with $[y(0),y'(0)] = [y^0,y^1]$ satisfies $y(T) = y'(T) = 0$, we find as a consequence of (6.5)

$$(y^0,\varphi'(0)) - (y^1,\varphi(0)) \;=\; -\int_0^T\!\!\int_\Gamma h(t,\sigma)\,B\frac{\partial\varphi}{\partial\nu}(t,\sigma)\;d\sigma\,dt$$

and by the Cauchy–Schwartz inequality we obtain (2.4). Therefore i) implies ii).

Step 2. For each $\varepsilon > 0$ we construct a bounded linear operator

$$\mathcal{L}_\varepsilon \;\in\; \mathcal{L}(V\times H, V'\times H)$$

in the following way: for any $[\varphi^0,\varphi^1] \in V\times H := \mathcal{H}$ we consider first the solution φ of (2.1) with initial data $[\varphi^0,\varphi^1]$. Then we consider the unique mild solution y of

$$y_{tt} - \Delta y = -\varepsilon\,\Delta\varphi \quad\text{in }\;(0,T)\times\Omega\,,\qquad y = -B^2\frac{\partial\varphi}{\partial\nu}\quad\text{on }\;(0,T)\times\partial\Omega\;,$$

$$y(T) = y'(T) = 0$$

and we set

$$\mathcal{L}_\varepsilon([\varphi^0,\varphi^1]) = [-y'(0), y(0)]$$

We find

$$\Big\langle \mathcal{L}_\varepsilon([\varphi^0,\varphi^1]),[\varphi^0,\varphi^1]\Big\rangle_{\mathcal{H}',\mathcal{H}} \;=\; (y(0),\varphi'(0)) - \langle y'(0),\varphi(0)\rangle$$
$$=\; \int_0^T\!\!\int_\Gamma B^2\,\frac{\partial\varphi}{\partial\nu}.\frac{\partial\varphi}{\partial\nu}(t,\sigma)\;d\sigma\,dt \;+\varepsilon\int_0^T |A^{\frac12}\varphi(t)|^2\,dt\;.$$

With $A=-\Delta$. On the other hand for any $T>0$

$$\int_0^T |A^{\frac{1}{2}}\varphi(t)|^2\,dt \;\geq\; c(T)\left\{|A^{\frac{1}{2}}\varphi(0)|^2+|\varphi'(0)|^2\right\} \;=\; c(T)\left\{|\varphi^0|_V^2+|\varphi^1|^2\right\}$$

with $c(T)>0$. Hence $\mathcal{L}_\varepsilon$ is coercive: $V\times H\to V'\times H$, and this implies $\mathcal{L}_\varepsilon(V\times H)=V'\times H$.

Step 3. As a consequence of step 2 there exists a pair $[\varphi^{0,\varepsilon},\varphi^{1,\varepsilon}]\in H\times V'$ such that the mild solution y_ε of

$$y_{tt}-\Delta y=-\varepsilon\,\Delta\varphi_\varepsilon \quad \text{in } (0,T)\times\Omega\,, \qquad y=-B^2\frac{\partial\varphi_\varepsilon}{\partial\nu} \quad \text{on } (0,T)\times\partial\Omega\ ,$$

with

$$[y_\varepsilon(0),y_\varepsilon'(0)]=[y^0,y^1]$$

satisfies

$$y(T)=y'(T)=0\ .$$

We find

$$\begin{aligned}(y^0,\varphi_\varepsilon'(0))-(y^1,\varphi_\varepsilon(0)) \;&=\; \int_0^T\!\!\int_\Gamma \left|B\frac{\partial\varphi_\varepsilon}{\partial\nu}(t,\sigma)\right|^2 d\sigma\,dt \,+\, \varepsilon\int_0^T |A^{\frac{1}{2}}\varphi_\varepsilon(t)|^2\,dt\\ &\leq\; C\left\{\int_0^T\!\!\int_\Gamma \left|B\frac{\partial\varphi_\varepsilon}{\partial\nu}(t,\sigma)\right|^2 d\sigma\,dt\right\}^{\frac{1}{2}}.\end{aligned}$$

In particular

$$\int_0^T\!\!\int_\Gamma \left|B\frac{\partial\varphi_\varepsilon}{\partial\nu}(t,\sigma)\right|^2 d\sigma\,dt \,+\, \varepsilon\int_0^T |A^{\frac{1}{2}}\varphi_\varepsilon(t)|^2\,dt \;\leq\; C^2\ .$$

Step 4. Convergence along a subsequence. From step 3 it is clear that

$$\sqrt{\varepsilon}\,\varphi_\varepsilon \quad \text{is bounded in } \ L^2(0,T;V')$$

and

$$h_\varepsilon=B\frac{\partial\varphi_\varepsilon}{\partial\nu} \quad \text{is bounded in } \ L^2(0,T;L^2(\Gamma))\ .$$

Along a subsequence, we may assume

$$h_\varepsilon \rightharpoonup h \quad \text{weakly in } \ L^2(0,T;L^2(\Gamma))\ .$$

Then clearly

$$B^2 \frac{\partial \varphi_\varepsilon}{\partial \nu} \rightharpoonup Bh \quad \text{weakly in} \quad L^2(0,T; L^2(\Gamma)) \ .$$

Step 5. Conclusion. By passing to the limit, it is clear that the solution y of (6.2) with $[y(0), y'(0)] = [y^0, y^1]$ and h as in step 4 satisfies $y(T) = y'(T) = 0$. The proof of Theorem 6.1 is now complete. ∎

We now state a variant of Theorem 4.1 devised for the case of boundary control.

Theorem 6.2. *Let $[\varphi^0, \varphi^1] \in D(A) \times V$ be such that for some $\lambda > 0$*

$$\text{(6.6)} \quad \forall [\psi^0, \psi^1] \in D(A) \times V \,, \quad \int_0^T \!\! \int_\Gamma \mathcal{B} \frac{\partial \varphi}{\partial \nu} \,.\, \mathcal{B} \frac{\partial \psi}{\partial \nu} \, d\sigma = \lambda \Big[(A\varphi^0, A\psi^0) + \langle A\varphi^1, \psi^1 \rangle \Big]$$

where φ and ψ are the solutions of (6.1) with respective initial data $[\varphi^0, \varphi^1]$ and $[\psi^0, \psi^1]$. Then the solution y of

$$y_{tt} - \Delta y = 0 \quad \text{in} \ (0,T) \times \Omega \,, \qquad y = -\frac{1}{\lambda} \mathcal{B}^2 \frac{\partial \varphi}{\partial \nu} \quad \text{on} \ (0,T) \times \partial \Omega$$

$$\text{(6.7)} \qquad y(0) = A\varphi^1 \,, \qquad y'(0) = -A^2 \varphi^0$$

satisfies $y(T) = y'(T) = 0$.

Proof: Essentially identical to that of Theorem 4.1. For the details cf. [11], proposition 2.2. ∎

The following result is a natural generalization of Theorem 2.3 from [11]. First we define $\mathcal{V} = D(A) \times V$ and $\mathcal{L} \in \mathcal{L}(\mathcal{V})$ by the formula

$$\forall [\varphi^0, \varphi^1] \in \mathcal{V}, \ \ \forall [\psi^0, \psi^1] \in \mathcal{V} \,, \quad \Big\langle \mathcal{L}([\varphi^0, \varphi^1]); [\psi^0, \psi^1] \Big\rangle_{\mathcal{V}} = \int_0^T \!\! \int_\Gamma \mathcal{B} \frac{\partial \varphi}{\partial \nu} \,.\, \mathcal{B} \frac{\partial \psi}{\partial \nu} \, d\sigma \ .$$

By the standard trace theorem, $\mathcal{L} \colon \mathcal{V} \to \mathcal{V}$ is *compact*. Let us denote by $\mathcal{N}$ the kernel of $\mathcal{L}$ and let $\Phi_n = [\varphi_n^0, \varphi_n^1]$ be an orthonormal Hilbert basis of $\mathcal{N}^\perp$ in $\mathcal{H} := V \times H$ made of eigenvectors associated to the non-increasing sequence λ_n of eigenvalues of $\mathcal{L}$ repeated according to multiplicity. Then we have

Theorem 6.3. *In order for $[y^0, y^1] \in H \times V'$ to be null-controllable under (6.2) at time T it is necessary and sufficient that the following set of two conditions is satisfied*

$$\text{(6.7)} \qquad \forall [\phi^0, \phi^1] \in \mathcal{N} \,, \qquad (y^0, \phi^1) = \langle y^1, \phi^0 \rangle \ ,$$

$$\sum_{n=1}^{\infty} \frac{\Big\{(y^0, \varphi_n^1) - \langle y^1, \varphi_n^0\rangle\Big\}^2}{\lambda_n} \;<\; \infty \;. \tag{6.8}$$

When these conditions are fulfilled, an exact control driving $[y^0, y^1]$ *to* $[0,0]$ *is given by the explicit formula*

$$h(t,\sigma) \;=\; -\sum_{n=1}^{\infty} \frac{(y^0, \varphi_n^1) - \langle y^1, \varphi_n^0\rangle}{\lambda_n}\; B\frac{\partial \varphi_n}{\partial \nu} \;. \tag{6.9}$$

Proof: Since it is a straightforward generalization of Theorem 2.3 from [11] and we already gave many similar arguments in this paper, the details are left to the reader. ∎

We conclude by recalling an example from [11].

Example 6.4. Let

$$\Omega = (0,\pi) \;.$$

We consider the problem

$$y_{tt} - y_{xx} = 0\,, \qquad y(t,0) = h(t), \quad y(t,\pi) = 0 \;. \tag{6.10}$$

For any $T \geq 2\pi$ and any $[y^0, y^1] \in H \times V' = L^2(\Omega) \times H^{-1}(\Omega)$ there exists $h \in L^2(0,T)$ such that the solution y of (6.10) with

$$y(0) = y^0\,, \qquad y_t(0) = y^1$$

satisfies $y(T) = y_t(T) = 0$.

In the special case

$$T = 2\pi$$

a control h is given explicitely by

$$h(t) \;=\; \frac{1}{2} \sum_{m=1}^{\infty} \Big(y_m^0 \sin mt - \frac{1}{m}\, y_m^1 \cos mt \Big)$$

with

$$y_m^0 = \frac{2}{\pi} \int_0^{\pi} y^0(x) \sin mx\, dx\,, \qquad y_m^1 = \frac{2}{\pi} \Big\langle y^1(x),\, \sin mx \Big\rangle_{V',V} \;. \;∎$$

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