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EVERY NILPOTENT OPERATOR FAILS TO DETERMINE THE COMPLETE NORM TOPOLOGY

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Abstract: We show that every nilpotent operator T, on an infinite-dimensional Banach space X, provides a decomposition of X into a direct sum of a finite number of subspaces with sufficiently connections. Finally we construct a complete norm on X that makes T continuous and not equivalent to the original norm on X.

1 – Introduction

The uniqueness of norm problem was initiated fifty years ago by C.E. Rickart [4] and his results were complemented by B.E. Johnson [3]. But the investigation of operators determining the complete norm topology in the context of Banach algebra is recent. It starts with the work of A.R. Villena [5] primarily for $C(K)$ spaces and uniform algebras, where it was shown that for a compact Hausdorff space K without isolated points and $f \in C(K)$, every complete norm on $C(K)$ which makes continuous the multiplication by f is equivalent to the supremum norm if and only if $\{\omega \in K \colon\ f(\omega) = \lambda\}$ has no interior points whenever λ lies in $\mathbb{C}$. Later K. Jarosz [1] generalizes the result of Villena by extending it to a larger class of algebras. Further interesting results have recently been established in [2].

This leaves the more general problem of investigating those bounded linear operators T on an infinite-dimensional Banach space $(X, \|\cdot\|)$, for which every complete norm $|\cdot|$ on X making continuous the operator T from $(X, |\cdot|)$ into

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$(X, |\cdot|)$ is automatically equivalent to $\|\cdot\|$. Such an operator is said to *determine the complete norm topology* of X. In the present paper, we will prove that a nilpotent bounded linear operator does not determine the complete norm topology of an arbitrary infinite-dimensional Banach space. Moreover we construct a complete norm on X making the operator continuous and not equivalent to the original norm.

2 – The result

In the following, let $(X, \|\cdot\|)$ be an infinite-dimensional Banach space over $\mathbb{R}$ or $\mathbb{C}$, and let $BL(X)$ denote the Banach algebra consisting of all bounded linear operators on X.

We begin with the following useful result on algebraically complementary linear subspaces of a Banach space.

Lemma 1. *Let $\{Y, Z\}$ be a pair of algebraically complementary linear subspaces of X such that Y is closed. If for $y \in Y$ and $z \in Z$ we define $|y+z| = \|y\| + \|z+Y\|$, then*

(i) *$|\cdot|$ is a complete norm on X;*

(ii) *$|\cdot|$ is equivalent to $\|\cdot\|$ if and only if Z is closed.*

Proof: (i) Clearly $|\cdot|$ is a norm in X. Let $\{x_n\}_n$ be any $|\cdot|$-Cauchy sequence in X. Then there exist $\|\cdot\|$-Cauchy sequences $\{y_n\}_n$ and $\{z_n+Y\}_n$ respectively in Y and X/Y such that $x_n = y_n + z_n$ for $n \in \mathbb{N}$. Since Y is closed, there exist $y \in Y$ and $u \in X$ such that $\{y_n\}_n$ converges to y and $\{z_n+Y\}_n$ converges to $u+Y$. Now it suffices to choose an element $z \in Z$ such that $u+Y{=}z+Y$ to obtain that $\{x_n\}_n$ is $|\cdot|$-convergent to $x = y+z$.

(ii) Suppose that the conditions of the lemma are fulfilled. Then Z is closed if and only if there exists a projection operator $p \in L(X)$ such that $\operatorname{Im} p = Y$. If $|\cdot|$ is equivalent to $\|\cdot\|$, then there exists $\alpha > 0$ such that $|\cdot| \le \alpha\|\cdot\|$. For $x = y+z$ with $y \in Y$ and $z \in Z$, set $p(x) = y$. Then is a linear operator defined on X, p is idempotent and $\operatorname{Im} p = Y$, $\ker p = Z$. Futhermore, $p \in BL(X)$, since

$$\|p(y+z)\| \;=\; \|y\| \;\le\; |y+z| \;\le\; \alpha\|y+z\|$$

that is, p is the bounded projection operator of X onto Y along Z. Thus $Z = \ker p$ is closed. Conversely, if Z is closed then it is easy to check that p is continuous and consequently Y and Z are now topologically complementary. Therefore there exists $\alpha > 0$ such that $\|y\| + \|z\| \leq \alpha \|y+z\|$ for every $y \in Y$ and $z \in Z$; hence

$$|y+z| \,=\, \|y\| + \|z+Y\| \,\leq\, \|y\| + \|z\| \,\leq\, \alpha \|y+z\| \;,$$

which means that $|\cdot|$ is equivalent to $\|\cdot\|$. ∎

In the same vein, we obtain the following result.

Lemma 2. *Let $\{Y_i\}_{i=1}^n$ be a family of closed subspaces of X and $\{X_i\}_{i=1}^n$ be a family of subspaces of X such that*

(i) $Y_1 \subsetneq Y_2 \subsetneq \cdots Y_{n-1} \subsetneq Y_n = X$;

(ii) $Y_{i+1} = Y_i \oplus X_{i+1}, \;\; 1 \leq i \leq n-1$;

(iii) $X_1 = Y_1$.

Then $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ and the norm on X defined by

$$|x_1 + x_2 + \cdots + x_n| \,=\, \|x_1\| + \sum_{i=2}^{n} \|x_i + Y_{i-1}\|$$

is complete. Actually $|\cdot|$ is equivalent to $\|\cdot\|$ if and only if each X_i is closed for $1 \leq i \leq n$.

Proof: Let $\{x_k\}_k$ be a $|\cdot|$-Cauchy sequence in X. Then there exist sequences $\{x_{k,i}\}_k$ in X_i such that $\{x_{k,1}\}_k$ is a $\|\cdot\|$-Cauchy sequence in X_1 and $\{x_{k,i+1} + Y_i\}_k$ is a $\|\cdot\|$-Cauchy sequence in Y_{i+1}/Y_i for all $i \in \{1,2,\ldots,n-1\}$. Since $X_1 = Y_1$ there exist $x_1 \in X_1$ and $u_{i+1} \in Y_{i+1}$ such that $\{x_{k,1}\}$ is $\|\cdot\|$-convergent to $x_1 \in X_1$ and $\{x_{k,i+1}\}_k$ is $\|\cdot\|$-convergent to $u_{i+1} + Y_i$ for all $i \in \{1,2,\ldots,n-1\}$. Finally by the argument used in Lemma 1, there exist $x_i \in X_i$ $(2 \leq i \leq n)$ such that $u_{i+1} + Y_i = x_{i+1} + Y_i$ for all $i \in \{1,2,\ldots,n-1\}$. Hence, the sequence $\{x_k\}_k$ is $|\cdot|$-convergent to the element $x_1 + x_2 + \cdots + x_n$, which means that $|\cdot|$ is complete on X.

Now if $|\cdot|$ is equivalent to $\|\cdot\|$, we suppose inductively that $X_1, X_2, \ldots, X_i$ are closed and we prove that X_{i+1} is also closed. Then there exists $\alpha > 0$ such that for all $x_j \in X_j$, $(1 \leq j \leq i)$

$$\|x_1\| + \|x_2\| + \cdots + \|x_i\| \,\leq\, \alpha \|x_1 + x_2 + \cdots + x_i\| \;,$$

then

$$\|x_1\| + \|x_2 + Y_1\| + \cdots + \|x_i + Y_{i-1}\| \,\leq\, \alpha \|x_1 + x_2 + \cdots + x_i\| \;,$$

hence, if $\beta = \max(1, \alpha)$, we obtain

$$|x_1 + x_2 + \cdots + x_i + x_{i+1}| \le \beta\Big(\|x_1 + x_2 + \cdots + x_i\| + \|x_{i+1} + Y_i\|\Big) .$$

Since by hypothesis $\|\cdot\|$ is equivalent to $|\cdot|$, Lemma 1 implies X_{i+1} is closed, which proves that X_i is closed for all $i \in \{1, 2, \ldots, n\}$. Now suppose that all the subspaces X_i $(1 \le i \le n)$ are closed. In this case the direct sum $X = X_1 \oplus X_2 \oplus \cdots \oplus X_n$ is topologic and then there exists $\alpha > 0$ such that for all $x_i \in X_i$, $(1 \le i \le n)$

$$\|x_1\| + \|x_2\| + \cdots + \|x_n\| \le \alpha\|x_1 + x_2 + \cdots + x_n\| ,$$

then

$$|x_1 + x_2 + \cdots + x_n| \le \|x_1\| + \|x_2\| + \cdots + \|x_n\| \le \alpha\|x_1 + x_2 + \cdots + x_n\| .$$

Hence $|\cdot|$ and $\|\cdot\|$ are equivalent. ∎

The following shows that a nilpotent operator provide an interesting decomposition of an infinite-dimensional Banach space.

Proposition 3. *Let $T \in BL(X)$ such that $T^{n+1} = 0$ and $T^n \ne 0$. Then there exists a family $\{X_i\}_{i=1}^{n+1}$ of subspaces of X such that:*

- **(i)** $X = X_1 \oplus X_2 \oplus \cdots \oplus X_{n+1}$, $X_1 = \ker T$;
- **(ii)** $T(X_{i+1}) \subseteq X_i$, $1 \le i \le n$;
- **(iii)** $\ker T^{i+1} = \ker T^i \oplus X_{i+1}$, $1 \le i \le n$.

Proof: Let $X = \ker T^n \oplus X_{n+1}$, where X_{n+1} is an algebraic complement of $\ker T^n$. It is easy to check that

$$T(X_{n+1}) \subseteq \ker T^n \quad \text{and} \quad T(X_{n+1}) \cap \ker T^{n-1} = \{0\} .$$

Then there exists a subspace X_n of X such that

$$T(X_{n+1}) \subseteq X_n \quad \text{and} \quad \ker T^n = \ker T^{n-1} \oplus X_n .$$

We have also $T(X_n) \cap \ker T^{n-2} = \{0\}$ and $T(X_n) \subseteq \ker T^{n-1}$, hence there exists a subspace X_{n-1} of X satisfying $T(X_n) \subseteq X_{n-1}$ and $\ker T^{n-1} = \ker T^{n-2} \oplus X_{n-1}$. Further application of the arguments used above gives a family $\{X_i\}_{i=2}^{n+1}$ of subspaces of X with the desired properties. ∎

We are now in a position to establish our main result.

Theorem 4. *A nilpotent operator on a Banach space does not determine the complete norm topology of the Banach space.*

Proof: Let $(X, \|\cdot\|)$ be a Banach space and $T \in BL(X)$ a nilpotent operator such that $T^{n+1} = 0$ and $T^n \neq 0$. Then there exists a family $\{X_i\}_{i=2}^{n}$ of subspaces of X verifying the properties of Proposition 3.

First case. Suppose that there exist $i_0 \in \{2, \ldots, n\}$ such that X_{i_0} is not closed, then since $X = \ker T \oplus X_2 \oplus \cdots \oplus X_{n+1}$, the norm $|\cdot|$ on X given by

$$|x_1 + x_2 + \cdots + x_{n+1}| \;=\; \|x_1\| + \sum_{i=2}^{n+1} \|x_i + \ker T^i\|$$

is a complete norm on X not equivalent to $\|\cdot\|$. However for all $x_i \in X_i$, $(1 \leq i \leq n{+}1)$ we have

$$\begin{aligned} |T(x_1 + x_2 + \cdots + x_{n+1})| \;&=\; |T(x_2) + T(x_3) + \cdots + T(x_{n+1})| \\ &=\; \|T(x_2)\| \;+ \sum_{i=3}^{n+1} \|T(x_i) + \ker T^{i-2}\| \;, \end{aligned}$$

and for $3 \leq i \leq n+1$ we have

$$\|T(x_i) + \ker T^{i-2}\| \;\leq\; \|T(x_i) + T(x)\| \;\leq\; \|T\|\,\|x_i + x\| \qquad \forall\, x \in \ker T^{i-1} \;;$$

hence

$$\|T(x_i) + \ker T^{i-2}\| \;\leq\; \|T\|\,\|x_i + \ker T^{i-1}\| \quad \text{ for } \; 3 \leq i \leq n{+}1 \;.$$

Therefore

$$\begin{aligned} |T(x_1 + x_2 + \cdots + x_{n+1})| \;&\leq\; \|T\| \left(\|x_2 + \ker T\| + \sum_{i=3}^{n+1} \|x_i + \ker T^{i-1}\| \right) \\ &\leq\; \|T\|\,|x_1 + x_2 + \cdots + x_{n+1}| \;. \end{aligned}$$

Then T is continuous from $(X, |\cdot|)$ to $(X, |\cdot|)$ and so T doesn't determine the complete norm topology of X.

Second case. Suppose that X_i is closed for all $i \in \{1, 2, \ldots, n{+}1\}$. Since $\ker T$ is infinite-dimensional, one can find a discontinuous linear functional φ_1 on $\ker T$

and $z \in X_{n+1}$ such that $\varphi_1(T^n(z)) = 1$. For $i \in \{2, 3, \ldots, n+1\}$, let φ_i be the linear functional on X_i given by

$$\varphi_{i+1} = \varphi_i \circ T_{|X_{i+1}}, \quad 1 \le i \le n ,$$

and $F_i \in L(X)$ defined by

$$F_i(x) = x - 2\,\varphi_i(x)\,T^{n-(i-1)}(z) \quad \forall\, x \in X_i, \;\; 1 \le i \le n .$$

It is easy to show that $\varphi_i(T^{n-(i-1)}(z)) = 1$ for all $i = 2, \ldots, n$. We claim that for all $1 \le i \le n$, F_i defines a linear bijection from X_i onto itself with $F_i^{-1} = F_i$. Indeed, fix i in $\{1, 2, \ldots, n\}$ and let x be an element in X_i such that $F_i(x) = x - 2\,\varphi_i(x)\,T^{n-(i-1)}(z) = 0$. Then applying φ_i, we obtain

$$0 = \varphi_i(x) - 2\,\varphi_i(x)\,\varphi_i(T^{n-(i-1)}(z)) = -\varphi_i(x) ,$$

and so $x = 0$. Now if $y = F_i(x) = x - 2\,\varphi_i(x)\,T^{n-(i-1)}(z)$, arguing as before, we deduce that $\varphi_i(y) = -\varphi_i(x)$ and that $x = y - 2\,\varphi_i(y)\,T^{n-(i-1)}(z)$ which is our claim. Consider for $1 \le i \le n+1$, the norms N_i on X_i and N on X defined by

$$N_i(x) = \|x - 2\,\varphi_i(x)\,T^{n-i+1}(z)\| \quad \forall\, x \in X_i ,$$

and

$$N(x_1 + x_2 + \cdots + x_{n+1}) = \sum_{i=1}^{n+1} N_i(x_i) \quad \forall\, x_i \in X_i, \;\; 1 \le i \le n+1 .$$

Since, the norm $\|\cdot\|$ is complete and for $1 \le i \le n$, F_i is an isometry from (X_i, N_i) onto $(X_i, \|\cdot\|)$, the norm N_i is also complete. Now

$$\begin{aligned} N\Big(T(x_1 + x_2 + \cdots + x_{n+1})\Big) &= N\Big(T(x_2) + \cdots + T(x_{n+1})\Big) \\ &= N_1(T(x_2)) + \cdots + N_n(T(x_{n+1})) . \end{aligned}$$

Since

$$\begin{aligned} N_i(T(x_{i+1})) &= \Big\|T(x_{i+1}) - 2\,\varphi_i\,(T(x_{i+1}))\,T^{n-i+1}(z)\Big\| \\ &= \Big\|T(x_{i+1}) - 2\,\varphi_{i+1}\,(x_{i+1})\,T^{n-i+1}(z)\Big\| \\ &\le \|T\|\,\Big\|x_{i+1} - 2\,\varphi_{i+1}\,(x_{i+1})\,T^{n+1-(i+1)}(z)\Big\| \\ &\le \|T\|\,N_{i+1}(x_{i+1}) , \end{aligned}$$

we obtain $N(T(x)) \leq \|T\| N(x)$. Hence T is continuous from (X, N) to (X, N), but since φ_1 is discontinuous, N_1 is not equivalent to $\|\cdot\|$ on $\ker T$ and then N is not equivalent to $\|\cdot\|$ on X, which shows that T does not determine the complete norm topology of X. ∎

It would be desirable to know if every bounded quasi-nilpotent operator fails to determine the complete norm topology of an infinite-dimensional Banach space.

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REFERENCES

[**1**] Jarosz, K. – Linear maps determining the norm topology, *Trans. Amer. Math. Soc.,* 353(2) (2000), 723–731.
[**2**] Jarosz, K. – Uniqueness of translation invariant norms, *J. Funct. Anal.,* 174(2) (2000), 417–429.
[**3**] Johnson, B.E. – The uniqueness of the (complete) norm topology, *Bull. Amer. Math. Soc.,* 73 (1967), 537–539.
[**4**] Rickart, C.E. – The uniqueness of norm problem in Banach algebras, *Ann. of Math.,* 51(2) (1950), 615–628.
[**5**] Villena, A.R. – Operators determining the the complete norm topology of $\mathcal{C}(\mathcal{K})$, *Studia Math.,* 124(2) (1997), 155–160.

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